Add 1st Order Examples

\[ \frac{dy}{dx} = u - u^2, \quad u(0) = 1 \]

\[ \int_{u=1}^{u} \frac{du}{u(u+1)} = \int_{x=0}^{x} \frac{dx}{x} \]

Use partial fractions to integrate

\[ \frac{1}{u(u+1)} = \frac{1}{u} - \frac{1}{u+1} \]

So

\[ \int \frac{du}{u(u+1)} = \log \left( \frac{u}{u+1} \right) \]

Near \( u = 1 \), \( \frac{u}{u+1} > 0 \)

So as long as \( u > 0 \) we have

\[ \log \left( \frac{u}{u+1} \right) - \log \left( \frac{1}{u+1} \right) = x - 0 \]

Solve for \( u \)

\[ u = \frac{e^x}{2 - e^{-x}} \quad \text{(over)} \]
Notice that this solution is pos for all \( x \leq \log(2) \). It does "blow up" as \( x \) approaches \( \log 2 \). No need to investigate the case when \( u < 0 \).

\[
2 \frac{du}{dx} = \sqrt{|u|} \quad \text{with} \quad u(0) = 0
\]

\[
\int_{0}^{u} \frac{du}{\sqrt{|u|}} = \int_{0}^{x} dx
\]

For \( u > 0 \)
\[
\int_{0}^{u} \frac{du}{\sqrt{u}} = \int_{0}^{x} \frac{du}{\sqrt{u}} = 2\sqrt{u}
\]

For \( u < 0 \)
\[
\int_{0}^{u} \frac{du}{\sqrt{-u}} = \int_{0}^{x} \frac{du}{\sqrt{-u}} = -2\sqrt{-u}
\]

So in case (1) we get \( u = \left(\frac{x}{2}\right)^2 \) (over) and in case (2) we get \( u = -\left(\frac{x}{2}\right)^2 \)
In case 1 \( \frac{dy}{dx} = \frac{x}{2} \) and \( \sqrt{|y|} = \frac{|x|}{2} \)
So this is only valid when \( x > 0 \)

In case 2 \( \frac{dy}{dx} = -\frac{x}{2} \) and \( \sqrt{|y|} = \frac{|x|}{2} \)
So this is only valid when \( x < 0 \)

So we have a solution valid for all \( x \) of the form

\[
U(x) = \begin{cases} 
(x/2)^2 & \text{when } x > 0 \\
-(x/2)^2 & \text{when } x < 0 
\end{cases}
\]

There is another solution to this IVP however it's

\[
U(x) = 0
\]

So this IVP has two solutions!

This can happen but only in very special instances.
\[3 \frac{du}{dx} = \sqrt{1-u} \quad u(0) = -1\]

As long as \( u < 0 \) we have

\[\frac{du}{dx} = \sqrt{1-u}\]

\[\int_{-1}^{u} \frac{du}{\sqrt{1-u}} = \int_{0}^{x} dx\]

\[-2\sqrt{1-u} + 2 = x - 0\]

Notice that \( u < 0 \) for \( x \) provided \(-\infty < x < 2\). Beyond \( x = 2 \), this is no longer valid. In fact, at \( x = 2 \) the solution "bifurcates" into two solutions just like what happens in exercise 2.
4 \ \frac{u \, du}{dx} = 2u - x

This is not separable or linear.
Let's see if it's exact.

\( (x-2u) + u \, du = 0 \)
\( A' = u \)
\( A'' = 0 \)
\( Axu = -2 \)
\( A'x = 0 \) Nuppe wot! exact either.

Maybe it's homogeneous.

\( \frac{du}{dx} = 2u - x \) let \( u = xv \)

\( \frac{d(xv)}{dx} = \frac{2xv - x}{xv} = \frac{2v - 1}{v} \)

\( \) Yeah! it's homog.

\( x \frac{dv}{dx} + v = \frac{2v - 1}{v} \)

\( \Rightarrow \quad x \frac{dv}{dx} = \frac{(v^2 - 2v + 1)}{v} = -\frac{(v - 1)^2}{v} \) (over)
\[ \int \frac{v}{(v-1)^3} dv = \int \frac{dx}{x} \]

\[ \int \frac{1}{v-1} + \frac{1}{(v-1)^2} = -\log(v) + C \]

\[ \log(v-1) = \frac{1}{v-1} = -\log(v) + C \]

\[ \log \left( \frac{u}{x} - 1 \right) = \frac{1}{\left( \frac{u}{x} - 1 \right)} = -\log(x) + C \]

Woops, my bad.

\[ \int (x+u) + (x+u^2) \frac{du}{dx} = 0 \]

Clearly not separable or linear, let's try exact.

\[ \Delta x = x+u \quad \Delta xy = 1 \quad \checkmark \text{yep!} \]
\[ \Delta u = x+u^2 \quad \Delta ux = 1 \quad \text{It's exact. (over)} \]
\[ A_x = x + u \implies A = \frac{x^2}{2} + u x + h(u) \]

\[ \implies A_u = x + h'(u) = x + u^2 \]

\[ \implies h'(u) = u^2 \implies h(u) = \frac{u^3}{3} \]

So our solution is given implicitly by

\[ A(x,u) = \frac{x^2}{2} + u x + \frac{u^3}{3} = c(u) \]

\[ 6 \frac{du}{dx} = \frac{x^2 + u^2}{2x^2} \quad u(u) = 2 \]

Not separable or linear.
Probably not exact either, but it is homogeneous.

\[ u = x v \]

\[ \frac{d(xv)}{dx} = \frac{x^2 + (xv)^2}{2x^2} = \frac{1 + v^2}{2} \]

\[ \frac{x dv + v}{dx} = \frac{1 - 2v + v^2}{2} \]

(over)
\[
\frac{1}{(v-1)^2} \, dv = \frac{1}{2} \, \frac{dx}{x} \\

u(1) = 2 \implies v(0) = 2 \\
\int_1^v \frac{1}{(v-1)^2} \, dv = \frac{1}{2} \int_1^x \frac{dx}{x} \\
-\frac{1}{v-1} \bigg|_1^v = \frac{1}{2} \left(\log(x) - 0\right) \\
-\frac{1}{v-1} = \frac{1}{2} \log x \\
\]

So \[-\frac{1}{v-1} + \frac{1}{2-1} = \frac{1}{2} \log x\]

Solve for \(u = xv + x\) and set
\[
\begin{align*}
\boxed{u} &= x \left(1 + \frac{1}{1 - \frac{1}{2} \log(x)}\right) \\
\end{align*}
\]

This solution is valid provided
\[1 - \frac{1}{2} \log x > 0 \implies x < e^2\]

at \(x = e^2\) the solution blows up.
\[ (x^2 + u^2) + (1+2ux) \frac{du}{dx} = 0 \]

\[ u(0) = 1 \]

Not separable or linear. Let's see if exact.

\[ L_x = x^2 + u^2 \quad L_u = 1 + 2xy \]

\[ L_xu = 2u \quad L_uu = 2u \quad \text{Yeah?} \]

\[ L_x = x^2 + u^2 \implies L = \frac{x^3}{3} + u^2x + h(u) \]

\[ L_u = 2ux + h'(u) = 1 + 2xy \]

\[ h'(u) = 1, \quad \text{so} \quad h(u) = u \]

\[ A = \frac{x^3}{3} + u^2x + u = \text{const.} \]

\[ u(0) = 1 \implies \frac{0^3}{3} + 1^2 \cdot 0 + 1 = \text{const.} \]

So \( u \) solves

\[ \frac{x^3}{3} + u^2x + u = 1 \]

\( (x^2 - 1) \)
I can solve this with the quad form
\[ x u^2 + u + \left( \frac{x^3 - 1}{3} \right) = 0 \]
\[ u = -1 \pm \sqrt{1 - 4x \left( \frac{x^3 - 1}{3} \right)} \]
\[ 2x \]
Take the branch that's defined around \( x = 0 \) (the initial point)
So
\[ u = -1 + \sqrt{1 - 4x \left( \frac{x^3 - 1}{3} \right)} \]
\[ 2x \]
It's interesting to note that for small \( x \), \( \sqrt{1 - \frac{x}{2}} \approx 1 - \frac{x}{2} \)
(use Taylor's Thm.) So near \( x = 0 \) see that
\[ u = -1 + \frac{2x(x^3 - 1)}{3} = 1 - \frac{x^3}{3} \]
\[ 2x \]
\[ \frac{x}{3} \]
\[ \text{tends to } 1 \text{ as } x \to 0 \]
(over)
This solution is valid provided
\[1 - 2x\left(\frac{x^3}{3} - 1\right) > 0.\]

\[\frac{dy}{dx} = xy + y^2, \quad y(0) = 1\]

Not separable, not linear, not exact.

Has not homogeneous either.

But it does fit into Bernoulli framework.

Try \( u = vy \)

\[vy^2 - 1 y' = xy^2 + y^2 x \]

\[y \frac{dy}{dx} = xy + \sqrt{x^2 - (x-1)} \]

\[= xy + \sqrt{x} + 1 \]

Take \( y = -1 \).

\[\frac{dv}{dx} = xy + 1 \quad \text{order linear over}\]

This is first order linear.
\[ \frac{dv}{dx} + xv = -1 \quad v(10) = \frac{1}{v(10)} = 1 \]

\[ \int_{x=0}^{x} \frac{d}{dx} \left( e^{x/2} v \right) = \int_{x=0}^{x} e^{x/2} \]

\[ e^{x/2} v(x) - e^{0/2} v(0) = \int_{0}^{x} e^{s/2} ds \]

\[ v(x) = e^{-x/2} \left( 1 - \int_{0}^{x} e^{s/2} ds \right) \]

So \[ u(x) = \frac{1}{v(x)} = \frac{e^{x/2}}{1 - \int_{0}^{x} e^{s/2} ds} \]

Note that near \( x = 0 \), \( u(x) \to 1 \)

But as \( \int_{0}^{x} e^{s/2} ds \to 1 \) the solution blows up. This is expected for such nonlinear problems.
\[ \frac{du}{dx} = u + xu^2, \quad u(0) = 1 \]

Again, let \( u = \frac{1}{v} \)

\[ \frac{1}{v} = \frac{1}{v} + \frac{x}{\sqrt{2}} \]

\[ -\frac{1}{\sqrt{2}} \frac{dv}{dx} = \frac{1}{v} + \frac{x}{\sqrt{2}} \]

So

\[ \frac{dv}{dx} + v = -x \]

\[ \int x \, dv + v \int x = -\int xe^x \]

\[ e^x v - 1 = - \left[ xe^x \bigg|_0^1 - \int_0^1 xe^x \right] \]

\[ = - [(xe^x - 0) - (e^1 - 1)] \]

\[ e^x v = \frac{1}{1 - xe^x + e^x - 1} \]
\[ v = 1 - x \Rightarrow u(x) = \frac{1}{1-x} \]

Again, as expected, solution blows up as \( x \to 1 \)

10 \[ \frac{du}{dx} + u = \sin x \]

\[ \int \frac{1}{e^x u} = \int e^x \sin x \]

\[ e^x u = C + \int e^x \sin x \]

Let \( I = \int e^x \sin x \)

\[ \int \frac{du}{uv} \int \frac{dv}{uv} \]

\[ = e^x \sin x - \int e^x \cos x \]

\[ = e^x \sin x - (e^x \cos x + \int e^x \sin x) \]

So \[ 2I = e^x \sin x - e^x \cos x \quad (\text{over}) \]
\[ u = Ce^x + \frac{1}{2}(\sin x - \cos x) \]

\[ e^x(u - \sin x) + e^x \frac{du}{dx} = 0 \]
\[ A_x = A_y = e^x \]
\[ A_{xy} = e^x \]
\[ A_{ux} = e^x \]

So this is exact.

Since \[ A_y = e^x \implies Q = ue^x + h(x) \]
\[ \Delta_x = u e^x + h'(x) = e^{2x}u - e^x \sin x \]
\[ h'(x) = e^x \sin x \]
\[ \Delta_y = -\frac{e^x}{2}(\sin x - \cos x) \]
\[ \lambda = ue^x + \frac{e^x}{2}(\sin x - \cos x) = \text{const} + \frac{x}{2} \]

This gives \[ u = \text{const} + e^x + \frac{1}{2}(\sin x - \cos x) \]

Just like the solution to exercise 10 above.