1a) $L(u) = u \frac{du}{dx} + u$

$L(cu) = cu \frac{du}{dx} + cu = c^2 u \frac{du}{dx} + cu \neq c \cdot L(u)$

for all $c \neq u$, NOT LINEAR

$L$ is first order

b) $L(u) = x \frac{du}{dx} + u$

$L(cu) = x \frac{d(cu)}{dx} + cu = c \left( x \frac{du}{dx} + u \right)$

$L(u_1 + u_2) = x \frac{d(u_1 + u_2)}{dx} + (u_1 + u_2)$

$= \left( x \frac{du_1}{dx} + u_1 + x \frac{du_2}{dx} + u_2 \right)$

$= L(u_1) + L(u_2)$

Linear

$L$ is first order
c) \[ L(y) = \frac{d^2 y}{dx^2} + e^x \frac{dy}{dx} \]
\[ L(cu) = \frac{d^2 (cu)}{dx^2} + e^x \frac{d(cu)}{dx} = c \left( \frac{d^2 y}{dx^2} + e^x \frac{dy}{dx} \right) \]
\[ = c \cdot L(y) \]
\[ L(u_1 + u_2) = \frac{d^2 (u_1 + u_2)}{dx^2} + e^x (u_1 + u_2) \]
\[ = \frac{d^2 u_1}{dx^2} + e^x u_1 + \frac{d^2 u_2}{dx^2} + e^x u_2 \]
\[ = L(u_1) + L(u_2) \]

Linear 2nd order

d) \[ L(u) = \frac{d^2 u}{dx^2} + \left( \frac{du}{dx} \right)^2 \]
\[ L(cu) = \frac{d^2 (cu)}{dx^2} + \left( \frac{d(cu)}{dx} \right)^2 \]
\[ = c \frac{d^2 y}{dx^2} + c^2 \left( \frac{dy}{dx} \right)^2 + c L(y) \]

for all \( c \) and \( y \), \( \boxed{\text{NOT LINEAR}} \)

\( L \) is 2nd order
201 \ \frac{dy}{dx} = u^2 - u \quad \text{This separable also it's a Bernoulli.}

Separable \quad \frac{dy}{u^2 - u} = dx

\frac{1}{u^2 - u} = \frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}

\frac{1}{u(u-1)} = \frac{(A + B)u - A}{u(u-1)}

- A = 1

B = -A

\frac{1}{u^2 - u} = -\frac{1}{u} + \frac{1}{u-1}

\int \frac{dy}{u^2 - u} = \log \left( \frac{u-1}{u} \right) = \int dx = x + c

\log \left( \frac{u-1}{u} \right) = e^{e^x}

\Rightarrow \quad \frac{u-1}{u} = e^{e^x} \Rightarrow e^{e^x} = e^{e^x}.
Solve for $u$

$u' = ce^x \cdot u$

$(1 - ce^x) u = 1$

By Bernoulli:

$\frac{du}{dx} + u = u^2$

$u = v^\gamma$

$\gamma v^{\gamma-1} \frac{dv}{dx} + v \gamma = v^{2\gamma}$

$\gamma \frac{dv}{dx} + v = v^{2\gamma} \cdot (\gamma-1)$

$\frac{dv}{dx} - v = -1$

$e^x \frac{d}{dx} (e^{-x} v) = -1$

$\int \frac{d}{dx} (e^{-x} v) = -e^{-x}$

$e^{-x} v = -e^{-x} + c$

$\frac{1}{u} = v = 1 + ce^x \quad \Rightarrow \quad \boxed{u = \frac{1}{1 + ce^x}}$

Same by redefining $c$. 

b) \[ x^2 \frac{du}{dx} + u^2 - xu = 0 \]

This is Homog but it's also Bernoulli. I'm only going to use Homog here.

\[ \frac{d}{dx} \left( \frac{u}{x} \right) = \frac{nu - u^2}{x^2} = \frac{wx - x^2}{x^2} \]

See this is Homog, let \( \frac{u}{x} = v \), then \( u = xv \).

\[ \frac{dv}{dx} = v - v^2 \]

\[ x \frac{dv}{dx} + v = v - v^2 \]

\[ \int \frac{dv}{v^2} = -\int \frac{dx}{x} = -\log(x) + C \]

\[ \frac{u}{x} = v = \frac{1}{\log x - C} \]

\[ u = \frac{x}{\log(x) - C} \]
3a) \[ \frac{dy}{dx} + \frac{1}{x} u = e^x \]

This is 1st order linear.

\[ e^{-\frac{1}{x}} \frac{d}{dx} \left( e^{\frac{1}{x}} u \right) = e^x \]

I'll take \( x > 0 \) here as well.

\[ \frac{1}{x} \frac{d}{dx} (xu) = e^x \]

by parts

\[ \int \frac{d}{dx} (xu) = \int xe^x = (x-1)e^x + C \]

\[ xu \Rightarrow u = \left( \frac{x-1}{x} \right) e^x + \frac{c}{x} \]

b) \[ xe^y \frac{dy}{dx} + (x+e^y) = 0 \]

But let's try exact.

\[ \text{NOT SEP} \quad \text{NOT HOMOG} \]

\[ \text{NOT LINEAR} \]

\[ \Lambda u = xe^y \quad \Lambda x = (x+e^y) \]

\[ \Lambda u_x = e^y \quad \Lambda x_u = e^y \]

Hey! \( H \)

is exact.
\( A_n = xe^n \)

\( A = xe^n + h(x) \quad \text{From equation} \)

\( A_x = e^n + h'(x) = x + e^n \)

\( \Rightarrow h'(x) = x \Rightarrow h(x) = \frac{x^2}{2} \)

Solution set is given by

\( A = xe^n + \frac{x^2}{2} = C \)

Solve for \( e^n \)

\( xe^n = \frac{C - \frac{x^2}{2}}{x} \)

\[ u = \log \left( \frac{C - \frac{x^2}{2}}{x} \right) \]

4) \[ \frac{dT}{dt} = k_b (T_A - T) \]

\( T_A = 70 \)

\( T(0) = 80 \quad T(1) = 75 \)

At what time \( t^* \) is \( T(t^*) = 98 \)?

Time he was killed.
\[
\begin{align*}
\frac{dT}{dt} + k_b T &= k_b T_0 \\
\int \frac{dT}{dt} e^{k_b t} &= \int 70 k_b e^{k_b t} \\
\Rightarrow e^{k_b t} T &= 70 e^{k_b t} + C \\
T(t) &= Ce^{-k_b t} + 70 \\
80 = T(0) &= C + 70 \Rightarrow C = 10 \\
75 &= T(1) = 10e^{-k_b} + 70 \\
\frac{75 - 70}{10} &= e^{-k_b} \\
\log\left(\frac{1}{2}\right) &= -k_b \\
\Rightarrow T(t) &= 10 \left(\frac{1}{2}\right)^t + 70 \\
98 = T(t_{\text{ens}}) &= 10 \left(\frac{1}{2}\right)^{t_{\text{ens}}} + 70 \Rightarrow \left(\frac{1}{2}\right)^{t_{\text{ens}}} = \frac{28}{10} \\
\Rightarrow t_{\text{ens}} &= \frac{\log(2.8)}{\log(1/2)} \approx 5 \text{ (neg number).}
\end{align*}
\]
5a) \[ \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = 0 \]

\[ r^2 - 2r + 1 = 0 \]

\[ r = \frac{2 \pm \sqrt{4 - 4 \cdot 1}}{2} = 1 \pm 0 \]

Double root!

\[ y(x) = e^x (C_1 + C_2x) \]

5b) \[ \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + 2y = 0 \]

\[ r^2 - 2r + 2 = 0 \]

\[ r = \frac{2 \pm \sqrt{4 - 4 \cdot 2}}{2} = 1 \pm \sqrt{-1} = 1 \pm i \]

Complex roots.

\[ y(x) = e^x (C_1 \cos x + C_2 \sin x) \]
(c) \[ \frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = 0 \]

\[ r^2 - 2r - 1 = 0 \]

\[ r = \frac{2 \pm \sqrt{4 + 4}}{2} = 1 \pm \sqrt{2} \]

distinct real roots

\[ y(x) = e^x (C_1 \cosh(\sqrt{2}x) + C_2 \sinh(\sqrt{2}x)) \]

d) \[ \frac{d^2y}{dx^2} - y \frac{dy}{dx} + 5y = 0 \]

\[ r^2 - 4r + 5 = 0 \]

\[ r = \frac{4 \pm \sqrt{16 - 4 \cdot 5}}{2} = 2 \pm i \]

\[ y(x) = e^{2x} (C_1 \cos x + C_2 \sin x) \]