Math 3331 Final Exam, Sanders Spring 2020

This exam has ten problems each worth 20 points with parts equally weighted. Your time limit is three hours.

(1) Please use lineless paper (i.e. Xerox paper) if you can.

(2) Add a cover sheet to the front of your exam solutions with your name, last name first, student id number, your photo id and your signature.

(3) Scan your cover sheet together with your subsequent solutions in numerical order.

1. We’ve considered four basic classes of first order ODE’s we can in principle solve. They are: (1) Separable, (2) Linear, (3) Homogeneous, (4) Exact. Classify each of the following but do not solve. An equation may be in more than one class (and you must list all classes if so), or it may be in none. No guessing – show your work!

   (a) \( x \frac{du}{dx} + u = 0 \)  
   (b) \( x \frac{du}{dx} + \sqrt{u^2 + x^2} = 0 \)  
   (c) \( \frac{du}{dx} + u + 1 = 0 \)  
   (d) \( (u + x + 1) \frac{du}{dx} + u + 2 = 0 \)

2. Find the explicit form general solution to each of the following.

   (a) \( \frac{du}{dx} + u = e^{-x} \)  
   (b) \( 2 \frac{du}{dx} = 3x^2u^3 \)

3. Find the general solution, an implicit form solution is OK, to each of the following.

   (a) \( x^3 \frac{du}{dx} = u^3 + x^2u \)  
   (b) \( (2x + u^2) + 2xu \frac{du}{dx} = 0 \)

4. A tank is initially filled with 20 gallons of fresh water. A salt brine solution, at three pounds of salt per one gallon of water, is piped into the tank at one gallon per minute. There is another pipe which will output one gallon per minute of the fully mixed water/brine solution. Let \( s(t) \) denote the amount of salt in the tank at time \( t \) measured in minutes.

   (a) Set up the first order initial value problem solved by \( s(t) \).

   (b) Solve for \( s(t) \).

5. Find the general solution of the following second order differential equations by using the given factorization.

   (a) \( \frac{d^2u}{dx^2} + 4 \frac{du}{dx} + 4u = 0 \).

   (b) \( \frac{d^2u}{dx^2} + 3 \frac{du}{dx} + 2u = 0 \).
6. Determine the **general solution** to each of the following by using the method of guessing.

(a) \( \frac{d^2u}{dx^2} - u = x^2 + 2x \)  
(b) \( \frac{d^2u}{dx^2} + u = \sin x \).

7. Solve the following inhomogeneous IVPs using Duhamel’s principle.

(a) \( \frac{d^2u}{dx^2} + u = 1, \quad u(0) = u'(0) = 0 \).

(b) \( \frac{d^2u}{dx^2} + \frac{1}{x} \frac{du}{dx} - \frac{1}{x^2}u = x, \quad u(1) = u'(1) = 0 \).

Hint for (b): The homogeneous equation is solved by \( u_h(x) = c_1 x + c_2 x^{-1} \). But note, you can’t use translation invariance (i.e. \( x - z \)) when you solve for \( c_1 \) and \( c_2 \) here.

8. Consider the matrix \( A = \begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & 0 \\ -2 & 0 & 3 \end{pmatrix} \).

(a) Determine the eigenvalues of \( A \).

(b) Determine all associated eigenvectors.

9. Use the matrices in this similarity transformation

\[
R^{-1}AR \equiv \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \equiv \Lambda
\]

to do the following.

(a) Compute \( e^{At} \) where \( A = \begin{pmatrix} -1 & 6 \\ -1 & 4 \end{pmatrix} \).

(b) Solve the initial value problem.

\[
\begin{align*}
\frac{du}{dt} &= -u + 6v, \quad u(0) = 1, \\
\frac{dv}{dt} &= -u + 4v, \quad v(0) = 2.
\end{align*}
\]

10. The matrix \( A = \begin{pmatrix} -1 & 9 \\ -1 & 5 \end{pmatrix} \) is **not** diagonalizable.

(a) Determine \( S \) such that \( S^{-1}AS = J \) where \( J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \).

(b) Compute \( e^{At} \).