①

Sanders
3331
E2 Sp25

1a Solve $\frac{d^2u}{dx^2} - 2\frac{du}{dx} + u = 0$

$u(0) = 1$

$u_x(0) = 2$

char poly is

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0 \Rightarrow r = 1 \pm 0$

General soln to the homog problem is

$u = c_1 e^x + c_2 x e^x$

For initial conds

$1 = u(0) = c_1 + 0$ $\Rightarrow c_1 = 1$

$2 = u_x(0) = c_1 + c_2$ $\quad c_2 = 1$

$$\boxed{u = e^x + xe^x}$$

b. $\frac{d^2u}{dx^2} + u = 0$

$u(1) = 1$

$u_x(1) = 2$

Char poly is $r^2 + 1 = 0$

$r = \pm i$

$u(x) = c_1 \cos x + c_2 \sin x$

But much better to use this

$u(x) = c_1 \cos(x-1) + c_2 \sin(x-1)$

$1 = u(1) = c_1$

$2 = u_x(1) = c_2$

$$\boxed{u(x) = \cos(x-1) + 2\sin(x-1)}$$

2 a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 1$

u_h solves $\frac{d^2u_h}{dx^2} + \frac{du_h}{dx} = 0$

$r^2 + r = 0 \quad r = 0, \ r = -1$

$\boxed{u_h = c_1 1 + c_2 e^{-x}}$ ← Do this once only,

$u_p = \alpha \cdot 1$ won't work bc 1 solves homog eqn.

$u_p = \alpha \cdot x$ will work

$u_p' = \alpha \qquad 0 + \alpha = 1 \quad \alpha = 1$

$u_p'' = 0$

$\boxed{u_g = c_1 + c_2 e^{-x} + x}$

b) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^x \qquad u_p = \alpha e^x$ will work,

$u_p' = \alpha e^x \quad u_p'' = \alpha e^x$

$(\alpha + \alpha) e^x = e^x \quad \alpha = \frac{1}{2}$

$\boxed{u_g = c_1 + c_2 e^{-x} + \frac{1}{2} e^x}$

③

c) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x}$ $y_p = \alpha e^{-x}$ won't work

B.C. e^{-x} solves homogen

problem

Instead try

$y_p = \alpha x e^{-x}$

$y_p' = \alpha(-xe^{-x} + e^{-x})$

$y_p'' = \alpha(xe^{-x} - 2e^{-x})$

$$y_p'' + y_p' = \alpha\left[(\cancel{xe^{-x}} - 2e^{-x}) + (-\cancel{xe^{-x}} + e^{-x})\right] = e^{-x}$$

$-\alpha e^{-x} = e^{-x}$ $\alpha = -1$

$$\boxed{y_g = c_1 + c_2 e^{-x} - xe^{-x}}$$

d) $\frac{d^2y}{dx^2} + \frac{dy}{dx} = \cos x$ try $y_p = \alpha \cos x + \beta \sin x$

$y_p' = -\alpha \sin x + \beta \cos x$

$y_p'' = -\alpha \cos x - \beta \sin x$

$$y_p'' + y_p' = (-\alpha \cos x - \beta \sin x) + (-\alpha \sin x + \beta \cos x) = \cos x$$

$-\alpha + \beta = 1$ $-\alpha - \beta = 0$ $\alpha = -\frac{1}{2}$ $\beta = \frac{1}{2}$

(over)

$$\boxed{u_g = c_1 + c_2 e^{-x} - \tfrac{1}{2}\cos x + \tfrac{1}{2}\sin(x)}$$

$f(z)=1$

3 a) $\dfrac{d^2u}{dx^2} - u = 1$ $\qquad u(x) = \displaystyle\int_0^x v(x,z)\, 1\, dz$

$u(0) = u_x(0) = 0$

v solves

$$\frac{d^2v}{dx^2} - v = 0$$

$0 = v(z)$

$1 = v_x(z)$

$r^2 - 1 = 0$

$r = \pm 1$

$v = c_1\cosh(x) + c_2 \sinh(x)$

But better to use this $\rightarrow$ $V = c_1 \cosh(x-z) + c_2 \sinh(x-z)$

$0 = V(z) = c_1$ $\qquad \boxed{V = \sinh(x-z)}$

$1 = V_x(z) = c_2$

$$u = \int_0^x \sinh(x-z)\, 1\, dz = -\cosh(x-z)\Big|_0^x$$

$$\boxed{u = -1 + \cosh(x)}$$

⑤

b) $\dfrac{d^2u}{dx^2} = x^2$

$u(1) = u_x(1) = 0$

$u = \int_1^x v(x,z)\, z^2\, dz$

$$\boxed{\dfrac{d^2v}{dx^2} = 0 \qquad \begin{array}{l} v(z) = 0 \\ v_x(z) = 1 \end{array}}$$

$r^2 = 0$

$v = c_1 + c_2 x$

Better to use this instead

$v = c_1 + c_2(x-z)$

$0 = c_1$

$1 = c_2$

$\Rightarrow v = x - z$

$u = \int_1^x (x-z) z^2\, dz$

$= \dfrac{xz^3}{3} - \dfrac{z^4}{4} \Big|_1^x$

$= \left(\dfrac{x^4}{3} - \dfrac{x^4}{4}\right) - \dfrac{x}{3} + \dfrac{1}{4}$

$$\boxed{u = \frac{1}{12} x^4 - \frac{x}{3} + \frac{1}{4}}$$

$\frac{4}{12} x^3$

Check $u(1) = \frac{1}{12} - \frac{4}{12} + \frac{3}{12} = 0$ ✓

$u_x(1) = \frac{1}{3} - \frac{1}{3} = 0$ ✓

$u_{xx} = x^2$ ✓

⑥

4 Write as first order systems.

a) $$\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = 0$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = \left(\frac{d^2y}{dt^2}\right) = -\frac{dy}{dt} + 2y = -v + 2y$$

$$\boxed{\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ -v+2y \end{pmatrix}}$$

b) $$\frac{d^2y}{dt^2} - y\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = \left(\frac{d^2y}{dt^2}\right) = y\frac{dy}{dt} = yv$$

$$\boxed{\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ yv \end{pmatrix}}$$

⑦

c) $$\frac{d^2y}{dt^2} = \left(\frac{dy}{dt}\right)^2 e^y$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = \left(\frac{d^2y}{dt^2}\right) = \left(\frac{dy}{dt}\right)^2 e^y = v^2 e^y$$

$$\boxed{\frac{d}{dt}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} v \\ v^2 e^y \end{pmatrix}}$$

d) $$\frac{d^3y}{dt^3} = \frac{dy}{dt} + 3y$$

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = w$$

$$\frac{dw}{dt} = \left(\frac{d^3y}{dt^2}\right) = \frac{dy}{dt} + 3y = v + 3y$$

$$\boxed{\frac{d}{dt}\begin{pmatrix} y \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ v+3y \end{pmatrix}}$$

⑧

5 let $A = \begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}$

a) Find evals

$$\det(A - \lambda I) = \det\begin{pmatrix} -3-\lambda & 2 \\ -4 & 3-\lambda \end{pmatrix}$$

$$= (\lambda+3)(\lambda-3) + 8 = \lambda^2 - 9 + 8 = \lambda^2 - 1 = 0$$

$$\boxed{\begin{aligned} \lambda &= -1 \\ \lambda &= +1 \end{aligned}}$$

b) Find evecs.

$\lambda = -1$ $\quad (A - \lambda I)\vec{r} = \vec{0}$

$$\left[\begin{array}{cc|c} -3+1 & 2 & 0 \\ -4 & 3+1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -2 & 2 & 0 \\ -4 & 4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \quad \boxed{\lambda = -1 \text{ has } r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$$

⑦

$\lambda = 1 \quad (A - \lambda I)\vec{r} = \vec{0}$

$$\left[\begin{array}{cc|c} -3-1 & 2 & 0 \\ -4 & 3-1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -4 & 2 & 0 \\ -4 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \boxed{\lambda = 1 \quad \vec{r} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

c) Compute e^{At}

Since A is diagonalizable

$$e^{At} = R e^{\Lambda t} R^{-1}$$

$$R = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \qquad R^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

$$e^{\Lambda t} = \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{+t} \end{pmatrix}$$

$$e^{At} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} e^{-t} & 0 \\ 0 & e^{t} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

(over)

(10)

$$= \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2e^{-t} & -e^{-t} \\ -e^{t} & e^{t} \end{pmatrix}$$

$$\boxed{e^{At} = \begin{pmatrix} 2e^{-t} - e^{t} & -e^{-t} + e^{t} \\ 2e^{-t} - 2e^{t} & -e^{-t} + 2e^{t} \end{pmatrix}}$$

d) Use (c) to solve $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boxed{\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} = e^{At}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -e^{-t} + e^{t} \\ -e^{-t} + 2e^{t} \end{pmatrix}}$$

$$\boxed{\begin{aligned} u(t) &= e^{t} - e^{-t} \\ v(t) &= 2e^{t} - e^{-t} \end{aligned}}$$