①

1 Find explicit soln to IVPs

Sanders
3331
Final
Sp25

a) $\boxed{\frac{du}{dt} + 2xu = e^{-x^2}}$

$u(0) = 2$

This is first order linear

$\boxed{e^{-x^2}\frac{d}{dx}(e^{x^2}u)} = e^{-x^2}$

$\int \frac{d}{dx}(e^{x^2}u) = \int 1$

$e^{x^2}u = c + x$

$u(x) = ce^{-x^2} + xe^{-x^2}$

$2 = u(0) = c + 0 \Rightarrow c = 2$

$\boxed{u(x) = 2e^{-x^2} + xe^{-x^2}}$

b) $\frac{du}{dx} - 2xu^2 = 0$

$u(0) = 1$

This is separable (over)

$$\int \frac{du}{u^2} = \int 2x \, dx$$

$$-\frac{1}{u} = x^2 + C$$

$$u = \frac{1}{C - x^2}$$

$$1 = u(0) = \frac{1}{C} \Rightarrow C = 1$$

$$\boxed{u(x) = \frac{1}{1 - x^2}}$$

2 Find general solution (implicit OK)

a) $(xu+1)\frac{du}{dx} + (\frac{1}{2}u^2 + 4x) = 0$

$$\Lambda_u \overset{(**)}{=} (xu+1) \qquad \Lambda_x = \frac{1}{2}u^2 + 4x \ (*)$$

$$\Lambda_{ux} = u \quad = \quad \Lambda_{xu} = u$$

This is exact.

$$\Lambda_u = xu + 1$$

$$\Rightarrow \Lambda = x\frac{u^2}{2} + u + h(x)$$

(over)

(3)

$$\Lambda_x = \frac{u^2}{2} + h'(x) = \boxed{\tfrac{1}{2}u^2 + 4x} \text{ from } *$$

$$h'(x) = 4x \Rightarrow h(x) = 2x^2$$

$$\boxed{\Lambda = x\frac{u^2}{2} + u + 2x^2 = \text{const.}}$$

For the fun of it, I'm gonna do it the other way

$$\Lambda_x = \tfrac{1}{2}u^2 + 4x$$

$$\Lambda = \tfrac{1}{2}u^2 x + 4\frac{x^2}{2} + g(u)$$

$$\Lambda_u = ux + g'(u) = \boxed{xu + 1} \text{ from } (**)$$

$$g'(u) = 1 \quad g(u) = u$$

$$\boxed{\Lambda = \tfrac{1}{2}u^2 x + 2x^2 + u = \text{const.}}$$

Same as above.

b) $$\frac{du}{dx} = \frac{u^2 - x^2}{xu} = \frac{(u/x)^2 - 1}{(u/x)} \text{ homog.}$$

$$\frac{u}{x} = v \qquad x\frac{dv}{dx} + v = \frac{v^2 - 1}{v}$$

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$$x\frac{dv}{dx} + \cancel{v} = \cancel{v} - \frac{1}{v}$$

This is sep-rable

$$\int v\,dv = -\int \frac{dx}{x}$$

$$\frac{v^2}{2} = -\log(x) + C$$

$$v = \pm\sqrt{C - 2\log(x)}$$

Want u, not v,

$$\frac{u}{x} = v \Rightarrow \boxed{u = \pm x\sqrt{C - 2\log(x)}}$$

3 $\frac{dT}{dt} = k_b(T_A - T)$ Newton's Law Cooling,

$T_A = 70$, $T(0) = 80$, $T(1) = 75$

Let's figure out k_b

$$\int_{T(0)=80}^{T(1)=75} \frac{dT}{T-70} = -\int_{t=0}^{t=1} k_b\,dt = -k_b$$

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$$\log(75-70) - \log(80-70) = -k_b$$

$$k_b = \log(10) - \log(5) = \log(2)$$

Let's say t^* is when he died

i.e. $T(t^*) = 98$

Again use

$$\int_{T(0)=80}^{T(t^*)=98} \frac{dT}{T-70} = -\int_{t=0}^{t=t^*} \log(2)\, dt$$

$$= -\log(2)\, t^*$$

$$\| $$

$$\log(98-70) - \log(80-70)$$

$$\|$$

$$\log(28/10) = \log(2.8)$$

$$\boxed{t^* = -\frac{\log(2.8)}{\log(2)}}$$

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Here's another way,

$$\frac{dT}{dt} + k_b T = k_b 70$$

$$e^{-k_b t} \frac{d}{dt}\left(e^{k_b t} T\right) = 70 k_b$$

Assume $k_b \neq 0$

$$\int \frac{d}{dt}\left(e^{k_b t} T\right) = \int 70 k_b e^{k_b t}$$

$$e^{k_b t} T = 70 e^{k_b t} + C$$

$$T = 70 + C e^{-k_b t}$$

$$80 = T(0) = 70 + C$$

$$75 = T(1) = 70 + C e^{-k_b} \Rightarrow \begin{array}{l} 10 = C \\ 5 = C e^{-k_b} \\ \hline \frac{1}{2} = e^{-k_b} \end{array}$$

$$\boxed{k_b = \log(2)} \Leftarrow \frac{1}{2} = e^{-k_b}$$

$$98 = T(t_*) = 70 + 10 e^{-k_b t_*}$$

$$\frac{28}{10} = e^{-k_b t_*}$$

$$\boxed{-\frac{\log(2.8)}{\log(2)} = t_*}$$

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4 Factor Then solve

a)

$\mathcal{L}(y) = \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y$ Const coeff find roots of char poly;

$r^2 - 2r + 1 = 0$

$\parallel$ $r = 1$ double,

$\left(\frac{d}{dx} - I\right)\left(\frac{dy}{dx} - y\right)$

Solve $\mathcal{L}(y) = 0 = \left(\frac{d}{dx} - I\right)\underbrace{\left(\frac{dy}{dx} - y\right)}_{v}$

$$0 = \frac{dv}{dx} - v$$

First order solve for v

$$\boxed{e^{x}\frac{d}{dx}\left(e^{-x}v\right) = 0} \Rightarrow v = c_1 e^{x}$$

$$v = \frac{dy}{dx} - y = e^{x}\frac{d}{dx}\left(e^{-x}y\right)$$

$\parallel$

$c_1 e^{x} \Rightarrow \int \frac{d}{dx}\left(e^{-x}y\right) = \int c_1$

$$e^{-x}y = c_1 x + c_2 \Rightarrow \boxed{y = c_1 x e^{x} + c_2 e^{x}}$$

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b) $\mathcal{L}(u) = \frac{d^2u}{dx^2} + \frac{du}{dx}$ (const coeff $\mathcal{L}$ char polynomial)

$$r^2 + r = 0 \Rightarrow r = 0 \quad r = -1$$

$$= \left(\frac{d}{dx} - 0I\right)\underbrace{\left(\frac{du}{dx} + u\right)}_{v} = e^{-x}$$

$$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x} + C_1$$

Now solve for u

$$\frac{du}{dx} + u = -e^{-x} + C_1 \; (= v)$$

$$= e^{-x}\frac{d}{dx}(e^{x}u) = -e^{-x} + C_1$$

$$\int \frac{d}{dx}(e^{x}u) = \int -1 + C_1 e^{x} = -x + C_1 e^{x} + C_2$$

$$= e^{x}u$$

So $\boxed{u = -xe^{-x} + C_1 + C_2 e^{-x}}$

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5 Use Duhamel to solve IVP

a) $\frac{d^2u}{dx^2} = \cos(x)$ $\Rightarrow$ $u = \int_0^x v(x,z)\cos(z)\,dz$

$u(0) = u_x(0) = 0$

where v solves

$$\begin{cases} \frac{d^2v}{dx^2} = 0 \\ v(z) = 0 \\ v_x(z) = 1 \end{cases}$$

$v = C_1(x-z) + C_2$

$0 = v(z) = C_2$

$1 = v_x(z) = C_1$

$\therefore v = (x-z)$

$$u = \int_0^x (x-z)\cos(z)\,dz$$

$$= \int_0^x x\cos(z)\,dz - \int_0^x \overset{u}{z}\,\overset{dv}{\cos(z)\,dz}$$

$$= x\sin(z)\Big|_0^x - \left[z\sin(z)\Big|_0^x - \int_0^x \sin(z)\,dz\right]$$

$$= \cancel{x\sin(x)} - \cancel{x\sin x} + -\cos(z)\Big|_0^x$$

$$\boxed{u = -\cos x + 1}$$

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b) $\frac{d^2u}{dx^2} - 2\frac{du}{dx} + u = 1$, $\Rightarrow u = \int_0^x v \, 1 \, dz$

$u(0) = u_x(0) = 0$

v solves

$r^2 - 2r + 1 = 0$

$v = c_1 e^x + c_2 x e^x$

But

$v = c_1 e^{x-z} + c_2 (x-z) e^{x-z}$

is better

$$\begin{cases} \frac{d^2v}{dx^2} - 2\frac{dv}{dx} + v = 0 \\ v(z) = 0 \\ v_x(z) = 1 \end{cases}$$

$0 = v(z)$
$1 = v_x(z)$ $\Rightarrow c_1 = 0 \quad c_2 = 1$ $\quad v = (x-z)e^{x-z}$

$u = \int_0^x (x-z) e^{x-z} \, 1 \, dz$

substitute
$x - z = w$
$-dz = dw$

$z = 0 \quad w = x$
$z = x \quad w = 0$

$-\int_x^0 w e^w dw = w e^w \Big|_0^x - \int_0^x e^w dw$

$$\boxed{u = x e^x - (e^x - 1)}$$

6 $A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{pmatrix}$

a) Find evals

$$\det\begin{pmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 0 \\ -1 & 0 & 1-\lambda \end{pmatrix}$$ cofactor

$$= (2-\lambda)\left[(1-\lambda)^2 - 1\right]$$

$$= (2-\lambda)\left[\lambda^2 - 2\lambda\right] = (2-\lambda)(\lambda-2)\lambda$$

$\boxed{\lambda = 0}$ simple

$\boxed{\lambda = 2}$ mult 2 eval,

b Find evecs

$$(A-\lambda I)\vec{r} = \vec{0}$$

$\lambda = 0$

$$\underset{\lambda=0}{}\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

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$r_3 = \alpha$, $r_2 = 0$, $r_1 - \alpha = 0$ $r_1 = \alpha$

$$\boxed{\lambda = 0 \text{ has evec } r = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

$(A - \lambda I)\vec{r} = \vec{0}$

$\lambda = 2$

$$\left[\begin{array}{ccc|c} 1-2 & 0 & -1 & 0 \\ 0 & 2-2 & 0 & 0 \\ -1 & 0 & 1-2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} +1 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$r_2 = \beta$ $r_3 = \alpha$

$r_1 + \alpha = 0$

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \begin{pmatrix} -\alpha \\ \beta \\ \alpha \end{pmatrix} = \alpha \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{\lambda = 2 \text{ has 2 evecs } r = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, r = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}$$

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7 $A = \begin{pmatrix} -1 & 9 \\ -1 & 5 \end{pmatrix}$

a) determine $\underline{\underline{S}}$ so that $S^{-1}AS = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

$$\det\begin{pmatrix} -1-\lambda & 9 \\ -1 & 5-\lambda \end{pmatrix} = (\lambda+1)(\lambda-5)+9$$

$$= \lambda^2 - 4\lambda + 4$$

$$= (\lambda-2)^2 = 0$$

$\lambda = 2$ find evec

$(A-\lambda I)r = 0$

$\lambda=2$ $\left[\begin{array}{cc|c} -3 & 9 & 0 \\ -1 & 3 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$\boxed{r = \begin{pmatrix} 3 \\ 1 \end{pmatrix}}$

The generator $\vec{g} \notin \operatorname{span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$

WLOG $\boxed{g = \begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$$(A-\lambda I)\vec{g} = \begin{pmatrix} -3 & 9 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$$

$$\boxed{S = \begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}}$$

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Check my work

Don't have to do This.

$S^{-1}AS = J$

$$\begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} -1 & 9 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} -6 & -1 \\ -2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \checkmark$$

b) Compute e^{At}

$$e^{At} = S e^{Jt} S^{-1}$$

$$e^{Jt} = \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}$$

$$e^{At} = e^{2t}\begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} -3 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} t & -1-3t \\ 1 & -3 \end{pmatrix}$$

$$= \boxed{e^{2t}\begin{pmatrix} -3t+1 & 9t \\ -t & 1+3t \end{pmatrix}}$$

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c Solve $\frac{du}{dt} = -u + 9v$ $\quad u(0)=1$

$\frac{dv}{dt} = -u + 5v$ $\quad v(0)=0$

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{At}\begin{pmatrix} u(0) \\ v(0) \end{pmatrix}$$

$$= e^{2t}\begin{pmatrix} -3t+1 & 9t \\ -t & 1+3t \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boxed{\begin{aligned} u(t) &= e^{2t}(-3t+1) \\ v(t) &= e^{2t}(-t) \end{aligned}}$$