①

Sanders
3331 E1
Sp 26

1 Classify

a) $x\frac{du}{dx} + u - x = 0$

$\frac{du}{dx} + \frac{1}{x}u = 1$ $\boxed{\text{First order linear}}$

$\frac{du}{dx} = 1 - \left(\frac{u}{x}\right)$ $\boxed{\text{Homogeneous}}$

$\frac{du}{dx} = 1 - \frac{u}{x}$ $\boxed{\text{NOT separable}}$

$A_u = x \quad A_x = u - x$

$A_{ux} = 1 \quad A_{xu} = 1$ $\boxed{\text{Exact}}$

b) $x^2\frac{du}{dx} + xu + x^2 = 0$

$x^2\frac{du}{dx} + xu = -x^2$ $\boxed{\text{First order linear}}$

$\frac{du}{dx} = -\frac{(xu + x^2)}{x^2}$ $\boxed{\text{NOT separable}}$

continue

(2)

$$\frac{du}{dx} = -\frac{u}{x} - 1$$ Homogeneous

$$x^2 \frac{du}{dx} + xu + x^2$$

$A_u = x^2$ $\qquad$ $A_x = xu + x^2$ $\qquad$ NOT EXACT

$A_{ux} = 2x$ $\qquad$ $A_{xu} = x$

c) $$x\frac{du}{dx} + u^2 - 1 = 0$$

Not linear

$$x\frac{du}{dx} = 1 - u^2 \qquad \frac{du}{1-u^2} = \frac{dx}{x}$$ Separable

$$\frac{du}{dx} = \frac{1-u^2}{x}$$ Not Homogeneous

$$x\frac{du}{dx} + u^2 - 1 = 0$$ NOT EXACT

$A_u = x$ $\qquad$ $A_x = u^2 - 1$

$A_{ux} = 1$ $\qquad$ $A_{xu} = 2u$

(3)

d) $(2y^2+x+1)\frac{dy}{dx} + y + 2x = 0$

$y^2\frac{dy}{dx}$ NOT Linear

$\frac{dy}{dx} = -\frac{(y+2x)}{2y^2+x+1}$ NOT Separable

NOT Homogeneous

$(2y^2+x+1)\frac{dy}{dx} + y+2x = 0$

$M_y = 2y^2+x+1$ $\quad$ $N_x = y+2x$

$M_{yx} = 1$ $\quad$ $N_{xy} = 1$ EXACT

2 Find Explicit Solution

a) $\frac{dy}{dx} + 2xy = e^{-x^2}$

First order linear method

$e^{-x^2}\frac{d}{dx}(e^{x^2}y) = e^{-x^2}$ (over)

(4)

$$\frac{d}{dx}(e^{x^2}u) = 1 \Rightarrow e^{x^2}u = x + c$$

$$\boxed{u(x) = ce^{-x^2} + xe^{-x^2}}$$

b) $\frac{du}{dx} - u(u-1) = 0$

Separable (also Bernoulli)

$$\int \frac{du}{u(u-1)} = \int dx$$

partial Fractions

$$\frac{A}{u} + \frac{B}{u-1} = \frac{1}{u(u-1)}$$

$$\frac{A(u-1) + Bu}{u(u-1)} \Rightarrow \begin{array}{l} A+B=0 \\ A=-1 \\ \Rightarrow B=1 \end{array}$$

$$\int \frac{du}{u(u-1)} = -\log(|u|) + \log(|u-1|) = x + c$$

(over)

$$\left|\frac{u-1}{u}\right| = e^c e^x$$

$$\frac{u-1}{u} = ce^x$$

$$u-1 = ce^x u$$

$$u(1-ce^x) = 1$$

$$\boxed{u(x) = \frac{1}{1-ce^x}}$$

3 Find Explicit solution

a) $3u^2x \frac{du}{dx} + (u^3+2x) = 0$

Hope it's exact because nothing else.

$A_u = 3u^2x$ $\quad A_x = u^3+2x$

$A_{ux} = 3u^2$ $\quad A_{xu} = 3u^2$

yeah!

$$A_u = 3u^2x$$

$$A = u^3x + h(x)$$

$$A_x = u^3 + h'(x) = u^3+2x$$

(over)

⑥ $h'(x) = 2x$ so $h(x) = x2$

$$\Delta = u^3 x + x^2 = C$$

Solve for y

$$u(x) = \sqrt[3]{\frac{C-y^2}{x}}$$

h) $u^2 x \frac{du}{dx} - (u^3 + x^3) = 0$

NOT linear, not sep $\quad \Delta_u = u^2 x \quad \Delta_x = -(u^3+x^3)$

$\Delta_{ux} = u^2 \neq \Delta_{xu} = -3u^2$

NOT EXACT

But

$$\frac{du}{dx} = \frac{u^3 + x^3}{u^2 x} = \frac{u}{x} + \left(\frac{x}{u}\right)^2$$

is homog.

$v = \frac{u}{x}$ i.e. $u = xv$

$$x \frac{dv}{dx} + v = v + \frac{1}{v^2}$$

$$x \frac{dv}{dx} = \frac{1}{v^2} \qquad \int v^2 dv = \int \frac{dx}{x}$$

(over)

(7)

$$\frac{v^3}{3} = \log|x| + C$$

$$v = \sqrt[3]{3\log|x| + C}$$

$$\begin{array}{c} \| \\ \frac{y}{x} \end{array} \quad \boxed{y(x) = x\sqrt[3]{3\log|x| + C}}$$

4 For depth of water in tank

$$\frac{dz}{dt} = -k\sqrt{z}$$

Also $z(0) = 4$ & $z(1) = 1$

a) Separable Solve for k,

$$\int_{z=4}^{z=1} \frac{dz}{\sqrt{z}} = \int_{t=0}^{t=1} -k\,dt = -k$$

$$\begin{array}{c} \| \\ 2\sqrt{z}\Big|_4^1 \end{array} = 2 - 4 \Rightarrow \boxed{k = 2}$$

8

b) For what t^* is $z(t^*) = 0$?

$$\int_{z=4}^{z=0} \frac{dz}{\sqrt{z}} = -\int_{t=0}^{t=t^*} 2\,dt \qquad \boxed{k=2}$$

$$2\sqrt{z}\Big|_4^0 = -2t^*$$

$$\| \; 0-4 \implies \boxed{t^* = 2}$$

5 Solve by using the factorization (only

a) $\frac{d}{dx}\underbrace{\left(\frac{du}{dx} - u\right)}_{v} = \frac{d^2u}{dx^2} - \frac{du}{dx} = 0$

$$\frac{dv}{dx} = 0 \implies v = c_1$$

$$\frac{du}{dx} - u = c_1 \qquad \int \frac{d}{dx}(e^{-x}u) = \int c_1 e^{-x}$$

$$e^x \frac{d}{dx}(e^{-x}u) = c_1 \qquad e^{-x}u = -c_1 e^{-x} + c_2$$

$$\boxed{u = c_1 + c_2 e^x}$$

9)

b) $\left(\frac{d}{dx} - I\right)\overbrace{\left(\frac{du}{dx} - u\right)}^{v}$

$$= \frac{d^2u}{dx^2} - 2\frac{du}{dx} + u = e^x$$

$$\frac{dv}{dx} - v = e^x$$

$$\shortparallel$$

$$e^x \frac{d}{dx}(e^{-x}v) = e^x \Rightarrow e^{-x}v = c_1 + x$$

$$v = c_1 e^x + x e^x$$

Solve for u

$$\frac{du}{dx} - u = v = c_1 e^x + x e^x$$

$$\shortparallel$$

$$e^x \frac{d}{dx}(e^{-x}u) \Rightarrow \int \frac{d}{dx}(e^{-x}u) = \int c_1 + x$$

$$e^{-x}u = c_1 x + c_2 + \frac{x^2}{2}$$

$$\boxed{u(x) = c_1 x e^x + c_2 e^x + \frac{x^2}{2} e^x}$$