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Sanders
3331 Sp26
Exam 2

1 Solve The IVPs

a) $$\frac{d^2u}{dx^2}+2\frac{du}{dx}+u=0$$

$u(0)=1, \quad u_x(0)=2$

method of guessing: Try $u=e^{rx}$

char polynomial

$r^2+2r+1=0$

$(r+1)^2=0$

$r=-1\pm 0$ (double root)

general soln is

$$\boxed{u=c_1e^{-x}+c_2xe^{-x}}$$

IVP will give the constants.

$1=u(0)=c_1e^{-0}+c_2\,0e^{-0}=c_1$

$2=u_x(0)=-c_1e^{-0}+c_2(-0e^{-0}+e^{-0})$

$=-c_1+c_2$

$c_1=1$
$c_2=3$

$$\boxed{u(x)=e^{-x}+3xe^{-x}}$$

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b) $$\frac{d^2u}{dx^2} + u = 0$$

$$u(1) = 1 \quad u_x(1) = 0$$

Get general soln from $u = e^{rx}$

$$r^2 + 1 = 0$$

$$r = \pm i$$

$$u(x) = C_1 \cos x + C_2 \sin x$$

But Because Cos and Sin at $x=1$

much better basis is

$$u(x) = C_1 \cos(x-1) + C_2 \sin(x-1)$$

$$1 = u(1) = C_1$$

$$0 = u_x(1) = C_2$$

$$\boxed{u(x) = \cos(x-1)}$$

If I use this basis I need

$$1 = u(1) = C_1 \cos(1) + C_2 \sin(1)$$

$$0 = u_x(1) = -C_1 \sin(1) + C_2 \cos(1)$$

you'll get $u(x) = \cos(1)\cos(x) + \sin(1)\sin(x)$

Same thing as above,

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2 $\mathcal{L}(u) = \frac{d^2u}{dx^2} - \frac{du}{dx}$ on all parts.

char poly $r^2 - r = 0$

$r(r-1) \Rightarrow \begin{matrix} r=0 \\ r=1 \end{matrix}$

homog soln is $\boxed{u_h(x) = C_1 + C_2 e^x}$

a) $\mathcal{L}(u) = e^{-x}$

Try $u_p(x) = \alpha e^{-x}$ and will work.

plug in and equate

$\alpha(e^{-x} + e^{-x}) = e^{-x}$

$\Rightarrow \alpha = 1/2$

$$\boxed{u(x) = C_1 + C_2 e^x + \tfrac{1}{2} e^{-x}}$$

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b) $\mathcal{L}(u) = x+1$

Normally try $u_p = \alpha_1 x + \alpha_0$ but This won't work because 1 solves the homog problem.

$$u_p = x(\alpha_1 x + \alpha_0) = \alpha_1 x^2 + \alpha_0 x$$
will work

$$u_p' = 2\alpha_1 x + \alpha_0$$
$$u_p'' = 2\alpha_1$$

$$\mathcal{L}(u_p) = 2\alpha_1 - (2\alpha_1 x + \alpha_0) = x+1$$
$$= (2\alpha_1 - \alpha_0) - 2\alpha_1 x$$

$$-2\alpha_1 = 1 \Rightarrow \alpha_1 = -1/2$$

$$2\alpha_1 - \alpha_0 = 1 \Rightarrow \alpha_0 = -2$$

$$u(x) = C_1 + C_2 e^x + x\left(-\tfrac{1}{2}x - 2\right)$$

(T)

c) $\mathcal{L}(u) = e^x$

$u_p = \alpha e^x$ won't work because it solves homog eqn. But

$u_p = \alpha x e^x$ will work

$u_p' = \alpha(xe^x + e^x)$

$u_p'' = \alpha(xe^x + 2e^x)$

$\mathcal{L}(u_p) = \alpha\left[(xe^x + 2e^x) - (xe^x + e^x)\right]$
$= e^x$

$\Rightarrow \alpha = 1$

$$\boxed{u(x) = C_1 + C_2 e^x + xe^x}$$

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d)

$\mathcal{L}(u) = \cos x$

Try $u_p = \alpha \cos x + \beta \sin x$ (will work).

$$u_p' = -\alpha \sin x + \beta \cos x$$
$$u_p'' = -\alpha \cos x - \beta \sin x$$

$$\mathcal{L}(u_p) = (-\alpha \cos x - \beta \sin x) - (-\alpha \sin x + \beta \cos x)$$
$$= \cos x$$

$$(-\alpha - \beta)\cos x + (-\beta + \alpha)\sin x$$

$$\begin{aligned} \alpha + \beta &= -1 \\ \alpha - \beta &= 0 \end{aligned} \Rightarrow \begin{aligned} \alpha &= -1/2 \\ \beta &= 1/2 \end{aligned}$$

$$u(x) = C_1 + C_2 e^x + -\frac{1}{2}\cos x - \frac{1}{2}\sin x$$

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3 Duhamel only!!!

a) $\begin{cases} \dfrac{d^2u}{dx^2} = x \\ u(0) = u_x(0) = 0 \end{cases}$

From Duhamel

$$u(x) = \int_0^x v(x,z) \underbrace{z}_{f(z)} dz$$

where $\begin{cases} \dfrac{d^2v}{dx^2} = 0 \\ v(z) = 0 \\ v_x(z) = 1 \end{cases}$

Char poly is
$r^2 = 0$
$r = 0 \pm 0$
$v = C_1 + C_2 x$
But much better to use
$v = C_1 + C_2(x - z)$

$0 = v(z) = C_1$
$1 = v_x(z) = C_2$

$\Longrightarrow v(x,z) = x - z$

$$u(x) = \int_0^x (x-z) z \, dz = x \frac{x^2}{2} - \frac{x^3}{3}$$

$$\boxed{u(x) = \frac{1}{6} x^3}$$

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b) $\frac{d^2 y}{dx^2} - 2\frac{dy}{dx} + y = e^x$

$y(0) = y_x(0) = 0$

$f(z)$

$$u(x) = \int_0^x v(x,z)\, e^z\, dz$$

$$\begin{cases} \frac{d^2 v}{dx^2} - 2\frac{dv}{dx} + v = 0 \\ v(z) = 0 \\ v_x(z) = 1 \end{cases}$$

$r^2 - 2r + 1 = 0$

$(r-1)^2 \qquad r = 1 \pm 0$

$v(x) = c_1 e^x + c_2 x e^x$

But much better to use

$v(x) = c_1 e^{x-z} + c_2 (x-z) e^{x-z}$

$0 = v(z) = c_1$

$1 = c_2$

$v(x,z) = (x-z) e^{x-z}$

$u(x) = \int_0^x (x-z) e^{x-z} e^z\, dz$

$= e^x \int_0^x x - z\, dz = e^x \left[x^2 - \frac{1}{2} x^2 \right]$

$$\boxed{u(x) = \frac{1}{2} x^2 e^x}$$

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4 write as 1st order system

a) $\frac{d^2u}{dt^2} + 2\frac{du}{dt} + 3u = 0$

$\frac{du}{dt} = v$

$\frac{dv}{dt} = \left(\frac{d^2u}{dt^2}\right) = -2\frac{du}{dt} - 3u$

$$\Rightarrow \boxed{\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ -2v-3u \end{pmatrix}}$$

b) $\frac{d^2u}{dt^2} - u\frac{du}{dt} + u^2 = 0$

$\frac{du}{dt} = v$

$\frac{dv}{dt} = \left(\frac{d^2u}{dt^2}\right) = u\frac{du}{dt} - u^2 = uv - u^2$

$$\boxed{\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ uv-u^2 \end{pmatrix}}$$

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c) $\frac{d^2u}{dt} = \left(\frac{du}{dt}\right)^4 + \sin(u)$

$\frac{du}{dt} = v$

$\frac{dv}{dt} = \left(\frac{d^2u}{dt^2}\right) = \left(\frac{du}{dt}\right)^4 + \sin(u) = v^4 + \sin(u)$

$$\boxed{\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} v \\ v^4 + \sin(u) \end{pmatrix}}$$

d) $\frac{d^3u}{dt^3} = \frac{d^2u}{dt^2} + u$

$\frac{du}{dt} = v$

$\frac{dv}{dt} = w = \left(\frac{d^2u}{dt^2}\right)$

$\frac{dw}{dt} = \left(\frac{d^2v}{dt^2} = \frac{d^3u}{dt^3}\right) = \frac{d^2u}{dt^2} + u = w + u$

$$\boxed{\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} v \\ w \\ w + u \end{pmatrix}}$$

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5 $A = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}$

a) evals: $\det\begin{pmatrix} -1-\lambda & 3 \\ -2 & 4-\lambda \end{pmatrix} = (\lambda-4)(\lambda+1)+6$

$$= 0$$

$$\lambda^2 - 3\lambda + 2$$

$$= (\lambda-1)(\lambda-2)$$

$\boxed{\lambda = 1, \lambda = 2}$

b) Find non-triv solns to $(A - \lambda I)\vec{r} = \vec{0}$

$\underline{\underline{\lambda = 1}}$ $\left[\begin{array}{cc|c} -1-(1) & 3 & 0 \\ -2 & 4-(1) & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -2 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$r_2 = \alpha$

$\alpha = 1$ WLOG

$-2r_1 + 3\alpha = 0$

$r_1 = \frac{3}{2}\alpha$

$\boxed{\vec{r} = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}}$

$\underline{\underline{\lambda = 2}}$ $\left[\begin{array}{cc|c} -1-(2) & 3 & 0 \\ -2 & 4-(2) & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$

$r_2 = \alpha$

$\vec{r} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

c) to compute e^{At}

$$R = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} \quad \text{instead of} \quad \begin{pmatrix} 3/2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$R^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$$

$$e^{At} = \underline{\underline{R e^{\Lambda t} R^{-1}}}$$

This is correct order!

$$= \underbrace{\begin{array}{cc} \lambda=1 & \lambda=2 \\ \end{array}\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}}_{R} \underbrace{\begin{pmatrix} e^{t} & 0 \\ 0 & e^{2t} \end{pmatrix}}_{e^{\Lambda t}} \underbrace{\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}}_{R^{-1}}$$

$$= \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} e^{t} & -e^{t} \\ -2e^{2t} & 3e^{2t} \end{pmatrix}$$

$$\boxed{= \begin{pmatrix} 3e^{t}-2e^{2t} & -3e^{t}+3e^{2t} \\ 2e^{t}-2e^{2t} & -2e^{t}+3e^{2t} \end{pmatrix}}$$

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d) to solve $\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -u+3v \\ -2u+4v \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$

$\begin{pmatrix} u(0) \\ v(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\begin{pmatrix} u \\ v \end{pmatrix} = e^{At}\vec{u}_0$

$= \begin{pmatrix} 3e^t - 2e^{2t} & -3e^t + 3e^{2t} \\ 2e^t - 2e^{2t} & -2e^t + 3e^{2t} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$$\boxed{\begin{aligned} u(t) &= 3e^t - 2e^{2t} \\ v(t) &= 2e^t - 2e^{2t} \end{aligned}}$$