Math 3333: Sequences, Subsequences and Limits

A *sequence* of points in a metric space $(\mathcal{M}, d)$ is a function from $\mathbb{N}$ into $\mathcal{M}$; that is for every $n \in \mathbb{N}$ the point denoted by x_n is an element of $\mathcal{M}$. A *subsequence* of a sequence x_n is given by $y_k = x_{n_k}$ where for every $k \in \mathbb{N}$ we have $n_k \in \mathbb{N}$ is a strictly increasing function of k. For example, x_2, x_5, x_{19}, x_{33}, $x_{123}, \ldots$ are the first five terms of a subsequence of x_n (here $n_1 = 2$, $n_2 = 5$, $n_3 = 19$, $n_4 = 33$, $n_5 = 123$) but x_4, x_6, x_3, x_{19}, $x_{61}, \ldots$ are not (because $n_3 = 3$ is not greater than $n_2 = 6$).

We say a sequence x_n is *convergent* if there is a point $x_* \in \mathcal{M}$ such that

$$\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} \text{ such that } d(x_n, x_*) < \epsilon \text{ whenever } n \geq N.$$

The notations '$x_n \to x_*$ as $n \to \infty$' or '$x_* = \lim_{n\to\infty} x_n$' are used interchangeably. Generally speaking, to check whether a given sequence is convergent requires explicitly knowing its limit.

A *Cauchy sequence*, say x_n, has the property that

$$\forall \epsilon > 0 \;\; \exists N \in \mathbb{N} \text{ such that } d(x_m, x_n) < \epsilon \text{ whenever } m \geq N \text{ and } n \geq N.$$

To check whether a given sequence is Cauchy does <u>not</u> require explicitly knowing its limit. Generally speaking however, a Cauchy sequence need not be convergent.

Here are two simple facts you should know how to prove. (These were done in class.)

• A convergent sequence is Cauchy. This is easy to see. According to the definition of a convergent sequence, there is an x_* such that $x_* = \lim_{n\to\infty} x_n$. Therefore for any given $\epsilon > 0$ there is an N such that $d(x_n, x_*) < \epsilon/2$ whenever $n \geq N$. So, for any $m \geq N$ and $n \geq N$ the triangle inequality gives

$$d(x_m, x_n) \leq d(x_m . x_*) + d(x_*, x_n) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which shows the given sequence is Cauchy.

• A Cauchy sequence is bounded. This is a bit trickier to see. Take $\epsilon = 1$ for example, and use the fact that the sequence is Cauchy to conclude there is an N such that $d(x_m . x_n) < 1$ for any $m \geq N$ and $n \geq N$. In particular this says we have $d(x_N, x_n) < 1$ for any $n \geq N$. Let $R = \max(d(x_N, x_1),\, d(x_N, x_2), \ldots, d(x_N, x_{N-1}),\, 1)$ and check that $d(x_N, x_n) \leq R$ for every $n = 1, 2, 3, \ldots$ Therefore, a Cauchy sequence is bounded.

A *complete metric space* is a metric space in which every Cauchy sequence is convergent. Obviously not every metric space is complete; consider $(\mathbb{Q}, |\cdot|)$ for example. Below, however, we will show the metric space $(\mathbb{R}, |\cdot|)$ is complete. Many metric spaces which arise in important applications are complete, this is just one example.

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1. Suppose a sequence x_n is a Cauchy sequence of points from a metric space. In addition suppose there is a subsequence x_{n_k} which converges to x_* as $k \to \infty$. Prove that x_n converges to x_* as $n \to \infty$. Hint: $d(x_n, x_*) \le d(x_n, x_{n_k}) + d(x_{n_k}, x_*)$, then on the first term use the fact that x_n is Cauchy and on the second use $x_* = \lim_{k\to\infty} x_{n_k}$.

2. Suppose $(\mathcal{M}, d)$ is a complete metric space and let $\mathcal{S} \subseteq \mathcal{M}$. (a) If $\mathcal{S}$ is closed prove the metric space $(\mathcal{S}, d)$ is complete. Note: You must show for $x_n \in \mathcal{S}$ and $x_* = \lim_{n\to\infty} x_n$ we must have $x_* \in \mathcal{S}$. (b) If $\mathcal{S}$ is not closed conclude the metric space $(\mathcal{S}, d)$ is not complete. Hint: Let $x_* \in \mathcal{M}$ be an accumulation point of $\mathcal{S}$ which is not in $\mathcal{S}$.

3. Suppose x_n is a convergent sequence with $x_* = \lim_{n\to\infty} x_n$. Let x_{n_k} denote any subsequence of x_n. Prove x_{n_k} is also convergent and $x_* = \lim_{k\to\infty} x_{n_k}$.

4. This exercise is a bit tricky. Let $\mathcal{K}$ be a compact set in a metric space. Suppose a given sequence of points satisfies $x_n \in \mathcal{K}$ for every $n \in \mathbb{N}$. (a) Show the following statement is impossible.

$$\forall\, x \in \mathcal{K} \text{ there is an } \epsilon_x > 0 \text{ such that } B(x, \epsilon_x) \text{ contains } x_n \text{ finitely often.}$$

Hint: If this were true then $\bigcup_{x\in\mathcal{K}} B(x, \epsilon_x)$ is an open cover of $\mathcal{K}$. A finite subcover therefore can not contain x_n for every $n \in \mathbb{N}$. (b) Use part (a) to conclude the following statement is true.

$$\exists\, x_* \in \mathcal{K} \text{ such that for any } \epsilon > 0 \text{ we have } B(x_*, \epsilon) \text{ contains } x_n \text{ infinitely often.}$$

(c) Use part (b) in order to come up with an algorithm to extract a subsequence x_{n_k} from the original sequence x_n such that $x_* = \lim_{k\to\infty} x_{n_k}$. Hint: Try something like this. For every $k \in \mathbb{N}$ show there is an $n_{k+1} > n_k$ such that $d(x_{n_{k+1}}, x_*) < 1/k$.

In what follows we restrict our attention to the metric space $(\mathbb{R}, |\cdot|)$. Please note that below when I talk about a convergent sequence of real numbers I'm talking about convergence with respect to the $|\cdot|$ metric.

A sequence of real numbers x_n is called *monotone increasing* when $x_{n+1} \ge x_n$ for every $n \in \mathbb{N}$. It's called *monotone decreasing* when $x_n \ge x_{n+1}$ for every $n \in \mathbb{N}$. A sequence is simply called *monotone* when it's either monotone increasing or monotone decreasing.

• A bounded monotone sequence of real numbers is convergent. The proof of this fact is very simple. Suppose x_n is monotone increasing (the other case is similar) and therefore bounded above. Define a set $\mathcal{S} \equiv \{x_n : n \in \mathbb{N}\}$. $\mathcal{S}$ is nonempty and bounded above, and so $b = \sup \mathcal{S}$ exists. The definition of the supremum tells us for any $\epsilon > 0$ there is an

$x_N \in \mathcal{S}$ such that $b - \epsilon < x_N \leq b$. Also, the fact that x_n is increasing tells us for any $n \geq N$ we have $b - \epsilon < x_N \leq x_n \leq b \Rightarrow |x_n - b| < \epsilon$. Therefore $\lim_{n\to\infty} x_n$ exists and its value is b.

The next result is very well known.

• Let x_n be a bounded sequence of real numbers. From this bounded sequence we can always extract a convergent subsequence x_{n_k}, i.e. $x_* = \lim_{k\to\infty} x_{n_k}$ for some $x_* \in \mathbb{R}$. This important fact is often referred to as the *Bolzano-Weierstrass Theorem.* I showed you the standard 'divide and conquer' proof in class – but probably not with great success. Use exercise 4 above to discover your own alternate proof.

• The metric space $(\mathbb{R}, |\cdot|)$ is complete. Here's the proof. A Cauchy sequence is bounded. Bolzano-Weierstrass says a bounded sequence of real numbers has a convergent subsequence. The result of exercise 1 above tells us a Cauchy sequence is convergent when it has a convergent subsequence. Therefore a Cauchy sequence of real numbers is convergent.

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5. Recall these standard limit theorems from calculus. Let x_n and y_n denote two convergent sequences of real numbers with say $x = \lim_{n\to\infty} x_n$ and $y = \lim_{n\to\infty} y_n$. Prove the following.

(a) $x_n + y_n$ is convergent and $x + y = \lim_{n\to\infty}(x_n + y_n)$.

(b) $x_n y_n$ is convergent and $xy = \lim_{n\to\infty}(x_n y_n)$.

If in addition $y_n \neq 0$ for every n and the limit $y \neq 0$

(c) x_n/y_n is convergent and $x/y = \lim_{n\to\infty}(x_n/y_n)$.

6. Here's more stuff from calculus. Suppose a_k denotes a sequence of real numbers. (a) Prove the sequence of *partial sums* $S_n = \sum_{k=1}^n |a_k|$ for $n = 1, 2, 3, \ldots$ is convergent if and only if S_n is bounded. Hint: Observe that here S_n is a nondecreasing sequence. (b) Given that $S_n = \sum_{k=1}^n a_k$ is convergent, show it is necessary that $\lim_{k\to\infty} a_k = 0$. Hint: S_n must be Cauchy, then consider $|S_n - S_{n-1}|$. (c) Given that $\sum_{k=1}^n |a_k|$ is bounded, show it must be true that $S_n = \sum_{k=1}^n a_k$ is convergent. Hint: WLOG take $n > m \geq 1$ and use $|S_n - S_m| = \left|\sum_{k=m+1}^n a_k\right| \leq \sum_{k=m+1}^n |a_k|$. (d) Give an example where $\sum_{k=1}^n a_k$ is bounded but not convergent.

7. Consider a sequence of real numbers x_n given by the iteration scheme: $x_1 = 1/4$ and for $n = 1, 2, 3, \ldots$ $x_{n+1} = 2(x_n - x_n^2)$. (a) Show $x_{n+1} - 1/2 = -2(x_n - 1/2)^2$. (b) Show $x_{n+1} - x_n = x_n(1 - 2x_n)$. (c) Use parts (a) and (b) to conclude x_n is nondecreasing and bounded above and is therefore convergent. (d) Determine the value of $x = \lim_{n\to\infty} x_n$. Answer: $x = 1/2$.

8. Suppose a real valued function $f(x)$ satisfies $|f(x) - f(y)| \le \frac{1}{2}|x - y|$ for all $x,\, y \in \mathbb{R}$. Consider a sequence of real numbers x_n given by the iteration scheme: $x_{n+1} = f(x_n)$ for $n = 1, 2, 3, \ldots$ where x_1 is given but arbitrary. (a) Show that

$$|x_{n+1} - x_n| \le \frac{1}{2}|x_n - x_{n-1}| \quad \Rightarrow \quad |x_n - x_{n-1}| \le \left(\frac{1}{2}\right)^{n-2} |x_2 - x_1|.$$

(b) Now show that for $n > m \ge 1$

$$x_n - x_m = \sum_{k=m+1}^{n} (x_k - x_{k-1}) \quad \Rightarrow \quad |x_n - x_m| \le \sum_{k=m+1}^{n} \left(\frac{1}{2}\right)^{k-2} |x_2 - x_1|.$$

(c) Recall the geometric series from calculus. Specifically

$$\sum_{k=1}^{n} \left(\frac{1}{2}\right)^{k} = 1 - \left(\frac{1}{2}\right)^{n}.$$

Use this and the estimate you obtained in part (b) to prove the sequence x_n is Cauchy and is therefore convergent. FYI. $x = \lim_{n\to\infty} x_n$ solves the equation $x = f(x)$. This exercise is the essence of what's called the Banach fixed point theorem.