Here I derive several identities found in Table 3.2 on page 147 of your text book. They're not in order, but all of the more difficult ones are here. Ones near the top are those I consider the easiest. As you go down they become more difficult.

3.40: $\nabla \times (\nabla \phi) = \mathbf{0}$

$$\begin{aligned}
\nabla \times (\nabla \phi) &= \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} \times \sum_j \frac{\partial \phi}{\partial x_j} \mathbf{e}_j \\
&= \sum_{i,j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} (\mathbf{e}_i \times \mathbf{e}_j) \equiv \mathbf{T}
\end{aligned}$$

Now, in term $\mathbf{T}$ above, reindex the sum so that $i \to j$, $j \to i$ to find

$$\mathbf{T} = \sum_{j,i} \frac{\partial^2 \phi}{\partial x_j \partial x_i} (\mathbf{e}_j \times \mathbf{e}_i) = \sum_{i,j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} (\mathbf{e}_j \times \mathbf{e}_i) .$$

Since $(\mathbf{e}_j \times \mathbf{e}_i) = -(\mathbf{e}_i \times \mathbf{e}_j)$, this shows

$$\mathbf{T} = -\sum_{i,j} \frac{\partial^2 \phi}{\partial x_i \partial x_j} (\mathbf{e}_i \times \mathbf{e}_j) = -\mathbf{T},$$

which implies $\mathbf{T} = \mathbf{0}$.

3.41: $\nabla \cdot (\nabla \times \mathbf{F}) = 0$

First, we write

$$\nabla \times \mathbf{F} = \sum_j \mathbf{e}_j \frac{\partial}{\partial x_j} \times \sum_k F_k \mathbf{e}_k = \sum_{j,k} \frac{\partial F_k}{\partial x_j} (\mathbf{e}_j \times \mathbf{e}_k).$$

Now, compute that

$$\begin{aligned}
\nabla \cdot (\nabla \times \mathbf{F}) &= \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot \sum_{j,k} \frac{\partial F_k}{\partial x_j} (\mathbf{e}_j \times \mathbf{e}_k) \\
&= \sum_{i,j,k} \frac{\partial^2 F_k}{\partial x_i \partial x_j} \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) \\
&= \sum_k \mathbf{e}_k \cdot \left(\sum_{i,j} \frac{\partial^2 F_k}{\partial x_i \partial x_j} (\mathbf{e}_i \times \mathbf{e}_j) \right) \equiv \sum_k \mathbf{e}_k \cdot (\mathbf{T}_k) ,
\end{aligned}$$

where $\mathbf{T}_k = \sum_{i,j} \frac{\partial^2 F_k}{\partial x_i \partial x_j} (\mathbf{e}_i \times \mathbf{e}_j)$. Reindex this sum so that $i \to j$, $j \to i$ to find

$$\mathbf{T}_k = \sum_{j,i} \frac{\partial^2 F_k}{\partial x_j \partial x_i} (\mathbf{e}_j \times \mathbf{e}_i) = \sum_{i,j} \frac{\partial^2 F_k}{\partial x_i \partial x_j} (\mathbf{e}_j \times \mathbf{e}_i).$$

Again, since $(\mathbf{e}_j \times \mathbf{e}_i) = -(\mathbf{e}_i \times \mathbf{e}_j)$, this shows

$$\mathbf{T}_k = -\sum_{i,j} \frac{\partial^2 F_k}{\partial x_i \partial x_j} (\mathbf{e}_i \times \mathbf{e}_j) = -\mathbf{T}_k,$$

which implies $\mathbf{T}_k = \mathbf{0}$ for every k. Therefore $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.

3.42: $\nabla \cdot (\nabla \phi_1 \times \nabla \phi_2) = 0$
Calculate that

$$\begin{aligned}\nabla \cdot (\nabla \phi_1 \times \nabla \phi_2) &= \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot \sum_{j,k} \frac{\partial \phi_1}{\partial x_j} \frac{\partial \phi_2}{\partial x_k} (\mathbf{e}_j \times \mathbf{e}_k) \\ &= \sum_{i,j,k} \frac{\partial}{\partial x_i} \left(\frac{\partial \phi_1}{\partial x_j} \frac{\partial \phi_2}{\partial x_k} \right) \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k).\end{aligned}$$

Use the product rule to split this last sum into two pieces:

$$\begin{aligned}&\sum_{i,j,k} \frac{\partial \phi_1}{\partial x_j} \frac{\partial^2 \phi_2}{\partial x_i \partial x_k} \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) + \sum_{i,j,k} \frac{\partial \phi_2}{\partial x_k} \frac{\partial^2 \phi_1}{\partial x_i \partial x_j} \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) \\ &\equiv I + II.\end{aligned}$$

The fact $\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = \mathbf{e}_j \cdot (\mathbf{e}_k \times \mathbf{e}_i)$ allows us to rewrite term I above as

$$I = \sum_j \frac{\partial \phi_1}{\partial x_j} \mathbf{e}_j \cdot \left(\sum_{i,k} \frac{\partial^2 \phi_2}{\partial x_i \partial x_k} (\mathbf{e}_k \times \mathbf{e}_i) \right) = \nabla \phi_1 \cdot (\nabla \times (\nabla \phi_2)),$$

and example (3.40) shows this is zero. Term II is shown to be zero similarly.

3.29: $\nabla \times (\phi \mathbf{F}) = \phi (\nabla \times \mathbf{F}) + \nabla \phi \times \mathbf{F}$
It should be easy by now to see that

$$\nabla \times (\phi \mathbf{F}) = \sum_{i,j} \frac{\partial}{\partial x_i} (\phi F_j) (\mathbf{e}_i \times \mathbf{e}_j).$$

Via the product rule, this is equal to

$$\begin{aligned} &\sum_{i,j} \phi \frac{\partial F_j}{\partial x_i} (\mathbf{e}_i \times \mathbf{e}_j) + \sum_{i,j} F_j \frac{\partial \phi}{\partial x_i} (\mathbf{e}_i \times \mathbf{e}_j) \\ &\equiv I + II. \end{aligned}$$

Term I is clearly seen to be equal to $\phi (\nabla \times \mathbf{F})$. Term II can be rewritten as

$$II = \sum_i \frac{\partial \phi}{\partial x_i} \mathbf{e}_i \times \sum_j F_j \mathbf{e}_j = \nabla \phi \times \mathbf{F}.$$

3.36: $\nabla \cdot (\mathbf{F} \times \mathbf{G}) = (\nabla \times \mathbf{F}) \cdot \mathbf{G} - \mathbf{F} \cdot (\nabla \times \mathbf{G})$
This one is pretty straight forward.

$$\begin{aligned} \nabla \cdot (\mathbf{F} \times \mathbf{G}) &= \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} \cdot \sum_{j,k} F_j G_k (\mathbf{e}_j \times \mathbf{e}_k) = \sum_{i,j,k} \frac{\partial}{\partial x_i} (F_j G_k) \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) \\ &= \sum_{i,j,k} \frac{\partial F_j}{\partial x_i} G_k \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) + \sum_{i,j,k} F_j \frac{\partial G_k}{\partial x_i} \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) \\ &\equiv I + II. \end{aligned}$$

In term I, use the fact that $\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = \mathbf{e}_k \cdot (\mathbf{e}_i \times \mathbf{e}_j)$ to write

$$\begin{aligned} I = \sum_{i,j,k} \frac{\partial F_j}{\partial x_i} G_k \mathbf{e}_k \cdot (\mathbf{e}_i \times \mathbf{e}_j) &= \sum_{i,j} \frac{\partial F_j}{\partial x_i} (\mathbf{e}_i \times \mathbf{e}_j) \cdot \sum_k G_k \mathbf{e}_k \\ &= (\nabla \times \mathbf{F}) \cdot \mathbf{G}. \end{aligned}$$

In term II, use the fact that $\mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) = \mathbf{e}_j \cdot (\mathbf{e}_k \times \mathbf{e}_i)$ to write

$$\begin{aligned} II = \sum_{i,j,k} F_j \frac{\partial G_k}{\partial x_i} \mathbf{e}_j \cdot (\mathbf{e}_k \times \mathbf{e}_i) &= \sum_j F_j \mathbf{e}_j \cdot \sum_{i,k} \frac{\partial G_k}{\partial x_i} (\mathbf{e}_k \times \mathbf{e}_i) \\ &= \mathbf{F} \cdot (-\nabla \times \mathbf{G}) = -\mathbf{F} \cdot (\nabla \times \mathbf{G}). \end{aligned}$$

3.37: $\nabla \times (\mathbf{F} \times \mathbf{G}) = (\nabla \cdot \mathbf{G})\mathbf{F} + (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} - (\nabla \cdot \mathbf{F})\mathbf{G}$

This one requires a fair amount of work. Recall that the $(\mathbf{F} \cdot \nabla)$ operator is defined by $\sum_i F_i \frac{\partial}{\partial x_i}$ and may act on vector valued functions as well as scalar valued functions.

$$\begin{aligned}\nabla \times (\mathbf{F} \times \mathbf{G}) &= \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} \times \sum_{j,k} F_j G_k (\mathbf{e}_j \times \mathbf{e}_k) \\ &= \sum_{i,j,k} \frac{\partial}{\partial x_i} (F_j G_k)\, \mathbf{e}_i \times (\mathbf{e}_j \times \mathbf{e}_k)\end{aligned}$$

Use the product rule and the fact that

$$\mathbf{e}_i \times (\mathbf{e}_j \times \mathbf{e}_k) = (\mathbf{e}_i \cdot \mathbf{e}_k)\mathbf{e}_j - (\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{e}_k$$

to obtain

$$\begin{aligned}&\nabla \times (\mathbf{F} \times \mathbf{G}) \\ &= \sum_{i,j,k} \left(F_j \frac{\partial G_k}{\partial x_i} + \frac{\partial F_j}{\partial x_i} G_k \right) ((\mathbf{e}_i \cdot \mathbf{e}_k)\mathbf{e}_j - (\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{e}_k) \\ &= \sum_{i,j,k} F_j \frac{\partial G_k}{\partial x_i} (\mathbf{e}_i \cdot \mathbf{e}_k)\mathbf{e}_j \quad (\equiv I) \\ &+ \sum_{i,j,k} \frac{\partial F_j}{\partial x_i} G_k (\mathbf{e}_i \cdot \mathbf{e}_k)\mathbf{e}_j \quad (\equiv II) \\ &- \sum_{i,j,k} F_j \frac{\partial G_k}{\partial x_i} (\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{e}_k \quad (\equiv III) \\ &- \sum_{i,j,k} \frac{\partial F_j}{\partial x_i} G_k (\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{e}_k \quad (\equiv IV).\end{aligned}$$

I will evaluate terms I and II. You do III and IV as an exercise.

$$I = \left(\sum_{i,k} \frac{\partial G_k}{\partial x_i} (\mathbf{e}_i \cdot \mathbf{e}_k) \right) \sum_j F_j \mathbf{e}_j = (\nabla \cdot \mathbf{G})\, \mathbf{F}$$

$$\begin{aligned}II &= \sum_{i,j} \frac{\partial F_j}{\partial x_i} \sum_k G_k (\mathbf{e}_i \cdot \mathbf{e}_k)\mathbf{e}_j = \sum_{i,j} G_i \frac{\partial F_j}{\partial x_i} \mathbf{e}_j \\ &= \sum_i G_i \frac{\partial}{\partial x_i} \left(\sum_j F_j \mathbf{e}_j \right) = (\mathbf{G} \cdot \nabla)\mathbf{F}\end{aligned}$$

3.38: $\nabla \times (\nabla \times \mathbf{F}) = \nabla (\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F}$

Recall that the Laplacian operator is defined by $\nabla^2 \equiv \sum_i \frac{\partial^2}{\partial x_i^2}$ and may act on vector valued functions as well as scalar valued functions.

$$\begin{aligned}\nabla \times (\nabla \times \mathbf{F}) &= \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} \times \sum_{j,k} \frac{\partial F_k}{\partial x_j} (\mathbf{e}_j \times \mathbf{e}_k) \\ &= \sum_{i,j,k} \frac{\partial^2 F_k}{\partial x_i \partial x_j} \mathbf{e}_i \times (\mathbf{e}_j \times \mathbf{e}_k)\end{aligned}$$

The formula stated earlier for the triple cross product allows us to write this as

$$\begin{aligned}&\sum_{i,j,k} \frac{\partial^2 F_k}{\partial x_i \partial x_j} ((\mathbf{e}_i \cdot \mathbf{e}_k)\mathbf{e}_j - (\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{e}_k) \\ =&\sum_{i,j,k} \frac{\partial^2 F_k}{\partial x_i \partial x_j} (\mathbf{e}_i \cdot \mathbf{e}_k)\mathbf{e}_j - \sum_{i,j,k} \frac{\partial^2 F_k}{\partial x_i \partial x_j} (\mathbf{e}_i \cdot \mathbf{e}_j)\mathbf{e}_k \\ &\equiv I - II.\end{aligned}$$

Term I can be rewritten as

$$I = \sum_j \mathbf{e}_j \frac{\partial}{\partial x_j} \left(\sum_{i,k} \frac{\partial F_k}{\partial x_i} (\mathbf{e}_i \cdot \mathbf{e}_k) \right) = \nabla (\nabla \cdot \mathbf{F}).$$

Term II can be rewritten as

$$II = \sum_{i,j} (\mathbf{e}_i \cdot \mathbf{e}_j) \frac{\partial^2}{\partial x_i \partial x_j} \left(\sum_k F_k \mathbf{e}_k \right) = \sum_i \frac{\partial^2}{\partial x_i^2} (\mathbf{F}) = \nabla^2 \mathbf{F}.$$

Last one, and it's probably the most difficult of the bunch. Here's a sketch. Give it a try.

3.39: $\nabla (\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla) \mathbf{G} + \mathbf{F} \times (\nabla \times \mathbf{G}) + (\mathbf{G} \cdot \nabla) \mathbf{F} + \mathbf{G} \times (\nabla \times \mathbf{F})$

First, write

$$\nabla (\mathbf{F} \cdot \mathbf{G}) = \sum_i \mathbf{e}_i \frac{\partial}{\partial x_i} \sum_{j,k} F_j G_k (\mathbf{e}_j \cdot \mathbf{e}_k) = \sum_{i,j,k} \frac{\partial}{\partial x_i} (F_j G_k) (\mathbf{e}_j \cdot \mathbf{e}_k) \mathbf{e}_i.$$

Use the product rule to write this as

$$\begin{aligned}&\sum_{i,j,k} F_j \frac{\partial G_k}{\partial x_i} (\mathbf{e}_j \cdot \mathbf{e}_k) \mathbf{e}_i + \sum_{i,j,k} G_k \frac{\partial F_j}{\partial x_i} (\mathbf{e}_j \cdot \mathbf{e}_k) \mathbf{e}_i \\ &\equiv I + II.\end{aligned}$$

Vector identity 1.30 on page 59, with $\mathbf{A} = \mathbf{e}_j$, $\mathbf{B} = \mathbf{e}_i$ and $\mathbf{C} = \mathbf{e}_k$, gives

$$(\mathbf{e}_j \cdot \mathbf{e}_k)\,\mathbf{e}_i = \mathbf{e}_j \times (\mathbf{e}_i \times \mathbf{e}_k) + (\mathbf{e}_j \cdot \mathbf{e}_i)\,\mathbf{e}_k.$$

Use this in term I above and manipulate to determine that

$$\sum_{i,j,k} F_j \frac{\partial G_k}{\partial x_i}\,(\mathbf{e}_j \cdot \mathbf{e}_k)\,\mathbf{e}_i = \mathbf{F} \times (\nabla \times \mathbf{G}) + (\mathbf{F} \cdot \nabla)\,\mathbf{G}.$$

Do the same with term II after reversing the roles of j and k.