Math 3335 Supplemental Notes and Homework 1

A vector space is composed of a set of vectors, say $\mathcal{V}$, and an associated scalar field, say $\mathcal{F}$. Below, let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ denote arbitrary vectors in $\mathcal{V}$, and let α and β denote arbitrary scalars from $\mathcal{V}$'s scalar field $\mathcal{F}$. Vector addition, $\mathbf{x}+\mathbf{y}$, and scalar multiplication, $\alpha\mathbf{x}$, must be defined for all vectors and scalars, and these operations must adhere to certain requirements.

The result of vector addition, $\mathbf{x}+\mathbf{y}$, is a vector in $\mathcal{V}$. Moreover:

(1) Vector addition must be associative. That is $\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z}$.

(2) Vector addition must commute. That is $\mathbf{x}+\mathbf{y} = \mathbf{y}+\mathbf{x}$.

(3) There is a unique vector $\mathbf{0} \in \mathcal{V}$ such that for every $\mathbf{x} \in \mathcal{V}$ we have $\mathbf{x}+\mathbf{0} = \mathbf{x}$. The vector $\mathbf{0}$ is called the additive identity.

(4) For every $\mathbf{x} \in \mathcal{V}$, there exists a $\tilde{\mathbf{x}} \in \mathcal{V}$ such that $\mathbf{x}+\tilde{\mathbf{x}} = \mathbf{0}$. The vector $\tilde{\mathbf{x}}$ is called the additive inverse for $\mathbf{x}$ and is denoted by $-\mathbf{x}$.

The result of scalar multiplication, $\alpha\mathbf{x}$, is a vector in $\mathcal{V}$. Moreover:

(5) Scalar multiplication must distribute with respect to vector addition. That is we must have $\alpha(\mathbf{x}+\mathbf{y}) = \alpha\mathbf{x}+\alpha\mathbf{y}$.

(6) Scalar multiplication must distribute with respect to field addition. That is we must have $(\alpha+\beta)\mathbf{x} = \alpha\mathbf{x}+\beta\mathbf{x}$.

(7) Scalar multiplication must be compatible with field multiplication. That is we must have $\alpha(\beta\mathbf{x}) = (\alpha\beta)\mathbf{x}$.

(8) If 1 is the scalar field's multiplicative identity, we must have $1\mathbf{x} = \mathbf{x}$.

For those interested, see the additional exercise I give you on page 4.

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1. Prove the following. Justify each step by stating which of the above properties was used. (a) $0\mathbf{x} = \mathbf{0}$, that is the scalar field's additive identity 0 times any vector $\mathbf{x}$ is the vector additive identity $\mathbf{0}$. (b) $-1\mathbf{x} = -\mathbf{x}$, that is the additive inverse of the scalar field's multiplicative identity -1 times a vector $\mathbf{x}$ is that vector's additive inverse $-\mathbf{x}$.

2. Suppose the set of vectors is composed of all real column matrices

$$\mathcal{V} = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \ x_1 \in \mathbb{R},\ x_2 \in \mathbb{R} \right\},$$

and the associated scalar field is $\mathbb{R}$. Define vector addition and scalar multiplication by

$$\mathbf{x}+\mathbf{y} \equiv \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}, \quad \alpha\mathbf{x} \equiv \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}.$$

Show this defines a vector space.

3. Suppose vectors are of the form above, and scalars, addition and multiplication are as given. However, this time let $\mathcal{V}$ be the following subsets.

(a) $\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \geq 0\right\}$ (b) $\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 + x_2 = 0\right\}$

(c) $\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 + x_2 = 1\right\}$ (d) $\left\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 = 0\right\}$

Which defines a vector space?

The vector space $\mathbb{R}^n$, $n = 2$ or 3,..., is composed of real column matrices

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

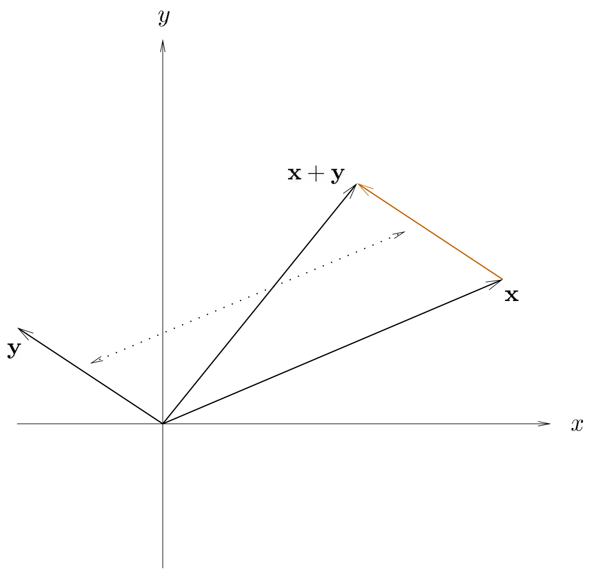
and scalars $\mathbb{R}$ with vector addition and scalar multiplication as defined for matrices. The *standard basis* for $\mathbb{R}^n$ is given by

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ \ldots, \ \mathbf{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}.$$

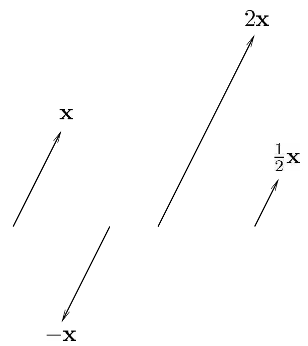
Therefore $\mathbf{x}$ can be written as the linear combination

$$\mathbf{x} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + \cdots + x_n\,\mathbf{e}_n.$$

Vectors, vector addition and scalar multiplication on $\mathbb{R}^2$ are graphically illustrated below.



Vectors $\mathbf{x}$, $\mathbf{y}$ and their vector sum $\mathbf{x}+\mathbf{y}$.



Scalar multiplication.

Here are some homework exercises from §1.7 of your text book. You may assume all described figures are two dimensional; (i.e. planar).

4. Exercise 7, page 23.

5. Exercise 8, page 23.

6. Exercise 9, page 24.

7. Exercise 10, page 24.

A norm on a vector space $\mathcal{V}$, say $n(\mathbf{x})$, is a well defined mapping from $\mathcal{V}$ into the nonnegative reals which satisfies the following properties. For ease of presentation, let's suppose $\mathcal{V}$'s scalar field is $\mathbb{R}$. (The phrase *real vector space* is used to specify that the scalar field is the set of real numbers.)

(1) For every $\mathbf{x} \in \mathcal{V}$, $n(\mathbf{x}) \geq 0$. Moreover $n(\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(2) For every $\mathbf{x} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$, $n(\alpha\mathbf{x}) = |\alpha|\, n(\mathbf{x})$.

(3) For every $\mathbf{x} \in \mathcal{V}$ and $\mathbf{y} \in \mathcal{V}$, $n(\mathbf{x}+\mathbf{y}) \leq n(\mathbf{x}) + n(\mathbf{y})$.

Item 3 is often called the *triangle inequality.* The norm of a vector $\mathbf{x}$ is often denoted by $||\mathbf{x}||$, not $n(\mathbf{x})$ as done above. I'll use this double bar notation fairly consistently. Your text book on the other hand uses $|\mathbf{x}|$ to denote the norm of a vector $\mathbf{x}$.

There are many different norms defined on the vector space $\mathbb{R}^n$. Here are but a few.

(1-norm) $\qquad ||\mathbf{x}||_1 \equiv |x_1| + \cdots + |x_n|$

(2-norm) $\qquad ||\mathbf{x}||_2 \equiv \sqrt{x_1^2 + \cdots + x_n^2}$

(∞-norm) $\qquad ||\mathbf{x}||_\infty \equiv \max(|x_1|, \ldots, |x_n|)$

The ∞-norm is often referred to as the max-norm for obvious reasons. The 2-norm is often referred to as the Euclidean-norm. Throughout this course, if a norm is left unsubscripted, it will refer to the 2-norm.

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8. Prove the two formulae given above for the 1-norm and the ∞-norm both satisfy all properties required of a norm. You may assume $n = 2$. (These make very good exam questions.)

9. Sketch the following regions in $\mathbb{R}^2$.

(a) $\{\mathbf{x} : ||\mathbf{x}||_1 \leq 1\}$ (b) $\{\mathbf{x} : ||\mathbf{x}||_2 \leq 1\}$ (c) $\{\mathbf{x} : ||\mathbf{x}||_\infty \leq 1\}$

10. In class, I will established the fact that the formula for the Euclidean norm given above is in fact a norm on the vector space $\mathbb{R}^2$. The hard part will be to verify property (3) – the triangle inequality. Use what I do in class to generalized this result to $\mathbb{R}^n$.

An Interesting Additional Exercise

1. Suppose we identify vectors $\mathbf{x}$ by $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ where x_1 and x_2 are real numbers. Now, suppose we define a crazy notion of vector addition by

$$\mathbf{x} + \mathbf{y} \equiv \begin{pmatrix} x_1 + y_1 - 1 \\ x_2 + y_2 - 1 \end{pmatrix},$$

and scalar multiplication by a real number α by

$$\alpha\mathbf{x} \equiv \begin{pmatrix} \alpha(x_1 - 1) + 1 \\ \alpha(x_2 - 1) + 1 \end{pmatrix}.$$

Show that the resulting system satisfies properties 1–8 listed on page 1. Explicitly calculate the additive identity vector, $\mathbf{0}$, and the additive inverse of $\mathbf{x}$, $-\mathbf{x}$.