An inner product on a real vector space \( \mathcal{V} \), for the moment say \( a(\mathbf{x}, \mathbf{y}) \), is a well defined mapping \( a: \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R} \) which satisfies the following properties.

(1) For every \( \mathbf{x} \in \mathcal{V} \), \( a(\mathbf{x}, \mathbf{x}) \geq 0 \). Moreover \( a(\mathbf{x}, \mathbf{x}) = 0 \) if and only if \( \mathbf{x} = \mathbf{0} \).

(2) For every \( \mathbf{x} \in \mathcal{V} \) and \( \mathbf{y} \in \mathcal{V} \), \( a(\mathbf{x}, \mathbf{y}) = a(\mathbf{y}, \mathbf{x}) \).

(3) For every \( \mathbf{x} \in \mathcal{V} \), \( \mathbf{y} \in \mathcal{V} \) and \( \mathbf{z} \in \mathcal{V} \) and every \( \alpha \in \mathbb{R} \) and \( \beta \in \mathbb{R} \) we must have that 
\[
a(\alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{z}) = \alpha a(\mathbf{x}, \mathbf{z}) + \beta a(\mathbf{y}, \mathbf{z}).
\]

Item 2 is often called symmetry. Item 3 says the inner product is linear in its first argument. Because of symmetry, it is also linear in its second argument. Such mappings are called bilinear. The inner product of vectors \( \mathbf{x} \) and \( \mathbf{y} \) is often simply denoted by \( (\mathbf{x}, \mathbf{y}) \).

1. Show the following both define an inner product on \( \mathbb{R}^2 \).
   
   (a) \( (\mathbf{x}, \mathbf{y}) = x_1y_1 + x_2y_2 \)  
   (b) \( (\mathbf{x}, \mathbf{y}) = 2x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2 \)

2. Let \( \mathbf{x}' \) and \( \mathbf{y}' \), both in \( \mathbb{R}^2 \), denote rotations of vectors \( \mathbf{x} \) and \( \mathbf{y} \). Specifically
   
   \[
   \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{x}' = \begin{pmatrix} \cos \theta x_1 - \sin \theta x_2 \\ \sin \theta x_1 + \cos \theta x_2 \end{pmatrix},
   \]
   and
   
   \[
   \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \mathbf{y}' = \begin{pmatrix} \cos \theta y_1 - \sin \theta y_2 \\ \sin \theta y_1 + \cos \theta y_2 \end{pmatrix}.
   \]

(a) Show that \( (\mathbf{x}', \mathbf{y}') = (\mathbf{x}, \mathbf{y}) \) for the inner product defined in part (a) from the previous exercise. (a) Show that in general \( (\mathbf{x}', \mathbf{y}') \neq (\mathbf{x}, \mathbf{y}) \) for the inner product defined in part (b).

The inner product defined in 1(a) is often called the Euclidean inner product on \( \mathbb{R}^2 \). It is also frequently called the dot product or scalar product and is specifically signified by the notation \( \mathbf{x} \cdot \mathbf{y} \). The dot product naturally generalizes in \( \mathbb{R}^n \) to

\[
x, \; y \in \mathbb{R}^n, \text{ then } \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n.
\]

The dot product will be, for the most part, the only inner product we consider for the remainder of this course.

3. Suppose \( \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \), \( \mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \) and \( \mathbf{z} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \). Calculate

(a) \( \mathbf{x} \cdot \mathbf{y} \)  
(b) \( \mathbf{y} \cdot \mathbf{z} \)  
(c) \( \mathbf{x} \cdot \mathbf{z} \)  
(d) \( (\mathbf{x} + \mathbf{z}) \cdot \mathbf{y} \)  
(e) \( \mathbf{y} \cdot (\mathbf{x} + \mathbf{z}) \)  
(f) \( \mathbf{x} \cdot 2\mathbf{y} \)

Answers: (a) 6, (b) \(-2\), (c) 0, (d) 4, (e) 4, (f) 12.
4. Suppose \( r_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, r_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \) and \( r_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \).

(a) Show that \( r_1 \cdot r_2 = r_1 \cdot r_3 = r_2 \cdot r_3 = 0 \).

(b) Use the dot product to determine the scalars \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) so that \( x = \alpha_1 r_1 + \alpha_2 r_2 + \alpha_3 r_3 \).

(Answer: \( \alpha_1 = \frac{1}{3}(x_1 + x_2 + x_3), \alpha_2 = \frac{1}{2}(x_1 - x_3) \) and \( \alpha_3 = \frac{1}{6}(x_1 - 2x_2 + x_3) \).)

5. Let’s generalize the previous exercise. Suppose in \( \mathbb{R}^n \) we have \( n \) nonzero vectors \( r_1, \ldots, r_n \) such that \( r_i \cdot r_j = 0 \) for all \( i \neq j \). Derive the identity

\[
x = \alpha_1 r_1 + \alpha_2 r_2 + \cdots + \alpha_n r_n,
\]

where the scalars are given by

\[
\alpha_1 = \frac{x \cdot r_1}{r_1 \cdot r_1}, \quad \alpha_2 = \frac{x \cdot r_2}{r_2 \cdot r_2}, \quad \ldots, \quad \alpha_n = \frac{x \cdot r_n}{r_n \cdot r_n}.
\]

The projection of a vector \( x \in V \) into a subspace \( S \) of \( V \) is denoted by \( P_S(x) \). The projection satisfies two defining properties:

1. \( P_S(x) \in S \).
2. \( (x - P_S(x)) \cdot z = 0 \) for every \( z \in S \).

Condition (2) says the displacement vector \( x - P_S(x) \) is perpendicular to the subspace \( S \).

The following method can be used to calculate \( P_S(x) \). Let \( \{b_1, \ldots, b_m\} \) denote a basis for \( S \). Condition (1) requires

\[
P_S(x) = \alpha_1 b_1 + \cdots + \alpha_m b_m,
\]

for some to be determined scalars \( \alpha_1, \ldots, \alpha_m \). Condition (2) is equivalent to requiring

\[
(x - P_S(x)) \cdot b_k = 0 \quad \text{for every basis vector} \quad b_k.
\]

This leads to \( m \) equations in \( m \) unknowns

\[
\alpha_1(b_1 \cdot b_1) + \cdots + \alpha_m(b_m \cdot b_1) = x \cdot b_1,
\]

\[
\vdots
\]

\[
\alpha_1(b_1 \cdot b_m) + \cdots + \alpha_m(b_m \cdot b_m) = x \cdot b_m.
\]

It can be shown that this matrix equation always has a solution for the unknown scalars \( \alpha_1, \ldots, \alpha_m \), (provided of course that \( \{b_1, \ldots, b_m\} \) is a basis for \( S \)), and the resulting linear combination \( \alpha_1 b_1 + \cdots + \alpha_m b_m \) is the projection of \( x \) into \( S \).

In the special event that the basis \( \{b_1, \ldots, b_m\} \) is an orthogonal basis for \( S \), that is \( b_i \cdot b_j = 0 \) for all \( i \neq j \) and \( b_i \cdot b_i > 0 \), the matrix equation is easily solved. Make sure you do this and compare the solution in this case to what you did in exercise 5 above!!!
6. Use only conditions (1) and (2) above to prove the following. (a) \( P_S(x) = x \) if and only if \( x \in S \). (b) The projection is unique. That is, if there are two projection vectors, say \( p_1 \) and \( p_2 \), which both satisfy conditions (1) and (2), then \( p_1 = p_2 \).

7. Determine the projection of the vector \( x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \) into the following subspaces of \( \mathbb{R}^3 \).

   (a) \( S = \text{span} \left \{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right \} \)
   (b) \( S = \text{span} \left \{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ -1 \end{pmatrix} \right \} \)
   (c) \( S = \text{span} \left \{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \end{pmatrix} \right \} \)
   (d) \( S = \text{span} \left \{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right \} \)

   Answers: (a) \( \frac{4}{3} (1, 1, 1)^T \), (b) \( \frac{1}{2} (3, 3, 2)^T \), (c) \( \frac{4}{3} (1, 1, 1)^T \), (d) \( (2, 1, 1)^T \).

8. Project the given vector \( x \in \mathbb{R}^4 \) into the given subspace \( S \subseteq \mathbb{R}^4 \).

   (a) \( x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} \), \( S = \text{span} \left \{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right \} \).
   (b) \( x = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix} \), \( S = \text{span} \left \{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right \} \).

   Answers: (a) and (b) \( (2, 1, 2, 1)^T \).

9. Consider a basis set \( \{ b_1, b_2, b_3 \} \) where

   \( b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \), \( b_2 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \), \( b_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \).

   (a) Find a nonzero vector in \( \text{span} \{ b_1, b_2 \} \) which is perpendicular to \( b_1 \). (b) Find a nonzero vector in \( \text{span} \{ b_1, b_2, b_3 \} \) which is perpendicular to all vectors in \( \text{span} \{ b_1, b_2 \} \).

   Answers: (a) Any nonzero vector parallel to \( (0, 1, -1, 0)^T \). (b) Any nonzero vector parallel to \( (1, 0, 0, -1)^T \). Note that these two vectors together with \( b_1 \) forms an orthogonal basis for \( \text{span} \{ b_1, b_2, b_3 \} \).

10. Suppose \( x \in V \) and \( z \in S \subseteq V \). (a) Show that \( ||x - z||^2 = ||x - P_S(x)||^2 + ||z - P_S(x)||^2 \). Hint: Write \( x - z = x - P_S(x) + P_S(x) - z \) and use the fact that \( P_S(x) - z \in S \). (b) From this conclude that \( ||x - z||^2 \) is minimized over \( z \in S \) when \( z = P_S(x) \). That is, the closest vector \( z \in S \) to the vector \( x \in V \) is the projection of \( x \) into \( S \).