Math 3335 Supplemental Notes and Homework 2

An inner product on a real vector space $\mathcal{V}$, for the moment say $a(\mathbf{x},\mathbf{y})$, is a well defined mapping $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ which satisfies the following properties.

(1) For every $\mathbf{x} \in \mathcal{V}$, $a(\mathbf{x},\mathbf{x}) \geq 0$. Moreover $a(\mathbf{x},\mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

(2) For every $\mathbf{x} \in \mathcal{V}$ and $\mathbf{y} \in \mathcal{V}$, $a(\mathbf{x},\mathbf{y}) = a(\mathbf{y},\mathbf{x})$.

(3) For every $\mathbf{x} \in \mathcal{V}$, $\mathbf{y} \in \mathcal{V}$ and $\mathbf{z} \in \mathcal{V}$ and every $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}$ we must have that $a(\alpha\mathbf{x} + \beta\mathbf{y},\mathbf{z}) = \alpha\, a(\mathbf{x},\mathbf{z}) + \beta\, a(\mathbf{y},\mathbf{z})$.

Item 2 is often called *symmetry*. Item 3 says the inner product is linear in its first argument. Because of symmetry, it is also linear in its second argument. Such mappings are called *bilinear*. The inner product of vectors $\mathbf{x}$ and $\mathbf{y}$ is often simply denoted by $(\mathbf{x},\mathbf{y})$.

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1. Show the following both define an inner product on $\mathbb{R}^2$.

(a) $(\mathbf{x},\mathbf{y}) = x_1y_1 + x_2y_2$ (b) $(\mathbf{x},\mathbf{y}) = 2x_1y_1 + x_1y_2 + x_2y_1 + 3x_2y_2$

2. Let $\mathbf{x}'$ and $\mathbf{y}'$, both in $\mathbb{R}^2$, denote rotations of vectors $\mathbf{x}$ and $\mathbf{y}$. Specifically

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \mathbf{x}' = \begin{pmatrix} \cos\theta\, x_1 - \sin\theta\, x_2 \\ \sin\theta\, x_1 + \cos\theta\, x_2 \end{pmatrix},$$

and

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \mathbf{y}' = \begin{pmatrix} \cos\theta\, y_1 - \sin\theta\, y_2 \\ \sin\theta\, y_1 + \cos\theta\, y_2 \end{pmatrix}.$$

(a) Show that $(\mathbf{x}',\mathbf{y}') = (\mathbf{x},\mathbf{y})$ for the inner product defined in part (a) from the previous exercise. (a) Show that in general $(\mathbf{x}',\mathbf{y}') \neq (\mathbf{x},\mathbf{y})$ for the inner product defined in part (b).

The inner product defined in 1(a) is often called the Euclidean inner product on $\mathbb{R}^2$. It is also frequently called the *dot product* or *scalar product* and is specifically signified by the notation $\mathbf{x} \cdot \mathbf{y}$. The dot product naturally generalizes in $\mathbb{R}^n$ to

$$\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n, \text{ then } \mathbf{x} \cdot \mathbf{y} \equiv x_1y_1 + x_2y_2 + \cdots + x_ny_n.$$

The dot product will be, for the most part, the only inner product we consider for the remainder of this course.

3. Suppose $\mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{y} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{z} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$. Calculate

(a) $\mathbf{x} \cdot \mathbf{y}$ (b) $\mathbf{y} \cdot \mathbf{z}$ (c) $\mathbf{x} \cdot \mathbf{z}$ (d) $(\mathbf{x} + \mathbf{z}) \cdot \mathbf{y}$ (e) $\mathbf{y} \cdot (\mathbf{x} + \mathbf{z})$ (f) $\mathbf{x} \cdot 2\mathbf{y}$

Answers: (a) 6, (b) -2, (c) 0, (d) 4, (e) 4, (f) 12.

4. Suppose $\mathbf{r}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\mathbf{r}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\mathbf{r}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$.
(a) Show that $\mathbf{r}_1 \cdot \mathbf{r}_2 = \mathbf{r}_1 \cdot \mathbf{r}_3 = \mathbf{r}_2 \cdot \mathbf{r}_3 = 0$. (b) Use the dot product to determine the scalars α_1, α_2 and α_3 so that $\mathbf{x} = \alpha_1\mathbf{r}_1 + \alpha_2\mathbf{r}_2 + \alpha_3\mathbf{r}_3$. (Answer: $\alpha_1 = \frac{1}{3}(x_1 + x_2 + x_3)$, $\alpha_2 = \frac{1}{2}(x_1 - x_3)$ and $\alpha_3 = \frac{1}{6}(x_1 - 2x_2 + x_3)$.)

5. Let's generalize the previous exercise. Suppose in $\mathbb{R}^n$ we have n nonzero vectors $\mathbf{r}_1, \ldots, \mathbf{r}_n$ such that $\mathbf{r}_i \cdot \mathbf{r}_j = 0$ for all $i \neq j$. Derive the identity

$$\mathbf{x} = \alpha_1\,\mathbf{r}_1 + \alpha_2\,\mathbf{r}_2 + \cdots + \alpha_n\,\mathbf{r}_n,$$

where the scalars are given by

$$\alpha_1 = \frac{\mathbf{x}\cdot\mathbf{r}_1}{\mathbf{r}_1\cdot\mathbf{r}_1},\ \alpha_2 = \frac{\mathbf{x}\cdot\mathbf{r}_2}{\mathbf{r}_2\cdot\mathbf{r}_2},\ \ldots,\ \alpha_n = \frac{\mathbf{x}\cdot\mathbf{r}_n}{\mathbf{r}_n\cdot\mathbf{r}_n}.$$

The projection of a vector $\mathbf{x} \in \mathcal{V}$ into a subspace $\mathcal{S}$ of $\mathcal{V}$ is denoted by $\mathbf{P}_{\mathcal{S}}(\mathbf{x})$. The projection satisfies two defining properties:

$$(1)\ \ \mathbf{P}_{\mathcal{S}}(\mathbf{x}) \in \mathcal{S}. \qquad (2)\ \ (\mathbf{x} - \mathbf{P}_{\mathcal{S}}(\mathbf{x}))\cdot\mathbf{z} = 0 \text{ for every } \mathbf{z} \in \mathcal{S}.$$

Condition (2) says the displacement vector $\mathbf{x} - \mathbf{P}_{\mathcal{S}}(\mathbf{x})$ is perpendicular to the subspace $\mathcal{S}$.

The following method can be used to calculate $\mathbf{P}_{\mathcal{S}}(\mathbf{x})$. Let $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ denote a basis for $\mathcal{S}$. Condition (1) requires

$$\mathbf{P}_{\mathcal{S}}(\mathbf{x}) = \alpha_1\mathbf{b}_1 + \cdots + \alpha_m\mathbf{b}_m,$$

for some to be determined scalars $\alpha_1, \ldots, \alpha_m$. Condition (2) is equivalent to requiring

$$(\mathbf{x} - \mathbf{P}_{\mathcal{S}}(\mathbf{x}))\cdot\mathbf{b}_k = 0 \text{ for every basis vector } \mathbf{b}_k.$$

This leads to m equations in m unknowns

$$\begin{aligned} \alpha_1(\mathbf{b}_1\cdot\mathbf{b}_1) + \cdots + \alpha_m(\mathbf{b}_m\cdot\mathbf{b}_1) &= \mathbf{x}\cdot\mathbf{b}_1 \\ &\vdots \\ \alpha_1(\mathbf{b}_1\cdot\mathbf{b}_m) + \cdots + \alpha_m(\mathbf{b}_m\cdot\mathbf{b}_m) &= \mathbf{x}\cdot\mathbf{b}_m. \end{aligned}$$

It can be shown that this matrix equation always has a solution for the unknown scalars $\alpha_1, \ldots, \alpha_m$, (provided of course that $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ is a basis for $\mathcal{S}$), and the resulting linear combination $\alpha_1\mathbf{b}_1 + \cdots + \alpha_m\mathbf{b}_m$ is the projection of $\mathbf{x}$ into $\mathcal{S}$.

In the <u>special</u> event that the basis $\{\mathbf{b}_1, \ldots, \mathbf{b}_m\}$ is an *orthogonal basis* for $\mathcal{S}$, that is $\mathbf{b}_i\cdot\mathbf{b}_j = 0$ for all $i \neq j$ and $\mathbf{b}_i\cdot\mathbf{b}_i > 0$, the matrix equation is easily solved. Make sure you do this and compare the solution in this case to what you did in exercise 5 above!!!

6. Use only conditions (1) and (2) above to prove the following. (a) $\mathbf{P}_{\mathcal{S}}(\mathbf{x}) = \mathbf{x}$ if and only if $\mathbf{x} \in \mathcal{S}$. (b) The projection is unique. That is, if there are two projection vectors, say $\mathbf{p}_1$ and $\mathbf{p}_2$, which both satisfy conditions (1) and (2), then $\mathbf{p}_1 = \mathbf{p}_2$.

7. Determine the projection of the vector $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ into the following subspaces of $\mathbb{R}^3$.

(a) $\mathcal{S} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$ (b) $\mathcal{S} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$

(c) $\mathcal{S} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$ (d) $\mathcal{S} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Answers: (a) $\frac{4}{3}(1,1,1)^T$, (b) $\frac{1}{2}(3,3,2)^T$, (c) $\frac{4}{3}(1,1,1)^T$, (d) $(1,2,1)^T$.

8. Project the given vector $\mathbf{x} \in \mathbb{R}^4$ into the given subspace $\mathcal{S} \subseteq \mathbb{R}^4$.

(a) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \quad \mathcal{S} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$

(b) $\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}, \quad \mathcal{S} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right\}.$

Answers: (a) and (b) $(2,1,2,1)^T$.

9. Consider a basis set $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ where

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}.$$

(a) Find a nonzero vector in $\text{span}\{\mathbf{b}_1, \mathbf{b}_2\}$ which is perpendicular to $\mathbf{b}_1$. (b) Find a nonzero vector in $\text{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ which is perpendicular to all vectors in $\text{span}\{\mathbf{b}_1, \mathbf{b}_2\}$. Answers: (a) Any nonzero vector parallel to $(0,1,-1,0)^T$. (b) Any nonzero vector parallel to $(1,0,0,-1)^T$. Note that these two vectors together with $\mathbf{b}_1$ forms an orthogonal basis for $\text{span}\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$.

10. Suppose $\mathbf{x} \in \mathcal{V}$ and $\mathbf{z} \in \mathcal{S} \subseteq \mathcal{V}$. (a) Show that $||\mathbf{x}-\mathbf{z}||^2 = ||\mathbf{x}-\mathbf{P}_{\mathcal{S}}(\mathbf{x})||^2 + ||\mathbf{z}-\mathbf{P}_{\mathcal{S}}(\mathbf{x})||^2$. Hint: Write $\mathbf{x}-\mathbf{z} = \mathbf{x}-\mathbf{P}_{\mathcal{S}}(\mathbf{x}) + \mathbf{P}_{\mathcal{S}}(\mathbf{x}) - \mathbf{z}$ and use the fact that $\mathbf{P}_{\mathcal{S}}(\mathbf{x}) - \mathbf{z} \in \mathcal{S}$. (b) From this conclude that $||\mathbf{x}-\mathbf{z}||^2$ is minimized over $\mathbf{z} \in \mathcal{S}$ when $\mathbf{z} = \mathbf{P}_{\mathcal{S}}(\mathbf{x})$. That is, the closest vector $\mathbf{z} \in \mathcal{S}$ to the vector $\mathbf{x} \in \mathcal{V}$ is the projection of $\mathbf{x}$ into $\mathcal{S}$.