Math 3335. The Dot and Cross Products

Throughout this course I will use the notation $\{\mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3\}$ to represent the so-called *standard basis* of $\mathbb{R}^3$. Specifically

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \Rightarrow \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1\,\mathbf{e}_1 + x_2\,\mathbf{e}_2 + x_3\,\mathbf{e}_3.$$

Let $\mathbf{x}$ and $\mathbf{y}$ be any two vectors from $\mathbb{R}^3$. The *dot product* of $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\mathbf{x}\cdot\mathbf{y}$, and it is scalar valued; i.e. $\mathbf{x}\cdot\mathbf{y}\in\mathbb{R}$. The *cross product* of $\mathbf{x}$ and $\mathbf{y}$ is denoted by $\mathbf{x}\times\mathbf{y}$, and it is vector valued; i.e. $\mathbf{x}\times\mathbf{y}\in\mathbb{R}^3$.

Both the dot and cross product are bilinear. That is, for any scalars $\alpha$ and $\beta$ and any vector $\mathbf{z}$ we have

$$\begin{aligned}
(\alpha\mathbf{x}+\beta\mathbf{y})\cdot\mathbf{z} &= \alpha\,(\mathbf{x}\cdot\mathbf{z}) + \beta\,(\mathbf{y}\cdot\mathbf{z}),\\
\mathbf{z}\cdot(\alpha\mathbf{x}+\beta\mathbf{y}) &= \alpha\,(\mathbf{z}\cdot\mathbf{x}) + \beta\,(\mathbf{z}\cdot\mathbf{y}),\\
(\alpha\mathbf{x}+\beta\mathbf{y})\times\mathbf{z} &= \alpha\,(\mathbf{x}\times\mathbf{z}) + \beta\,(\mathbf{y}\times\mathbf{z}),\\
\mathbf{z}\times(\alpha\mathbf{x}+\beta\mathbf{y}) &= \alpha\,(\mathbf{z}\times\mathbf{x}) + \beta\,(\mathbf{z}\times\mathbf{y}).
\end{aligned}$$

Therefore

$$\mathbf{x}\cdot\mathbf{y} = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j\,(\mathbf{e}_i\cdot\mathbf{e}_j) \quad \text{and} \quad \mathbf{x}\times\mathbf{y} = \sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j\,(\mathbf{e}_i\times\mathbf{e}_j).$$

This tells us that the dot product is uniquely defined by nine scalars, $(\mathbf{e}_i\cdot\mathbf{e}_j)$, and the cross product is uniquely defined by nine vectors, $(\mathbf{e}_i\times\mathbf{e}_j)$.

A fundamental physical principle states that laws of physics should never depend on the particular orientation of the observer. Mathematically this principle is equivalent to the concept of rotational invariance. It is therefore natural to ask that both the dot product and cross product be rotationally invariant. Surprisingly, this added requirement will essentially determine both uniquely.

---

1. A *proper rotation matrix* $R$ satisfies $R^T R = I$ and has $\det(R) = +1$ so that it preserves orientation. Consider the following three matrices.

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \quad R_2 = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \quad R_3 = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Verify each $R$ above is a proper rotation. (b) Determine the eigenvalues for each. (c) Investigate the action of each when applied to $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$.

2. Prove that the product of any two rotation matrices is a rotation matrix.

---

Rotation invariance for the dot product says

$$(R\mathbf{x} \cdot R\mathbf{y}) = (\mathbf{x} \cdot \mathbf{y})$$

for all vectors $\mathbf{x}$ and $\mathbf{y}$ and any rotation matrix $R$. Now what can we conclude from this? Let $R$ be a $90°$ rotation which takes $\mathbf{e}_1$ to $\mathbf{e}_2$ to see $(\mathbf{e}_1 \cdot \mathbf{e}_1) = (R\mathbf{e}_1 \cdot R\mathbf{e}_1) = (\mathbf{e}_2 \cdot \mathbf{e}_2)$. Similarly see that $(\mathbf{e}_1 \cdot \mathbf{e}_1) = (\mathbf{e}_3 \cdot \mathbf{e}_3)$ and so

$$(\mathbf{e}_1 \cdot \mathbf{e}_1) = (\mathbf{e}_2 \cdot \mathbf{e}_2) = (\mathbf{e}_3 \cdot \mathbf{e}_3).$$

Next, let $R$ be a $180°$ rotation which leaves $\mathbf{e}_1$ fixed and takes $\mathbf{e}_2$ to $-\mathbf{e}_2$. This gives $(\mathbf{e}_1 \cdot \mathbf{e}_2) = (R\mathbf{e}_1 \cdot R\mathbf{e}_2) = (\mathbf{e}_1 \cdot -\mathbf{e}_2) = -(\mathbf{e}_1 \cdot \mathbf{e}_2)$, which implies we must have $(\mathbf{e}_1 \cdot \mathbf{e}_2) = 0$. Similar $180°$ rotations yield

$$(\mathbf{e}_i \cdot \mathbf{e}_j) = 0 \quad \text{for every } i \neq j.$$

Therefore, up to a free multiplicative constant, e.g. $(\mathbf{e}_1 \cdot \mathbf{e}_1)$, a bilinear and rotationally invariant scalar product necessarily takes the form

$$\begin{aligned} \mathbf{x} \cdot \mathbf{y} = &= \sum_{i=1}^{3} \sum_{j=1}^{3} x_i y_j \, (\mathbf{e}_i \cdot \mathbf{e}_j) \\ &= x_1 y_1 + x_2 y_2 + x_3 y_3. \end{aligned} \tag{1}$$

The right hand side of formula (1) gives what is called the dot product.

---

3. Let the dot product $\mathbf{x} \cdot \mathbf{y}$ be given by formula (1) above. Its form was deduced as a necessary consequence of rotation invariance. Here you will show it is in fact invariant under any rotation. (a) Regard vectors $\mathbf{x}$ and $\mathbf{y}$ as $3 \times 1$ column matrices. Verify that $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$. (b) Use this to see $R\mathbf{x} \cdot R\mathbf{y} = \mathbf{x}^T R^T R\mathbf{y}$ and then conclude the dot product is invariant under any rotation. (c) The Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^3$ is given by $||\mathbf{x}|| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. Prove that $||R\mathbf{x}|| = ||\mathbf{x}||$ for any rotation $R$.

4. Suppose $\mathbf{x} = x_1 \mathbf{e}_1$ with $x_1 > 0$ and $\mathbf{y} = y_1 \mathbf{e}_1 + y_2 \mathbf{e}_2$ with $\mathbf{y} \neq \mathbf{0}$. (a) By plane trigonometry conclude that $||\mathbf{x}||||\mathbf{y}|| \cos\theta = \mathbf{x} \cdot \mathbf{y}$ where $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$. (b) Use rotational invariance to conclude this formula is valid for any two nonzero three dimensional vectors $\mathbf{x}$ and $\mathbf{y}$.

---

Deducing the form of the cross product is a bit more involved compared to what was done for the dot product. We again ask that the rotation invariance property be necessarily satisfied, but here since the cross product is vector valued, this property requires

$$(R\mathbf{x} \times R\mathbf{y}) = R\,(\mathbf{x} \times \mathbf{y})$$

for any proper rotation matrix $R$. First, for any $i$ we must have $(\mathbf{e}_i \times \mathbf{e}_i) = \mathbf{0}$. To see this, let $R$ have $\mathbf{e}_i$ as its axis of rotation (i.e. $R\mathbf{e}_i = \mathbf{e}_i$) to see that since

$$(\mathbf{e}_i \times \mathbf{e}_i) = (R\mathbf{e}_i \times R\mathbf{e}_i) = R(\mathbf{e}_i \times \mathbf{e}_i)$$

we must have $(\mathbf{e}_i \times \mathbf{e}_i) = \alpha\mathbf{e}_i$ for some scalar $\alpha$. (For this particular $R$ we have $R\mathbf{x} = \mathbf{x}$ $\Leftrightarrow$ $\mathbf{x} = \alpha\mathbf{e}_i$ since the eigenspace associated to eigenvalue $\lambda = 1$ is spanned by $\mathbf{e}_i$.) Now use this and rotate by $180°$ with rotation axis perpendicular to $\mathbf{e}_i$ to see

$$\alpha\mathbf{e}_i = (\mathbf{e}_i \times \mathbf{e}_i) = (-\mathbf{e}_i \times -\mathbf{e}_i) = -\alpha\mathbf{e}_i \quad \Rightarrow \quad \alpha\mathbf{e}_i = \mathbf{0}.$$

To deduce the form of the off-diagonal terms consider first $(\mathbf{e}_1 \times \mathbf{e}_2)$ and a $180°$ rotation $R$ about $\mathbf{e}_3$. This gives

$$(\mathbf{e}_1 \times \mathbf{e}_2) = (-\mathbf{e}_1 \times -\mathbf{e}_2) = (R\mathbf{e}_1 \times R\mathbf{e}_2) = R(\mathbf{e}_1 \times \mathbf{e}_2) \quad \Rightarrow \quad (\mathbf{e}_1 \times \mathbf{e}_2) = \beta\mathbf{e}_3$$

for some scalar $\beta$. This and a $90°$ rotation $R$ also gives

$$(\mathbf{e}_2 \times \mathbf{e}_1) = (R\mathbf{e}_1 \times -R\mathbf{e}_2) = -(R\mathbf{e}_1 \times R\mathbf{e}_2) = -\beta R\mathbf{e}_3 = -\beta\mathbf{e}_3.$$

Rotate these two relations by $90°$ about $\mathbf{e}_1$ and then again about $\mathbf{e}_2$ to finally get

$$\begin{aligned}
(\mathbf{e}_1 \times \mathbf{e}_2) &= -(\mathbf{e}_2 \times \mathbf{e}_1) = \beta\mathbf{e}_3, \\
(\mathbf{e}_1 \times \mathbf{e}_3) &= -(\mathbf{e}_3 \times \mathbf{e}_1) = -\beta\mathbf{e}_2, \\
(\mathbf{e}_2 \times \mathbf{e}_3) &= -(\mathbf{e}_3 \times \mathbf{e}_2) = \beta\mathbf{e}_1.
\end{aligned}$$

Therefore, up to a free multiplicative constant, e.g. $\beta$ above, a bilinear and rotationally invariant vector product necessarily takes the form

$$\begin{aligned}
\mathbf{x} \times \mathbf{y} &= \sum_{i=1}^{3}\sum_{j=1}^{3} x_i y_j \, (\mathbf{e}_i \times \mathbf{e}_j) \\
&= (x_2y_3 - x_3y_2)\,\mathbf{e}_1 - (x_1y_3 - x_3y_1)\,\mathbf{e}_2 + (x_1y_2 - x_2y_1)\,\mathbf{e}_3.
\end{aligned} \tag{2}$$

The right hand side of formula (2) gives what is called the cross product.

---

5. A cofactor expansion easily establishes the formula

$$\det\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = (\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}.$$

(a) Use properties of the determinant to prove $(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z} = (\mathbf{z} \times \mathbf{x}) \cdot \mathbf{y} = (\mathbf{y} \times \mathbf{z}) \cdot \mathbf{x}$.
(b) Also prove $(M\mathbf{x} \times M\mathbf{y}) \cdot M\mathbf{z} = \det(M)\,((\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z})$ for any $3 \times 3$ matrix $M$.
(c) Prove the cross product is rotationally invariant. That is, $(R\mathbf{x} \times R\mathbf{y}) = R(\mathbf{x} \times \mathbf{y})$ for any proper rotation matrix $R$. Hint: Use part (b).

6. We showed above that $(\mathbf{e}_i \times \mathbf{e}_j) = -(\mathbf{e}_j \times \mathbf{e}_i)$ for every $i$ and $j$. (a) Use this to prove $(\mathbf{x} \times \mathbf{y}) = -(\mathbf{y} \times \mathbf{x})$ for any two vectors $\mathbf{x}$ and $\mathbf{y}$. (b) Use this to conclude $\mathbf{x} \times \mathbf{x} = \mathbf{0}$ for any vector $\mathbf{x}$. (c) Prove that $\mathbf{x} \times \mathbf{y}$ is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$. Hint: Use 5(a) and part (b) of this exercise.

For the two problems below, define

$$\mathbf{a} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{c} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \ \mathbf{d} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \ \mathbf{e} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

7. Compute the following.

(a) $\mathbf{a} \times \mathbf{b}$ (b) $\mathbf{c} \times \mathbf{d}$ (c) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}$ (d) $(\mathbf{c} \times \mathbf{d}) \cdot \mathbf{e}$

8. Compute the following.

(a) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ (b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ (c) $(\mathbf{c} \times \mathbf{d}) \times \mathbf{e}$ (d) $\mathbf{c} \times (\mathbf{d} \times \mathbf{e})$