Math 3335. Some Vector Identities

Here several important vector identities will be derived from the following basic formula

$$(1) \qquad (\mathbf{e}_i \times \mathbf{e}_j) \cdot (\mathbf{e}_m \times \mathbf{e}_n) = (\mathbf{e}_i \cdot \mathbf{e}_m)(\mathbf{e}_j \cdot \mathbf{e}_n) - (\mathbf{e}_i \cdot \mathbf{e}_n)(\mathbf{e}_j \cdot \mathbf{e}_m)$$

where $\mathbf{e}_i$, $\mathbf{e}_j$, $\mathbf{e}_m$ and $\mathbf{e}_n$ are any four standard basis vectors.

Let me first derive this formula. Observe that both the left side and the right side of (1) take on only three possible values: $+1$, -1 and 0. Now check that the left side

$$\begin{aligned} &(\mathbf{e}_i \times \mathbf{e}_j) \cdot (\mathbf{e}_m \times \mathbf{e}_n) = +1 \\ &\iff \quad i \neq j \text{ and } m = i \text{ and } n = j, \end{aligned}$$

and the right side

$$\begin{aligned} &(\mathbf{e}_i \cdot \mathbf{e}_m)(\mathbf{e}_j \cdot \mathbf{e}_n) - (\mathbf{e}_i \cdot \mathbf{e}_n)(\mathbf{e}_j \cdot \mathbf{e}_m) = +1 \\ &\iff \quad i = m \text{ and } j = n \text{ and } i \neq n \text{ or } j \neq m. \end{aligned}$$

Since these two index sets are equal, that is since

$$\{i \neq j \text{ and } m = i \text{ and } n = j\} = \{i = m \text{ and } j = n \text{ and } i \neq n \text{ or } j \neq m\},$$

formula (1) is proved valid for all such indices. Next check that the left side

$$\begin{aligned} &(\mathbf{e}_i \times \mathbf{e}_j) \cdot (\mathbf{e}_m \times \mathbf{e}_n) = -1 \\ &\iff \quad i \neq j \text{ and } m = j \text{ and } n = i, \end{aligned}$$

and the right side

$$\begin{aligned} &(\mathbf{e}_i \cdot \mathbf{e}_m)(\mathbf{e}_j \cdot \mathbf{e}_n) - (\mathbf{e}_i \cdot \mathbf{e}_n)(\mathbf{e}_j \cdot \mathbf{e}_m) = -1 \\ &\iff \quad i \neq m \text{ or } j \neq n \text{ and } i = n \text{ and } j = m. \end{aligned}$$

Again, we have

$$\{i \neq j \text{ and } m = j \text{ and } n = i\} = \{i \neq m \text{ or } j \neq n \text{ and } i = n \text{ and } j = m\}.$$

and so formula (1) is proved valid for all of these indices. For all other indices, both the left side and right side of (1) are zero. Therefore, formula (1) is established for every $\mathbf{e}_i$, $\mathbf{e}_j$, $\mathbf{e}_m$, $\mathbf{e}_n$ from the set of standard basis vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

We can now generalize formula (1). Let $\mathbf{a} = \sum_{i=1}^3 a_i \mathbf{e}_i$, $\cdots$, $\mathbf{d} = \sum_{n=1}^3 d_n \mathbf{e}_n$, and use linearity of the two products to write

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \left(\sum_{i=1}^3 a_i \mathbf{e}_i \times \sum_{j=1}^3 b_j \mathbf{e}_j \right) \cdot \left(\sum_{m=1}^3 c_m \mathbf{e}_m \times \sum_{n=1}^3 d_n \mathbf{e}_n \right)$$

$$= \left(\sum_{i,j} a_i b_j \left(\mathbf{e}_i \times \mathbf{e}_j \right) \right) \cdot \left(\sum_{m,n} c_m d_n \left(\mathbf{e}_m \times \mathbf{e}_n \right) \right)$$

$$= \sum_{i,j,m,n} a_i b_j c_m d_n \left(\left(\mathbf{e}_i \times \mathbf{e}_j \right) \cdot \left(\mathbf{e}_m \times \mathbf{e}_n \right) \right).$$

From our basic formula (1) we see this is equal to

$$\sum_{i,j,m,n} a_i b_j c_m d_n \left(\left(\mathbf{e}_i \cdot \mathbf{e}_m \right) \left(\mathbf{e}_j \cdot \mathbf{e}_n \right) - \left(\mathbf{e}_i \cdot \mathbf{e}_n \right) \left(\mathbf{e}_j \cdot \mathbf{e}_m \right) \right)$$

$$= \left(\sum_{i,j,m,n} a_i b_j c_m d_n \left(\mathbf{e}_i \cdot \mathbf{e}_m \right) \left(\mathbf{e}_j \cdot \mathbf{e}_n \right) \right) - \left(\sum_{i,j,m,n} a_i b_j c_m d_n \left(\mathbf{e}_i \cdot \mathbf{e}_n \right) \left(\mathbf{e}_j \cdot \mathbf{e}_m \right) \right)$$

$$= \left(\sum_{i,m} a_i c_m \left(\mathbf{e}_i \cdot \mathbf{e}_m \right) \sum_{j,n} b_j d_n \left(\mathbf{e}_j \cdot \mathbf{e}_n \right) \right) - \left(\sum_{i,n} a_i d_n \left(\mathbf{e}_i \cdot \mathbf{e}_n \right) \sum_{j,m} b_j c_m \left(\mathbf{e}_j \cdot \mathbf{e}_m \right) \right),$$

and by linearity of the dot product this is equal to

$$(\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\,(\mathbf{b} \cdot \mathbf{c}).$$

Therefore, for any four vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ from $\mathbb{R}^3$ we have shown

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\,(\mathbf{b} \cdot \mathbf{c}). \tag{2}$$

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1. Use formula (2) and the fact that $\mathbf{a} \cdot \mathbf{b} = ||\mathbf{a}||||\mathbf{b}|| \cos\theta$ to derive

$$||\mathbf{a} \times \mathbf{b}|| = ||\mathbf{a}||||\mathbf{b}||| \sin\theta|$$

where θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$.

2. Regard vectors $\mathbf{a}$ and $\mathbf{b}$ as two adjacent sides of a parallelogram P_2. Use the previous exercise to show that $||\mathbf{a} \times \mathbf{b}||$ gives the area of P_2.

3. Regard vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as three adjacent edges of a parallelepiped P_3. Show that $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$ gives the volume of P_3.

4. In exercise 5(a) of your previous homework you proved

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}$$

for any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. (I'll refer to these as the cyclic identity.)
(a) Use this and formula (2) above to derive

$$\mathbf{a} \cdot \big(\mathbf{b} \times (\mathbf{c} \times \mathbf{d})\big) = \mathbf{a} \cdot \big((\mathbf{b} \cdot \mathbf{d})\,\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{d}\big).$$

(b) Conclude from part (a) and a change of letters that

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}.$$

This is sometimes called the *bac-cab* formula.

5. Use the fact that $(\mathbf{b} \times \mathbf{a}) = -(\mathbf{a} \times \mathbf{b})$ and the bac-cab formula from the previous exercise to derive

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\, \mathbf{a}.$$

6. The so-called *tripple scalar product*, $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$, is denoted in your textbook by

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] \equiv (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}.$$

Observe that because of the cyclic identity, you don't have to remember the positions of the dot and cross, i.e. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$. Just keep the order straight and make sure the product makes sense and results in a scalar.
Use exercise 4 to derive the formula

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}, \mathbf{b}, \mathbf{d}]\, \mathbf{c} - [\mathbf{a}, \mathbf{b}, \mathbf{c}]\, \mathbf{d}.$$

7. Use exercise 5 to derive

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a}, \mathbf{c}, \mathbf{d}]\, \mathbf{b} - [\mathbf{b}, \mathbf{c}, \mathbf{d}]\, \mathbf{a}.$$

8. The *Kronecker delta* symbol, $\delta_{i,j}$, and the *Levi-Civita epsilon* symbol, $\epsilon_{i,j,k}$, are given by

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases} \qquad \epsilon_{i,j,k} = \begin{cases} +1 & \text{if } (i,j,k) \text{ is an even permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ is an odd permutation of } (1,2,3) \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that

$$(\mathbf{e}_i \cdot \mathbf{e}_j) = \delta_{i,j} \quad \text{and} \quad (\mathbf{e}_i \times \mathbf{e}_j) \cdot \mathbf{e}_k = \epsilon_{i,j,k}.$$

(b) Show that

$$\mathbf{e}_i \times \mathbf{e}_j = \sum_{k=1}^{3} \epsilon_{i,j,k}\, \mathbf{e}_k.$$

(c) Show that the basic formula (1) can be rewritten in terms of ϵ and δ

$$\sum_{k=1}^{3} \epsilon_{i,j,k}\, \epsilon_{m,n,k} = \delta_{i,m}\delta_{j,n} - \delta_{i,n}\delta_{j,m}.$$