Math 3335. Integration on Curves, Surfaces and Volumes

1. Determine, if possible, a scalar potential ϕ so that $\nabla\phi = \mathbf{F}$ for the following.

(a) $\mathbf{F}(x,y,z) = (y+z)\,\mathbf{e}_x + (x+z)\,\mathbf{e}_y + (x+y)\,\mathbf{e}_z$

(b) $\mathbf{F}(x,y,z) = \dfrac{1}{(x^2+z^2)^2}\left(2xy\,\mathbf{e}_x - (x^2+z^2)\,\mathbf{e}_y + 2yz\,\mathbf{e}_z\right)$

(c) $\mathbf{F}(x,y,z) = (2x+y)\,\mathbf{e}_x + (x+2y+z)\,\mathbf{e}_y + y\,\mathbf{e}_z$

(d) $\mathbf{F}(x,y,z) = (2xy+z^2)\,\mathbf{e}_x + (x^2+2yz)\,\mathbf{e}_y + (y^2+2xz+2z)\,\mathbf{e}_z$

2. A helix is parameterized by $\mathbf{x}(t) = \cos t\,\mathbf{e}_x + \sin t\,\mathbf{e}_y + t\,\mathbf{e}_z$. Compute the length of one of its loops, say from $t=0$ to $t=2\pi$. Answer: $2\pi\sqrt{2}$.

3. Consider a plane spiral parameterized by $\mathbf{x}(t) = e^{-t}\cos t\,\mathbf{e}_x + e^{-t}\sin t\,\mathbf{e}_y$. Compute the spiral's length from $\mathbf{x}_0 = \mathbf{e}_x$ (i.e. $t=0$) to $\mathbf{x}_1 = \mathbf{0}$ (i.e. $t=\infty$). Answer: $\sqrt{2}$.

4. Reparameterize the curves given in exercises 2 and 3 above in terms of arclength s. Use $s(t) = \int_0^t ||\mathbf{v}(\tau)||\,d\tau$ to determine the inverse $t(s)$.

5. Compute the line integral $\int_\Gamma \mathbf{F}\cdot d\mathbf{x}$ where the vector function $\mathbf{F}(x,y,z) = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z$ and the path Γ is given as follows.

(a) Γ is the helix in exercise 2 above. (b) Γ is the spiral in exercise 3 above.

The vector function given here satisfies $\mathbf{F} = \nabla\phi$ where $\phi(x,y,z) = \frac{1}{2}\left(x^2+y^2+z^2\right)$. Check the values you got for your integrals by computing $\phi(\mathbf{x}_1) - \phi(\mathbf{x}_0)$ where $\mathbf{x}_0$ and $\mathbf{x}_1$ are the endpoints of the appropriate path Γ.

6. Consider a 3–d solid region R given by $\{(x,y,z) : x^2+4y^2 \le z \le 1\}$. Visualize this as a parabolic bowl with elliptical cross sections cut off at the top by the plane $z=1$. The boundary of R is composed of two surfaces; $S_{side} = \{(x,y,z) : z = x^2+4y^2 \;\&\; z \le 1\}$ and $S_{top} = \{(x,y,z) : x^2+4y^2 \le 1 \;\&\; z = 1\}$. Both boundaries can be regarded as mappings from the 2–d parameter set $\mathcal{C} = \{(x,y) : x^2+4y^2 \le 1\}$ into 3–d by

$$\mathbf{F}_{side}(x,y) = x\,\mathbf{e}_x + y\,\mathbf{e}_y + (x^2+4y^2)\,\mathbf{e}_z \quad \Rightarrow \quad S_{side} = \mathbf{F}_{side}(\mathcal{C}),$$

$$\text{and} \qquad \mathbf{F}_{top}(x,y) = x\,\mathbf{e}_x + y\,\mathbf{e}_y + 1\,\mathbf{e}_z \quad \Rightarrow \quad S_{top} = \mathbf{F}_{top}(\mathcal{C}).$$

With this particular parameterization, verify the following.

(a) $d\mathbf{A}_{side} = (-2x\,\mathbf{e}_x - 8y\,\mathbf{e}_y + 1\,\mathbf{e}_z)\,dx\,dy$ (b) $d\mathbf{A}_{top} = \mathbf{e}_z\,dx\,dy$

Now consider a vector field $\mathbf{F}(x,y,z) = y\,\mathbf{e}_x + z\,\mathbf{e}_y + x\,\mathbf{e}_z$ and compute that its curl is given by $\nabla \times \mathbf{F} = -\mathbf{e}_x - \mathbf{e}_y - \mathbf{e}_z$. With this, use cartesian coordinates (i.e. $(x,y) \in \mathcal{C}$) to compute the following surface integrals.

(c) $\displaystyle\iint_{S_{side}} (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$ (d) $\displaystyle\iint_{S_{top}} (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$

A closed curve, say Γ, is the boundary for both surfaces S_{side} and S_{top}. Orient Γ in the counter–clockwise direction when looking down the z–axis.

(e) Compute $\displaystyle\oint_\Gamma \mathbf{F} \cdot d\mathbf{x}$ and compare to values obtained in parts (c) and (d).

(f) Finally, compute the volume of R, $\displaystyle\iiint_R dV$, using cartesian coordinates (in which obviously $dV = dx\,dy\,dz$) and iterated integrals. (Answer: $\pi/4$.)

7. Here we repeat the previous exercise, but this time using elliptical–cylindrical coordinates

$$x(r,\theta,\zeta) = 2r\cos\theta, \quad y(r,\theta,\zeta) = r\sin\theta, \quad z(r,\theta,\zeta) = \zeta.$$

Here both boundaries can be regarded as mappings from the <u>rectangular</u> 2–d parameter set $\mathcal{P} = \{(r,\theta) : 0 \le r \le 1/2,\ 0 \le \theta \le 2\pi\}$ into 3–d by

$$\begin{aligned} &\mathbf{F}_{side}(r,\theta) = 2r\cos\theta\,\mathbf{e}_x + r\sin\theta\,\mathbf{e}_y + 4r^2\,\mathbf{e}_z \quad \Rightarrow \quad S_{side} = \mathbf{F}_{side}(\mathcal{P}), \\ \text{and} \quad &\mathbf{F}_{top}(r,\theta) = 2r\cos\theta\,\mathbf{e}_x + r\sin\theta\,\mathbf{e}_y + 1\,\mathbf{e}_z \quad \Rightarrow \quad S_{top} = \mathbf{F}_{top}(\mathcal{P}). \end{aligned}$$

With this particular parameterization, verify the following.

(a) $d\mathbf{A}_{side} = (-4r\cos\theta\,\mathbf{e}_x - 8r\sin\theta\,\mathbf{e}_y + 1\,\mathbf{e}_z)\,2r\,dr\,d\theta$ (b) $d\mathbf{A}_{top} = \mathbf{e}_z\,2r\,dr\,d\theta$

Finally, repeat parts (c), (d) and (f) from exercise 6 using these coordinates. Note that in these coordinates $dV = 2r\,dr\,d\theta\,d\zeta$.

The previous two exercises exemplify *Stokes's theorem* (as stated in calculus). That is

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \oint_\Gamma \mathbf{F} \cdot d\mathbf{x},$$

where the curve Γ is the properly oriented boundary of the surface S. Next we give an example of the *divergence theorem*. This important theorem says

$$\iiint_R (\nabla \cdot \mathbf{F})\,dV = \iint_S \mathbf{F} \cdot d\mathbf{A},$$

where the 3–d solid region R has S as its boundary surface which is oriented so that its normal vector points outwards to R.

8. Use the region R defined in exercise 6 and the elliptical-cylindrical coordinates given in exercise 7 to compute

(a) $\displaystyle\iiint_R (\nabla \cdot \mathbf{F})\, dV$ (b) $\displaystyle\iint_S \mathbf{F} \cdot d\mathbf{A}$

where $\mathbf{F}(x, y, z) = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z$.

9. Consider the hemispherical region R given by $\{(x, y, z) : x^2 + y^2 + z^2 \leq 1 \ \&\ z \geq 0\}$. Its boundary surfaces can be parameterized by $(x, y) \in \mathcal{C} \equiv \{(x, y) : x^2 + y^2 \leq 1\}$

(c–c)
$$\mathbf{F}_{top}(x, y) = x\,\mathbf{e}_x + y\,\mathbf{e}_y + \left(\sqrt{1 - (x^2 + y^2)}\right)\mathbf{e}_z$$
$$\mathbf{F}_{bot}(x, y) = x\,\mathbf{e}_x + y\,\mathbf{e}_y + 0\,\mathbf{e}_z$$
$$S_{top} = \mathbf{F}_{top}(\mathcal{C}) \qquad S_{bot} = \mathbf{F}_{bot}(\mathcal{C}),$$

or perhaps more simply in polar form $(r, \theta) \in \mathcal{P} \equiv \{(r, \theta) : 0 \leq r \leq 1 \ \&\ 0 \leq \theta \leq 2\pi\}$

(c–p)
$$\mathbf{F}_{top}(r, \theta) = r\cos\theta\,\mathbf{e}_x + r\sin\theta\,\mathbf{e}_y + \left(\sqrt{1 - r^2}\right)\mathbf{e}_z$$
$$\mathbf{F}_{bot}(r, \theta) = r\cos\theta\,\mathbf{e}_x + r\sin\theta\,\mathbf{e}_y + 0\,\mathbf{e}_z$$
$$S_{top} = \mathbf{F}_{top}(\mathcal{P}) \qquad S_{bot} = \mathbf{F}_{bot}(\mathcal{P}),$$

or even more simply in spherical form $(\phi, \theta) \in \mathcal{S} \equiv \{(\phi, \theta) : 0 \leq \phi \leq \pi/2 \ \&\ 0 \leq \theta \leq 2\pi\}$

(c–s)
$$\mathbf{F}_{top}(\phi, \theta) = \cos\phi\cos\theta\,\mathbf{e}_x + \cos\phi\sin\theta\,\mathbf{e}_y + \sin\phi\,\mathbf{e}_z$$
$$\mathbf{F}_{bot}(\phi, \theta) = \cos\phi\cos\theta\,\mathbf{e}_x + \cos\phi\sin\theta\,\mathbf{e}_y + 0\,\mathbf{e}_z$$
$$S_{top} = \mathbf{F}_{top}(\mathcal{S}) \qquad S_{bot} = \mathbf{F}_{bot}(\mathcal{S}).$$

Use the coordinates (c–p) to evaluate the following surface integrals.

$\mathbf{F}(x, y, z) = x\,\mathbf{e}_x + y\,\mathbf{e}_y + z\,\mathbf{e}_z$

(a) $\displaystyle\iint_{S_{top}} \mathbf{F} \cdot d\mathbf{A}$ (b) $\displaystyle\iint_{S_{bot}} \mathbf{F} \cdot d\mathbf{A}$.

Use the coordinates (c–s) to evaluate the following surface integrals.

$\mathbf{F}(x, y, z) = y\,\mathbf{e}_x + z\,\mathbf{e}_y + x\,\mathbf{e}_z$

(c) $\displaystyle\iint_{S_{top}} (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$ (d) $\displaystyle\iint_{S_{bot}} (\nabla \times \mathbf{F}) \cdot d\mathbf{A}$.

10. Consider the 2–d coordinates

$$x = x(u, v) \equiv u^2 - v^2, \quad y = y(u, v) \equiv 2uv,$$

and a 2–d solid region R defined by $\{(x(u, v), y(u, v)) : 0 \leq u \leq 1 \ \&\ 0 \leq v \leq 1\}$.

(a) Sketch the region R. (b) Show that $dV = (4u^2 + 4v^2)\, du\, dv$.

(c) Compute that the volume (area) of R is $\displaystyle\iint_R dV = 8/3$.