MATH 3335 Final Exam. Sanders Spring 2018

This exam has 10 problems, and all 10 problems will be graded. Use my supplied paper only. Return your solution sheets with the problems in order. Put your name, **last name first**, and **student id number** on each solution sheet you turn in. Each problem is worth 20 points with parts equally weighted unless otherwise indicated.

1. Consider the following three vectors:

$$\mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

(a) Project $\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ into span $\{\mathbf{a}, \mathbf{b}\}$. (b) Project $\mathbf{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ into span $\{\mathbf{b}, \mathbf{c}\}$.

2. A norm on a vector space $\mathcal{V}$ must satisfy the triangle inequality, that is $||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$ for any two vectors $\mathbf{x} \in \mathcal{V}$, $\mathbf{y} \in \mathcal{V}$. Show the following formula for norms on $\mathcal{V} = \mathbb{R}^2$, $\mathbf{x} = (x_1, x_2)^t$, both satisfy the triangle inequality.

(a) $||\mathbf{x}|| \equiv |x_1| + |x_2|$. (b) $||\mathbf{x}|| \equiv \max(\,2\,|x_1|,\, 3\,|x_2|\,)$.

(For any two real numbers a and b, you may assume $|a + b| \leq |a| + |b|$.)

3. Consider the three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ given in problem 1 above. Explicitly compute the following quantities.

(a) $\mathbf{a} \cdot \mathbf{c}$ (c) $\mathbf{a} \times \mathbf{b}$

(b) $(\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}$ (d) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

4. Assume the following identity for arbitrary vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ in $\mathbb{R}^3$:

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})\,(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})\,(\mathbf{b} \cdot \mathbf{c}).$$

Use this along with the cyclic identity to deduce the following.

(a) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\,\mathbf{a}$.

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\,\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\,\mathbf{c}$.

(You may derive e.g. (b) from (a), but at least one must be deduced from what's given.)

5. Determine whether the given scalar function is differentiable at the origin, $(x, y) = (0, 0)$.

(a) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}} & \text{if } x^2 + y^2 > 0 \\ 0 & \text{if } x^2 + y^2 = 0. \end{cases}$ (b) $f(x,y) = \begin{cases} \dfrac{x^4 + y^4}{x^2 + y^2} & \text{if } x^2 + y^2 > 0 \\ 0 & \text{if } x^2 + y^2 = 0. \end{cases}$

6. Suppose $x(r,\theta) = r\cos\theta$ and $y(r,\theta) = 2r\sin\theta$. (Please note the $2r$.) Now, regard r and θ (implicitly) as functions of x and y.

(a) Use the chain rule to deduce $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{pmatrix} \begin{pmatrix} r_x & r_y \\ \theta_x & \theta_y \end{pmatrix}$.

(b) Solve this system to determine the partial derivatives r_x, r_y, θ_x and θ_y.

7. Compute the following closed loop line integrals, $\oint_\Gamma \mathbf{F}\cdot d\mathbf{x}$, given the following.

(a) $\mathbf{F} = x\,\mathbf{e}_y$ and Γ parametereized by $x = \cos\theta,\ y = 2\sin\theta,\ z = 4,\ 0 \le \theta \le 2\pi$.

(b) $\mathbf{F} = y\,\mathbf{e}_z$ and Γ parametereized by $x = \cos\theta,\ y = \sin\theta,\ z = \cos\theta,\ 0 \le \theta \le 2\pi$.

8. Consider an elliptical parabolic surface S given by $z = 4x^2 + y^2$ cut-off by the plane $z = 4$. Let vector field $\mathbf{F} = x\,\mathbf{e}_y$. Compute

$$\iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\, da,$$

where the surface normal $\mathbf{n}$ points <u>inward</u> toward the z-axis. (Recall the notation $\mathbf{n}\,da$ means the same as $d\mathbf{A}$.)

Hint: I used $x = r\cos\theta$, $y = 2r\sin\theta$, $z = 4r^2$ with $0 \le \theta \le 2\pi$ and $0 \le r \le 1$. Check your answer with the answer you got in 7(a) above.

9. Derive the following identities where below ϕ denotes a smooth scalar field and $\mathbf{F}$ denotes a smooth vector field.

(a) $\nabla\cdot(\nabla\times\mathbf{F}) = 0$ (b) $\nabla\times(\phi\mathbf{F}) = \phi\nabla\times\mathbf{F} + \nabla\phi\times\mathbf{F}$

10. Recall the gradient operator ∇ in a d-dimensional orthogonal coordinate system $\mathbf{x}(\mathbf{u})$ was shown to be given by

$$\nabla = \sum_{j=1}^{d} \mathbf{e}_{u_j} \frac{1}{||\mathbf{x}_{u_j}||}\frac{\partial}{\partial u_j}, \quad \text{where for each } j \ \ \mathbf{e}_{u_j} \equiv \mathbf{x}_{u_j}/||\mathbf{x}_{u_j}||.$$

Also recall cylindrical coordinates are given by

$$x(r,\theta,\zeta) = r\cos\theta,\ y(r,\theta,\zeta) = r\sin\theta,\ z(r,\theta,\zeta) = \zeta.$$

(a) (5 points) Compute that in cylindrical coordinates we have

$$\mathbf{e}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \quad \mathbf{e}_\zeta = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad \nabla = \mathbf{e}_r\frac{\partial}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial}{\partial\theta} + \mathbf{e}_\zeta\frac{\partial}{\partial\zeta}.$$

(b) (10 points) With $\mathbf{F} = f_r\,\mathbf{e}_r + f_\theta\,\mathbf{e}_\theta + f_\zeta\,\mathbf{e}_\zeta$, use the results above to derive

$$\nabla\cdot\mathbf{F} = \frac{\partial f_r}{\partial r} + \frac{1}{r}f_r + \frac{1}{r}\frac{\partial f_\theta}{\partial\theta} + \frac{\partial f_\zeta}{\partial\zeta}.$$

(c) (5 points) Show $\nabla^2 f = \dfrac{\partial^2 f}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial f}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 f}{\partial\theta^2} + \dfrac{\partial^2 f}{\partial\zeta^2}$.