Limits and Continuity on $\mathbb{R}^n$

One of the most profound concepts in mathematics is the notion of the limit. It has puzzled introductory calculus students for years. Let f be a real valued function, $f\colon \mathbb{R}^n \to \mathbb{R}$, where below we will use the shorthand notation $f(\mathbf{x})$ to signify $f(x_1, x_2, \ldots, x_n)$. Let $\mathbf{x}_0 \in \mathbb{R}^n$. What is meant by writing the *limit*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L$$

is the following: For any number $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(\mathbf{x}) - L| < \epsilon \text{ whenever } 0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta.$$

Generally, the number δ depends on ϵ. (Sometimes I'll write δ_ϵ to emphasize this fact.)

The main weakness of this ϵ-δ definition of the limit is that it is nonconstructive in nature. It does not tell you what the limit L is, it only gives you a means to check whether or not $L = \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})$. Nevertheless, there are very good reasons why the ϵ-δ definition has stood the test of time. It's not too hard to understand, and it makes establishing the major limit theorems a straight-forward endeavor.

Here are two Calculus I ($n = 1$) examples.

Define a function $c(x)$ by

$$c(x) = \begin{cases} \cos(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Does the limit $\lim_{x\to 0} c(x)$ exist? It does **not**, and here's why. Assume for the moment that the limit did exist, say $c = \lim_{x\to 0} c(x)$. This would say (taking $\epsilon = 1$) we must have $|c(x) - c| < 1$ whenever $0 < |x - 0| < \delta$ for some $\delta > 0$. However, no matter how small $\delta > 0$ is, there is a positive odd integer, n_o, such that $x_o \equiv \frac{1}{n_0 \pi} < \delta$ and a positive even integer, n_e, such that $x_e \equiv \frac{1}{n_e \pi} < \delta$. Notice that $c(x_o) = -1$ and $c(x_e) = +1$. But there is no number c satisfying both $|(-1) - c| < 1$ and $|(+1) - c| < 1$. Therefore, the assumption that the limit existed is impossible.

Next define a function $p(x) \equiv x\, c(x)$ where $c(x)$ is as defined in the previous paragraph. Does the limit $\lim_{x\to 0} p(x)$ exist? It does, its limit value is zero, and here's why. For $x \neq 0$, notice that $|p(x) - 0| = |x \cos(1/x)| \leq |x|$. So, whenever $0 < |x - 0| < \delta_\epsilon \equiv \epsilon$, we have $|p(x) - 0| \leq |x - 0| < \delta_\epsilon = \epsilon$. That is, for any $\epsilon > 0$ (no matter how small) we have

$$|p(x) - 0| < \epsilon \text{ whenever } 0 < |x - 0| < \epsilon \equiv \delta_\epsilon.$$

All of the basic limit theorems from Calculus I extend to $\mathbb{R}^n$. For example

$$\text{If } \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = F \text{ and } \lim_{\mathbf{x}\to\mathbf{x}_0} g(\mathbf{x}) = G,$$

$$\text{then } \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) + g(\mathbf{x}) = F + G,\ \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x})\, g(\mathbf{x}) = F\, G, \text{ etc.}$$

Their proofs are trivially adapted to $\mathbb{R}^n$ by making one simple observation:

$$\text{If } 0 < \delta_1 \le \delta_2 \quad \text{then} \quad \{\mathbf{x} : 0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta_1\} \subseteq \{\mathbf{x} : 0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta_2\}.$$

That is, a ball centered at $\mathbf{x}_0$ with radius $\delta_1 > 0$ is contained inside a ball centered at the same point with larger radius δ_2.

Here's a remark intended for the more interested student. The norm used above to define the n-dimensional limit was not explicitly stated. However, since all norms on finite dimensional vector spaces are equivalent, it doesn't matter which norm is used. That is, whether or not the limit exists and/or what its particular value is does not depend on the norm. Sadly, this fact does not generalized to infinite dimensional normed vector spaces.

Next, I'm going to show you by example how to evaluate a two dimensional limit by using your first year calculus skills.

For the first example, consider the function

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

To determine a possible value of $\lim_{(x,y)\to(0,0)} f(x,y)$ (assuming it exists) we'll use polar coordinate rays, $x = r\cos\theta$ and $y = r\sin\theta$, and let $r \downarrow 0$. Plugging in, we get

$$\lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta) = \lim_{r\downarrow 0} \frac{(r\cos\theta)^2 - (r\sin\theta)^2}{(r\cos\theta)^2 + (r\sin\theta)^2} = \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

If the limit as $(x,y) \to (0,0)$ were to exist, it must not depend on the path. But here it does; i.e. it depends on the ray angle θ. Therefore, for this function the limit as $(x,y) \to (0,0)$ does <u>not</u> exist.

This example illustrates the following fact, a fact you are expected to know. If

$$\lim_{r\downarrow 0} f(x_0 + r\cos\theta, y_0 + r\sin\theta)$$

does <u>not</u> exist, or yields a limit which <u>varies</u> with θ, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

does <u>not</u> exist.

The second example is more interesting primarily because it has a limit. Consider

$$u(x,y) = \begin{cases} \dfrac{x\sin(x)\cosh(y) + y\cos(x)\sinh(y)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0). \end{cases}$$

The complex analysis student might recognize this as the real part of $\sin(z)/z$ when $z \neq 0$.

Again, assuming the limit of u exists as $(x, y) \to (0, 0)$

$$\begin{aligned}\lim_{(x,y)\to(0,0)} u(x,y) &= \lim_{r\downarrow 0} u(r\cos\theta, r\sin\theta) \\ &= \lim_{r\downarrow 0}\left(\cos\theta\,\frac{\sin(r\cos\theta)}{r}\cosh(r\sin\theta) + \sin\theta\cos(r\cos\theta)\,\frac{\sinh(r\sin\theta)}{r}\right)\end{aligned}$$

Using first year calculus techniques, compute that as $r \downarrow 0$

$$\frac{\sin(r\cos\theta)}{r} \to \cos\theta, \quad \cosh(r\sin\theta) \to 1,$$
$$\cos(r\cos\theta) \to 1, \quad \frac{\sinh(r\sin\theta)}{r} \to \sin\theta,$$

and these together yield

$$\lim_{r\downarrow 0} u(r\cos\theta, r\sin\theta) = \cos^2\theta + \sin^2\theta = 1.$$

So, if the limit of $u(x, y)$ were to exist as $(x, y) \to (0, 0)$, u would have to tend to one. You'll be asked to show in an exercise below that indeed $u(x, y) \to 1$ as $(x, y) \to (0, 0)$.

This example illustrates another fact you are expected to know. If

$$\lim_{r\downarrow 0} f(x_0 + r\cos\theta, y_0 + r\sin\theta)$$

<u>exist</u> and is <u>constant</u> in θ, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y)$$

<u>may</u> exist, and if it did, the two limits (i.e. $r \downarrow 0$ and $(x, y) \to (x_0, y_0)$) would have the same value.

The third example I give here is intended to show you it is possible for

$$\lim_{r\downarrow 0} f(x_0 + r\cos\theta, y_0 + r\sin\theta) = \mathit{const} \text{ in } \theta, \text{ but } \lim_{(x,y)\to(x_0,y_0)} f(x,y) \ \underline{\text{fails}} \text{ to exist.}$$

Consider the following $2d$ real valued function

$$f(x,y) = \begin{cases} 1 & \text{if } y = x^2 \\ 0 & \text{otherwise.} \end{cases}$$

Try to visualize this. It has value one on the parabola, $y = x^2$, but is zero everywhere else. Now, let's consider $f(r\cos\theta, r\sin\theta)$ for various fixed angles θ. Along any ray with $\pi \le \theta \le 2\pi$, (in the third or fourth quadrant), $f(r\cos\theta, r\sin\theta) = 0$ for every $r > 0$. Therefore, for these angles $\lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta) = 0$. Along the vertical ray, $\theta = \pi/2$, we again have $f(r\cos\theta, r\sin\theta) = 0$ for every $r > 0$. Finally, for rays with $0 < \theta < \pi/2$ or $\pi/2 < \theta < \pi$ notice that

$$f(r\cos\theta, r\sin\theta) = \begin{cases} 1 & \text{if } r = \sin\theta/\cos^2\theta > 0 \\ 0 & \text{for all other } r > 0. \end{cases}$$

Make sure you understand why this also says $\lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta) = 0$ for every θ in these ranges. Therefore, for every ray angle θ, we have shown

$$\lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta) = 0.$$

However, it's easy to see the limit of this function can not possibly exist as $(x,y) \to (0,0)$. Check that for any $\delta > 0$, there are points in the ball

$$\{(x,y) : 0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta\}$$

with $f(x,y) = 0$ but also points with $f(x,y) = 1$.

I'm going to close this $2d$ limit discussion by stating a little theorem. Suppose a possible limit is identified by computing

$$\lim_{r\downarrow 0} f(x_0 + r\cos\theta, y_0 + r\sin\theta) = L,$$

where L does not depend on the ray angle θ. If

$$\lim_{r\downarrow 0} \sup_{\theta} |f(x_0 + r\cos\theta, y_0 + r\sin\theta) - L| = 0,$$

then the limit of $f(x,y)$ exists as $(x,y) \to (x_0, y_0)$, and the limit value is L. I'll be glad to prove this in class if requested.

The fact that the full 2-d limit may not exist even when it has a unique limit along rays is somewhat pathological. It is essentially the same idea that a sequence may not *converge uniformly* when it has a *pointwise limit.* (You should touch on these concepts when you take intermediate analysis.) I don't want to dwell on this distinction here however. Give it some thought, but not too much. <u>I don't plan to test you on it</u>.

We'll close this set of homework notes by briefly discussing the concept of continuity.

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is *continuous* at a point $\mathbf{x}_0$ if for any number $\epsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(\mathbf{x}) - f(\mathbf{x}_0)| < \epsilon \text{ whenever } ||\mathbf{x} - \mathbf{x}_0|| < \delta.$$

It should be obvious how closely related the concepts of the limit and continuity are. In fact,

$$f(\mathbf{x}) \text{ is continuous at } \mathbf{x}_0 \quad \Longleftrightarrow \quad f(\mathbf{x}_0) = \lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}).$$

Clearly, the functions $f(\mathbf{x}) = \mathit{const}$ and $f(\mathbf{x}) = x_i$ for each coordinate direction $1 \le i \le n$ are continuous at every point $\mathbf{x}_0 \in \mathbb{R}^n$.

Also, the following facts are established for n-dimensional real valued functions in exactly the same way as done in Calculus I for 1-dimensional functions.

If $f(\mathbf{x})$ and $g(\mathbf{x})$ are both continuous at $\mathbf{x}_0$ then

$$f(\mathbf{x}) + g(\mathbf{x}) \text{ and } f(\mathbf{x})\, g(\mathbf{x}) \text{ are continuous at } \mathbf{x}_0.$$

In addition if $f(\mathbf{x}_0) \neq 0$ then

$$1/f(\mathbf{x}) \text{ is continuous at } \mathbf{x}_0 .$$

From these facts, conclude that any rational polynomial

$$f(\mathbf{x}) = \frac{p(\mathbf{x})}{q(\mathbf{x})}, \quad p \text{ and } q \text{ are polynomials in } x_1, \ldots, x_n,$$

is continuous at any point $\mathbf{x}_0$ where $q(\mathbf{x}_0) \neq 0$. For example

$$f(x,y,z) = \frac{3x^2 + z^3 - 7xyz}{x^2 + 2y^2 + 6}$$

is continuous everywhere in $\mathbb{R}^3$ since the denominator here is never zero.

Finally, suppose $f(\mathbf{x})$ is continuous at $\mathbf{x}_0 \in \mathbb{R}^n$, and suppose $\alpha : \mathbb{R} \to \mathbb{R}$ is continuous at $y_0 \equiv f(\mathbf{x}_0) \in \mathbb{R}$. Then the composition

$$\alpha(f(\mathbf{x})) \text{ is continuous at } \mathbf{x}_0 .$$

Again, the proof of this fact is just like the proof you saw in first year calculus.

This last result allows us to take any member from the catalogue of first year calculus 1-d continuous functions, e.g. e^x, $\sin(x)$, $\cos(x)$, etc, and extend them to $\mathbb{R}^n$. For example

$$f(x,y) = \begin{cases} \log(|x^2 - 3xy| + 1) + \cos(x)/\sin(y) & \text{when } y \neq n\pi \\ 0 & \text{when } y = n\pi. \end{cases}$$

is continuous everywhere except possibly when y is an integer multiple of π.

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1. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ has a limit F as $\mathbf{x}$ tends to $\mathbf{x}_0$. Prove there is a $\delta > 0$ such that $|f(\mathbf{x})| < 1 + |F|$ for any $\mathbf{x}$ satisfying $0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta$. Hint: Let $\epsilon = 1$.

2. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}$ has a <u>nonzero</u> limit F as $\mathbf{x}$ tends to $\mathbf{x}_0$. Prove there is a $\delta > 0$ such that $|f(\mathbf{x})| > \frac{1}{2}|F|$ for any $\mathbf{x}$ satisfying $0 < ||\mathbf{x} - \mathbf{x}_0|| < \delta$. Hint: Let $\epsilon = \frac{1}{2}|F|$.

3. The following limits do <u>not</u> exist. Explain why.

(a) $$\lim_{(x,y)\to(0,0)} f(x,y) \text{ when } f(x,y) = \begin{cases} +1 & \text{if } x - y \geq 0 \\ -1 & \text{if } x - y < 0. \end{cases}$$

(b) $$\lim_{(x,y)\to(0,0)} f(x,y) \text{ when } f(x,y) = \begin{cases} \dfrac{x}{\sqrt{x^2 + y^2}} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(c) $$\lim_{(x,y)\to(2,1)} f(x,y) \text{ when } f(x,y) = \begin{cases} \dfrac{x}{\sqrt{(x-2)^2 + (y-1)^2}} & \text{if } (x,y) \neq (2,1) \\ 0 & \text{if } (x,y) = (2,1). \end{cases}$$

(d) $$\lim_{(x,y)\to(1,0)} f(x,y) \text{ when } f(x,y) = \begin{cases} \dfrac{(x-1)\,y}{4(x-1)^2 + y^2} & \text{if } (x,y) \neq (1,0) \\ 0 & \text{if } (x,y) = (1,0). \end{cases}$$

4. Consider functions $f(x,y)$ which, for $(x,y) \neq (0,0)$, are defined as given. Each has a limit as $(x,y) \to (0,0)$. Determine their limit values.

(a) $f(x,y) = \dfrac{x^3 + xy^2 + x^2 + y^2}{x^2 + y^2}$ (b) $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$

(c) $f(x,y) = \dfrac{e^{x^2+y^2} - 1}{x^2 + y^2}$ (d) $f(x,y) = \dfrac{e^x + e^y - x - y - 2}{x^2 + y^2}$

5. Recall $u(x,y)$ from the example discussed earlier in the text

$$u(x,y) = \begin{cases} \dfrac{x \sin(x) \cosh(y) + y \cos(x) \sinh(y)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0). \end{cases}$$

There we showed

$$\lim_{r \downarrow 0} u(r\cos\theta, r\sin\theta) = 1.$$

Here you will show that in fact $\lim_{(x,y)\to(0,0)} u(x,y) = 1$.
Don't worry. This will <u>not</u> be on your exam.

(a) It can be shown that for $0 < \kappa \leq 1$ we have $|\cosh(\kappa r) - 1| \leq |\cosh r - 1|$ and when $0 < r \leq \pi$ $|\sin(\kappa r)/(\kappa r) - 1| \leq |\sin r/r - 1|$. Use these to show that for r in this range

$$\sup_\theta \left| \cos\theta \, \frac{\sin(r\cos\theta)}{r} \cosh(r\sin\theta) - \cos^2\theta \right| \leq |\cosh r - 1| + \left| \frac{\sin r}{r} - 1 \right|.$$

(b) It can be shown that for $0 < \kappa \leq 1$ we have $|\sinh(\kappa r)/(\kappa r) - 1| \leq |\sinh r/r - 1|$ and when $0 < r \leq \pi/2$ $|\cos \kappa r - 1| \leq |\cos r - 1|$. Use these to show that for r in this range

$$\sup_\theta \left| \sin\theta \cos(r\cos\theta) \, \frac{\sinh(r\sin\theta)}{r} - \sin^2\theta \right| \leq |\cos r - 1| + \left| \frac{\sinh r}{r} - 1 \right|.$$

(c) Use parts (a) and (b) to conclude

$$\lim_{r\downarrow 0} \sup_\theta |u(r\cos\theta, r\sin\theta) - 1| = 0 \quad \text{which implies} \quad \lim_{(x,y)\to(0,0)} u(x,y) = 1.$$

(d) Is $u(x,y)$ continuous for every $(x,y) \in \mathbb{R}^2$?

6. Consider $v(x,y)$ given by

$$v(x,y) = \begin{cases} \dfrac{x \cos(x) \sinh(y) - y \sin(x) \cosh(y)}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

It has a limit as $(x,y) \to (0,0)$. (a) What is the value of this limit? (b) Is $v(x,y)$ continuous for every $(x,y) \in \mathbb{R}^2$?
FYI: This is the imaginary part of $\sin(z)/z$ for complex $z \neq 0$