1. A complex function $f$ is **analytic** on an open set $\Omega \subseteq \mathbb{C}$ if $f'(z)$ exists for every $z \in \Omega$. Determine points at which the following are not analytic.

   (a) $\frac{1}{z - 2 + 3i}$
   (b) $\frac{iz^3 + 2z}{z^2 + 1}$
   (c) $\frac{3z - 1}{z^2 + z + 4}$
   (d) $z^2(2z^2 - 3z + 1)^{-2}$

2. A complex function $f$ is said to be **entire** if it is analytic on the whole complex plane $\mathbb{C}$. Suppose $f$ and $g$ below are entire. Decide whether the following statements in general are true or not.

   (a) $f^3$ is entire.
   (b) $fg$ is entire.
   (c) $f/g$ is entire.
   (d) $5f + ig$ is entire.
   (e) $f(1/z)$ is entire.
   (f) $f(g)$ is entire.

   Give a counter example when not. You need not prove when true.

Recall the Cauchy-Riemann conditions. If analytic $f(z) = u(x, y) + iv(x, y)$, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$ 

3. Suppose $f(z) = u(x, y) + iv(x, y)$ is analytic. Show the following are true.

   (a) $(u_x)^2 + (v_x)^2 = (u_y)^2 + (v_y)^2$
   (b) $u_x u_y + v_x v_y = 0$
   (c) $u_{xx} + u_{yy} = 0$
   (d) $v_{xx} + v_{yy} = 0$

4. Is it possible for $f(z) = u(x, y) + iv(x, y)$ for the given functions $u$ and $v$? If so, determine $f(z)$. Hint: To determine $f(z)$ consider $u(x, 0) + iv(x, 0)$.

   (a) $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$
   (b) $u(x, y) = e^{-y} \cos x$, $v(x, y) = e^{-y} \sin x$
   (c) $u(x, y) = x^2 + y^2$, $v(x, y) = 2xy$
   (d) $u(x, y) = e^y \cos x$, $v(x, y) = e^y \sin x$

5. Do the same as in the previous exercise.

   (a) $u(x, y) = \frac{x}{x^2 + y^2}$, $v(x, y) = \frac{y}{x^2 + y^2}$
   (b) $u(x, y) = e^x(x \cos y + y \sin y)$, $v(x, y) = e^x(y \cos y + x \sin y)$
   (c) $u(x, y) = \frac{x}{x^2 + (y + 1)^2}$, $v(x, y) = -\frac{(y + 1)}{x^2 + (y + 1)^2}$
   (d) $u(x, y) = e^x(x \cos y - y \sin y)$, $v(x, y) = e^x(y \cos y + x \sin y)$
6. Sketch the given open regions \( \Omega \subseteq \mathbb{C} \) in the complex plane.
   (a) \( \Omega = \{ z \in \mathbb{C} : 0 < \Re(z) < 1 \} \)
   (b) \( \Omega = \{ z \in \mathbb{C} : 0 < \Im(z) < 1 \} \)
   (c) \( \Omega = \{ z \in \mathbb{C} : |z| < 1 \} \)
   (d) \( \Omega = \{ z \in \mathbb{C} : 0 < \arg(z) < \pi/4 \} \)

7. Sketch the given open regions \( \Omega \subseteq \mathbb{C} \) in the complex plane.
   (a) \( \Omega = \{ z \in \mathbb{C} : 0 < \Re(z) < 1 \text{ and } \Im(z) > 0 \} \)
   (b) \( \Omega = \{ z \in \mathbb{C} : 0 < \Im(z) < 1 \text{ and } \Re(z) < 1 \} \)
   (c) \( \Omega = \{ z \in \mathbb{C} : |z| < 1 \text{ and } 0 < \arg(z) < \pi/4 \} \)
   (d) \( \Omega = \{ z \in \mathbb{C} : |z| > 1 \text{ and } \Im(z) > 0 \} \)

8. Consider the open set \( \Omega = \{ z \in \mathbb{C} : |z| < 2 \text{ and } 0 < \arg(z) < \pi/2 \} \). We will use the notation \( \partial \Omega \) to denote the boundary of \( \Omega \). (FYI: In terms of set operations, we can write the boundary of the open set \( \Omega \) as \( \partial \Omega = \Omega^c \cap \overline{\Omega} \).) Let \( f(z) = z^3 \). In the complex plane, sketch the image of \( \partial \Omega \) under the map \( f \).

9. Consider the open set \( \Omega = \{ z \in \mathbb{C} : |z| > 2 \text{ and } 0 < \arg(z) < \pi/2 \} \). Let \( f(z) = 1/z \). In the complex plane, sketch the image of \( \partial \Omega \) under the map \( f \).

10. Consider the open set \( \Omega = \{ z \in \mathbb{C} : 0 < \Re(z) < \pi/2 \text{ and } \Im(z) > 0 \} \). Let \( f(z) = e^{iz} \).
    In the complex plane, sketch the image of \( \partial \Omega \) under the map \( f \).

You can find a dictionary of conformal maps at
You might enjoy looking these over.