Math 3364: Homework 4

1. Recall from calculus: Along a smooth path $\Gamma(x_0, x_1)$ in $\mathbb{R}^2$ which joins $x_0$ to $x_1$ we have
\[
\int_{\Gamma(x_0, x_1)} \nabla \phi (x) \cdot dx = \int_{\Gamma(x_0, x_1)} (\phi_x \, dx + \phi_y \, dy) = \phi(x_1) - \phi(x_0).
\]
Use this together with Cauchy-Riemann to prove that along a smooth path $\Gamma(z_0, z_1)$ in $\mathbb{C}$ which joins $z_0$ to $z_1$ we have
\[
\int_{\Gamma(z_0, z_1)} f'(z) \, dz = f(z_1) - f(z_0).
\]

2. Recall Green’s theorem from calculus: If $S$ is a region in the plane with a smooth simple boundary $\Gamma$ (positively oriented), and if $f(x, y) e_x + g(x, y) e_y$ is a smooth 2-d vector field on $S$, then the following line integral and surface integral are equal.
\[
\oint_{\Gamma} f \, dx + g \, dy = \iint_{S} \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \, dx \, dy.
\]
Suppose $u(x, y)$ and $v(x, y)$ are two real functions which satisfy the Cauchy-Riemann conditions. Use this and Green’s theorem to deduce the following.

(a) $\oint_{\Gamma} u \, dx - v \, dy = 0$.  
(b) $\oint_{\Gamma} v \, dx + u \, dy = 0$.

(c) Conclude that when $f = u + iv$ we have $\oint_{\Gamma} f(z) \, dz = 0$.

3. Let $\Gamma(z_0, z_1)$ denote a smooth path in an open and simply connected subset $\Omega \subseteq \mathbb{C}$ which starts at $z_0$ and ends at $z_1$. Suppose $f$ is a given continuous complex function on $\Omega$ whose integral is always path independent; i.e. for any two points $z_0$ and $z_1$ in $\Omega$ and any two smooth paths $\Gamma_1 = \Gamma_1(z_0, z_1)$ and $\Gamma_2 = \Gamma_2(z_0, z_1)$ joining $z_0$ to $z_1$ we have $\int_{\Gamma_1} f(\zeta) \, d\zeta = \int_{\Gamma_2} f(\zeta) \, d\zeta$. The assumption of path independence allows us to define the following function of $z$:
\[
F(z) = \int_{\Gamma(z_0, z)} f(\zeta) \, d\zeta
\]
for any $z \in \Omega$ where $z_0$ is a given constant point in $\Omega$. (a) Observe that $F(z_0) = 0$.

(b) Let $z \in \Omega$ and $z + \Delta z \in \Omega$. Observe that
\[
F(z + \Delta z) - F(z) = \int_{\Gamma(z, z + \Delta z)} f(\zeta) \, d\zeta.
\]

(c) Take a particular straight-line path for $\Gamma(z, z + \Delta z)$ parameterized by $z(t) = z + t\Delta z$ for $t \in [0, 1]$; $(z(t) \in \Omega$ provided $|\Delta z|$ is small enough). Conclude that
\[
F(z + \Delta z) - F(z) = \Delta z \int_{0}^{1} f(z + t\Delta z) \, dt.
\]
(d) From this conclude that \( F \) is analytic in \( \Omega \) and is in fact the antiderivative of \( f \); i.e. \( dF(z)/dz = f(z) \).

4. Let \( \Gamma \) be a straight line in the complex plane going from \( z_0 = 1 + i \) to \( z_1 = 2 + 2i \).
   (a) Parameterize \( \Gamma \) by \( z(t) = z_0 + t(z_1 - z_0) \), \( t \) runs from 0 to 1, and compute \( \int_\Gamma z^2\,dz = \int_0^1 (z(t))^2 z'(t)\,dt \).
   (b) Use exercise 1 to compute this integral without parameterizing the line.

5. Let \( \Gamma \) be a 1/4 circle in the complex plane going from \( z_0 = 1 \) to \( z_1 = i \).
   (a) Parameterize \( \Gamma \) by \( z(t) = e^{it} \), \( t \) runs from 0 to \( \pi/4 \), and compute \( \int_\Gamma z^2\,dz = \int_0^{\pi/4} (z(t))^2 z'(t)\,dt \).
   (b) Use exercise 1 to compute this integral without parameterizing the circle.

6. Cauchy’s integral formula says that when \( f \) is analytic on an open and simply connected set \( \Omega \subseteq \mathbb{C} \), and \( \Gamma \) is a positively oriented simple path in \( \Omega \) which surrounds a point \( z \), then
   \[
   2\pi if(z) = \oint_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta.
   \]
   (a) Use induction to deduce
   \[
   \frac{d^k}{dz^k} \left( \frac{1}{\zeta - z} \right) = k! \frac{1}{(\zeta - z)^{k+1}}
   \]
   provided \( z \neq \zeta \).
   (b) Use this identity and Cauchy’s integral formula to derive the following generalize Cauchy formula
   \[
   2\pi if^{(k)}(z) = k! \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{k+1}}\,d\zeta,
   \]
   where \( f^{(k)}(z) \) denotes the \( k \)th derivative of \( f \) at \( z \).

7. Let \( \Gamma_R \) denote the positively oriented circle, centered at \( z = 0 \) with radius \( R > 0 \). Use Cauchy’s integral formula and partial fractions to determine the values of the following integrals.
   (a) \( \oint_{\Gamma_R} \frac{z}{z^2 - 1}\,dz \)
   (b) \( \oint_{\Gamma_R} \frac{e^z}{z^2 - 1}\,dz \)
   Please note that your answer will depend on \( R \). The case here when \( R = 1 \) is special. Don’t worry about this now.

8. Let \( \Gamma_R \) be as given in the previous exercise. Determine the values of the following integrals.
   (a) \( \oint_{\Gamma_R} \frac{z}{(z-1)^2}\,dz \)
   (b) \( \oint_{\Gamma_R} \frac{e^z}{(z-1)^2}\,dz \)
9. Let $\Gamma_R$ be as given in exercise 7. Determine the values of the following integrals.

(a) \[ \oint_{\Gamma_R} \frac{1}{z^2(z-1)} \, dz \]
(b) \[ \oint_{\Gamma_R} \frac{\cos z}{z^2(z-1)} \, dz \]

10. Suppose $f(z)$ is analytic in an open ball $B_\delta(z_0) \equiv \{z : |z - z_0| < \delta\}$ for some $\delta > 0$. Use the generalized Cauchy formula to conclude there is a constant $M_\rho$ such that

\[ |f^{(n)}(z_0)| \leq n! \frac{M_\rho}{\rho^n} \]

for any given $\rho$ satisfying $0 < \rho < \delta$. Hint: Parameterize $\Gamma$ by $z_0 + \rho e^{i\theta}$ and take $M_\rho = \max_{\zeta \in \overline{B}_\rho(z_0)} |f(\zeta)|$.  
