

①

$$1 \quad C = \{z : |z| = 1\}$$

Cauchy's Integral
Formulae

a) $\oint_C \frac{e^z}{z-0} dz$ e^z is analytic inside C
and 0 inside C

$$\therefore = 2\pi i e^0 = \boxed{2\pi i}$$

b) $\oint_C \frac{z+2}{(z-0)^2} dz$ $z+2$ is analytic
inside C and 0
inside C

$$\therefore = \frac{2\pi i}{1!} \frac{d}{dz}(z+2) \Big|_{z=0}$$

$$= \boxed{2\pi i}$$

c) $\oint_C \frac{e^z}{(z-0)(z-2)} dz$

$e^z/(z-2)$ analytic inside
 C and 0 inside.

$$= 2\pi i \frac{e^z}{z-2} \Big|_{z=0}$$

$$= \boxed{-\pi i}$$

d) $\oint_C \frac{\sin(z)}{(z-0)^2} dz$

$\sin(z)$ is analytic
inside C & 0 inside

$$= \frac{2\pi i}{1!} \frac{d}{dz} \sin(z) \Big|_{z=0} = \boxed{2\pi i}$$

②

$$2a) \oint_{|z|=4} \frac{z^2+5}{z-3} dz$$

z^2+5 analytic in $|z| \leq 4$
and 3 inside $|z|=4$

$$= 2\pi i (z^2+5) \Big|_{z=3} = \boxed{28\pi i}$$

$$b) \oint_{|z-2|=1} \frac{1}{(z-2)(z+2)} dz$$

$\frac{1}{z+2}$ is analytic in $|z-2| \leq 1$
and 2 inside

$$= 2\pi i \frac{1}{z+2} \Big|_{z=2} = \boxed{\frac{\pi i}{2}}$$

$$c) \oint_{|z|=2} \frac{\sin(z)}{(z-1)^3} dz$$

$\sin(z)$ is analytic
in $|z|=2$ and
 $z=1$ is inside.

$$= \frac{2\pi i}{2!} \frac{d^2}{dz^2} \sin(z) \Big|_{z=1}$$

$$= \boxed{-\pi i \sin(1)}$$

③

$$3a) \oint_{|z|=1} \left(z + \frac{1}{z} \right) \frac{dz}{z}$$

$$= \oint_{|z|=1} \frac{z^2 + 1}{(z-0)^2} dz$$

$z^2 + 1$ is analytic in $|z| < 1$
and $z=0$ inside

$$= \frac{2\pi i}{2!} \frac{d}{dz} (z^2 + 1) \Big|_{z=0} = \boxed{0}$$

b) parameterize $|z|=1$ by $z = e^{i\theta}$
 $-\pi \leq \theta \leq \pi$

$$\oint_{|z|=1} \left(z + \frac{1}{z} \right) \frac{dz}{z}$$

$$= \int_{-\pi}^{\pi} (e^{i\theta} + e^{-i\theta}) \frac{ie^{i\theta}}{e^{i\theta}} d\theta$$

$$= 2 \int_{-\pi}^{\pi} \frac{e^{i\theta} + e^{-i\theta}}{2} i d\theta = 2i \int_{-\pi}^{\pi} \cos(\theta) d\theta$$

$$= 0 \text{ from (a)}$$

$$\text{So } \boxed{\int_{-\pi}^{\pi} \cos(\theta) d\theta = 0}$$

(4)

4 Evaluate $\int_{-\pi}^{\pi} \cos^{2n}(\theta) d\theta$

Consider $\oint_{|z|=1} \left(\frac{z + 1/z}{2} \right)^{2n} \frac{dz}{iz} = I$

First parameterize $|z|=1$ $z = e^{i\theta}$
 $-\pi < \theta \leq \pi$

to get

$$\begin{aligned} I &= \int_{-\pi}^{\pi} \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right)^{2n} \frac{ie^{i\theta} d\theta}{ie^{i\theta}} \\ &= \int_{-\pi}^{\pi} \cos^{2n}(\theta) d\theta \end{aligned}$$

Now to get a number for I I'll
 expand integrand with binomial theorem

$$\begin{aligned} \frac{\left(z + \frac{1}{z} \right)^{2n}}{2^{2n}} \frac{1}{iz} &= \sum_{k=0}^{2n} \binom{2n}{k} z^{2n-k} z^{-k} \\ &= \frac{1}{i 2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} z^{2n-2k-1} \quad (\text{over}) \end{aligned}$$

⑤ Recall $\oint_{|z|=1} z^n dz = \begin{cases} 0 & n \neq -1 \\ 2\pi i & \text{when } n = -1 \end{cases}$

So $\oint \frac{1}{i 2^{2n}} \sum_{k=0}^{2n} \binom{2n}{k} z^{2n-2k-1} dz$

$1 \neq -1 \Rightarrow 0 \quad \forall k \text{ except when}$
 $2n-2k-1 = -1$
 $\Rightarrow \underline{\underline{k=n}}$

$= \frac{1}{i 2^{2n}} \binom{2n}{n} 2\pi i = I$

Let's simplify.

$I = \frac{2\pi}{2^{2n}} \frac{(2n)!}{(2n-n)! n!}$ Binomial
coeff

$= 2\pi \frac{(2 \cdot 4 \cdot \dots \cdot 2n) (1 \cdot 3 \cdot 5 \cdot \dots \cdot 2n-1)}{2^n n! \cdot 2^n n!}$

$= 2\pi \frac{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \dots \cdot (2n)}$

⑥

5) f is bounded & entire

a) From Cauchy integral formula

$$f'(z) = \frac{1!}{2\pi i} \oint_{\Gamma_R} \frac{f(\zeta)}{(z-\zeta)^2} d\zeta$$

where take $\Gamma_R = \{\zeta : |z-\zeta|=R\}$,

Parameterize $\zeta = z + Re^{i\theta}$
 $-\pi < \theta \leq \pi$

So

$$f'(z) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(z + Re^{i\theta})}{(Re^{i\theta})^2} iRe^{i\theta} d\theta$$

$$= \frac{1}{2\pi R} \int_{-\pi}^{\pi} f(z + Re^{i\theta}) e^{-i\theta} d\theta$$

So use Δ -inequality

Say $|f(z)| \leq M$

$$|f'(z)| \leq \frac{1}{2\pi R} \int_{-\pi}^{\pi} |f(z + Re^{i\theta})| d\theta$$

$$= \frac{M}{R}$$

⑦ So for mb $z \in \mathbb{C}$ we have

$$|f'(z)| \leq \frac{M}{R} \quad \text{for } m > R > 0 \quad (b + R \rightarrow \infty)$$

$$\Rightarrow \left(f'(z) = 0 \quad \forall z \in \mathbb{C} \right)$$

$$b) \quad \int_{\gamma_0^z} f'(z) dz = f(z) - f(0) \quad \text{FTC}$$

$$\parallel \text{ part (a)} \\ \int_{\gamma_0^z} 0 dz = 0$$

So $f(z) = f(0)$ for every $z \in \mathbb{C}$.

$$6 \quad \text{let } p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$$

where $n \geq 1$.

a) Use backward Δ -inequality

$$|p(z)| \geq |z^n| - (|a_{n-1}|z^{n-1}| + \dots + |a_0|)$$

$$\text{let } z = Re^{i\theta} \quad (\text{over})$$

(12)

So

$$|P(z)| \geq R^n \left[1 - \left(\frac{|a_{n-1}|}{R} + \dots + \frac{|a_0|}{R^n} \right) \right]$$

Take R large enough so that $1 - [\dots] \geq \frac{1}{2}$

and larger if necessary s.t. $\frac{1}{2} R^n > 1$

$$\therefore |P(z)| \geq 1 \quad \forall |z| \geq R,$$

b) For the purpose of contradiction,
Suppose $P(z)$ is never zero,

$\Rightarrow \frac{1}{P(z)}$ is continuous and diff'ble

for all $z \in \mathbb{C}$.

But on $|z| \leq R \Rightarrow \left| \frac{1}{P(z)} \right| \leq M$

Since continuous function on $|z| \leq R$ is bounded.

$$\therefore \left| \frac{1}{P(z)} \right| \leq \max(M, 1) \quad \forall z \in \mathbb{C}.$$

$\Rightarrow \frac{1}{P(z)}$ is (bounded & entire) $\Rightarrow \frac{1}{P(z)} = \text{const.}$ (over)

⑪

But $n > 1$ $\frac{1}{p(z)}$ is clearly not constant,

This contradicts $p(z)$ is never zero,

7/ Fact 1 If f is analytic on S then $|f|$ is continuous, Because

$$||f(z_1)| - |f(z_2)|| \leq |f(z_1) - f(z_2)|$$

(This is the backward Δ -inequality.)

Fact 2 From Cauchy

$$f(z_*) = \frac{1}{2\pi i} \oint_{|z-z_*|=p} \frac{f(z)}{z-z_*} dz$$

$$\Rightarrow |f(z_*)| = \left| \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{f(z_* + pe^{i\theta})}{pe^{i\theta}} ipe^{i\theta} d\theta \right|$$

$$= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} f(z_* + pe^{i\theta}) d\theta \right|$$

$$\therefore |f(z_*)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(z_* + pe^{i\theta})| d\theta \quad (\text{over})$$

⑫

Define $R = \text{open region inside } \Gamma$,

The set $\bar{R} \equiv R \cup \Gamma$ is closed and bounded and $|f(z)|$ is continuous

on $\bar{R} \Rightarrow \max_{z \in \bar{R}} |f(z)| = M$ is

attained on $\bar{R}$, i.e. $|f(z^*)| = M$.

For the purpose of contradiction,

assume $M > \max_{z \in \Gamma} |f(z)|$. If

this is impossible we must have

$M = \max_{z \in \Gamma} |f(z)|$, i.e. $|f|'$ max

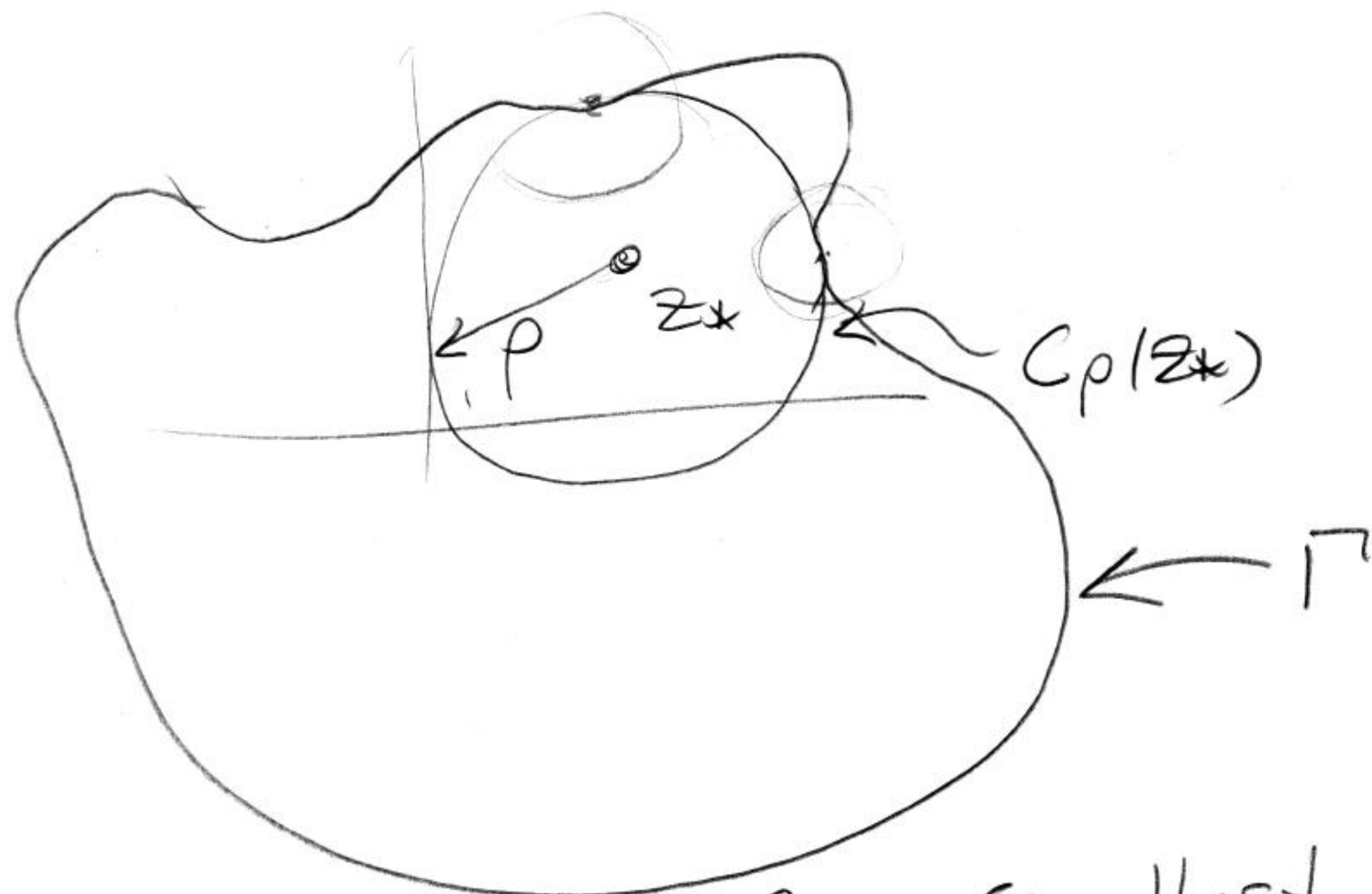
is attained on Γ .

To this goal, let $z^* \in R$ where

$|f(z^*)| = M > \max_{z \in \Gamma} |f(z)|$. Let

(*) $\varepsilon \equiv M - \max_{z \in \Gamma} |f(z)| > 0$. (over)

(12)



Let $\rho > 0$ be the smallest $\rho \exists$
 $C_\rho(z_*)$ touches Γ , see fig above.

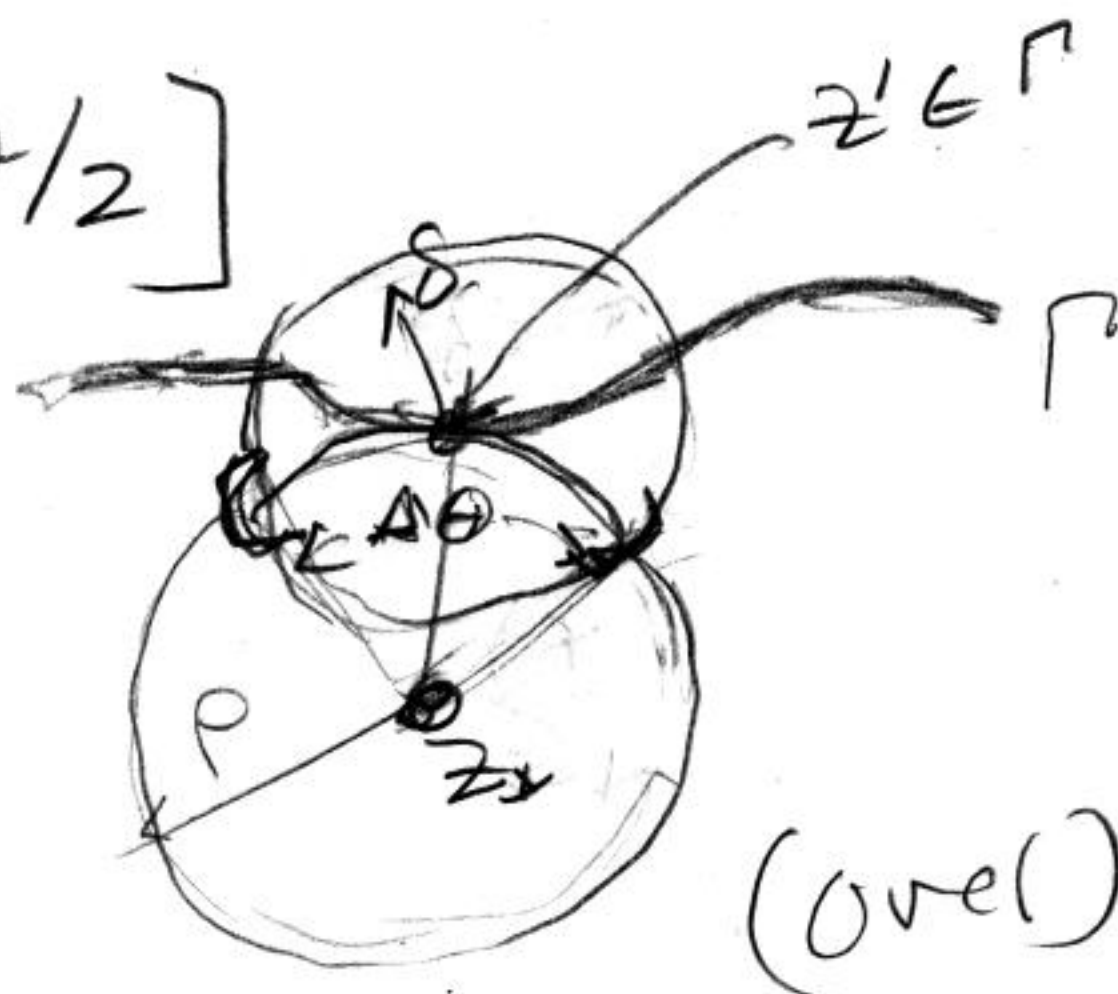
From Fact 2 we have

$$(*) M = |f(z_*)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(z_* + \rho e^{i\theta})| d\theta$$

Again by continuity $\exists \Delta\theta > 0$

$$\exists |f(z_* + \rho e^{i\theta})| \leq M - \frac{\varepsilon}{2} \text{ on}$$

$$\theta \in [\theta_* - \Delta\theta/2, \theta_* + \Delta\theta/2]$$



use continuity of $|f|$
assumption (*)

$$\begin{aligned} & |f(z_* + \rho e^{i\theta})| \\ & \leq |f(z_* + \rho e^{i\theta}) - f(z')| + |f(z')| \\ & \leq \varepsilon/2 + M - \varepsilon = M - \varepsilon/2 \end{aligned}$$

(14) and Thus $\leq \gamma \leq$ (use $(*)$)

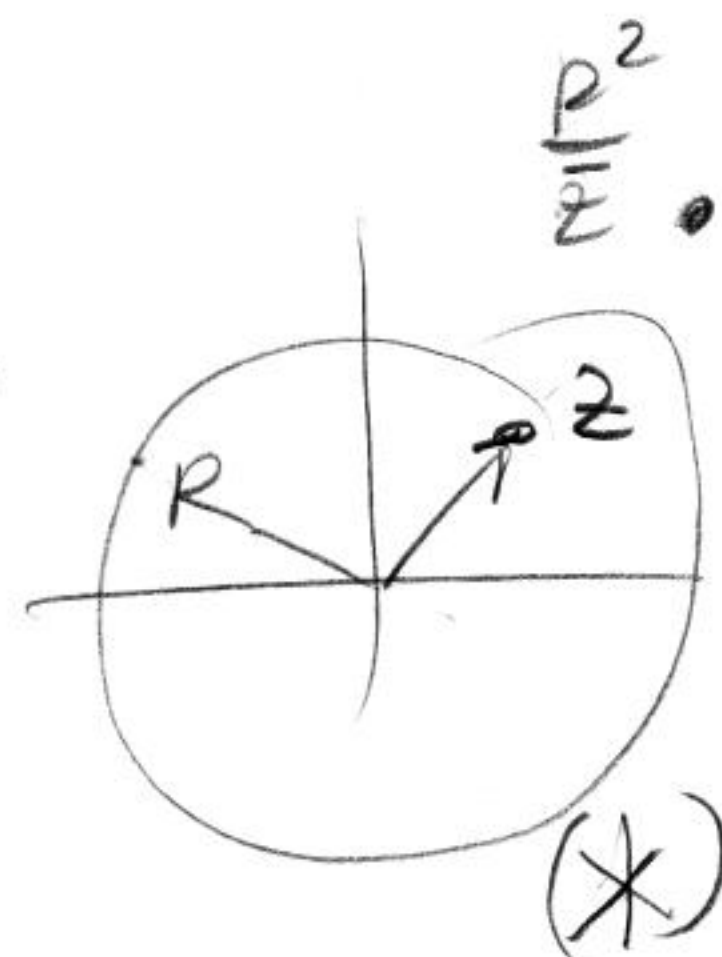
$$M \leq \frac{1}{2\pi} \left(M(2\pi - \Delta\theta) + \left(M - \frac{\varepsilon}{2}\right) \Delta\theta \right) \\ = M - \frac{\varepsilon}{4\pi} \Delta\theta.$$

Clearly, $(*)$ is impossible.

So we conclude $M = \max_{z \in \Gamma} |f(z)|$.

8) From Cauchy,

$(|z| < R)$ $f(z) = \frac{1}{2\pi i} \oint_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z} d\zeta$



But also

$$0 = \frac{1}{2\pi i} \oint_{|\zeta|=R} \frac{f(\zeta)}{\zeta - R^2/\bar{z}} d\zeta$$

because $R^2/\bar{z} \notin \{z : |z| \leq R\}$. (see $*$)

i.e. $\frac{f(\zeta)}{\zeta - R^2/\bar{z}}$ is analytic in $|\zeta| \leq R$.

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put these two together

$$(a) f(z) = \frac{1}{2\pi i} \oint_{|z|=R} \left(\frac{1}{z-z} - \frac{1}{z-R^2/\bar{z}} \right) f(z) dz$$

$$= \frac{1}{2\pi i} \oint_{|z|=R} \frac{(z - R^2/\bar{z})}{(z-z)(z-R^2/\bar{z})} f(z) dz$$

plug in $z = re^{i\theta}$, $r < R$

and parametrize $z = Re^{i\varphi}$, $-\pi < \varphi \leq \pi$

$$f(re^{i\theta}) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{(re^{i\theta} - \frac{R^2}{r}e^{-i\theta})}{(Re^{i\varphi} - re^{i\theta})(Re^{i\varphi} - \frac{R^2}{r}e^{-i\theta})} iRe^{i\varphi} f(Re^{i\varphi}) d\varphi$$

$$= \frac{1}{2\pi i} \int_{-\pi}^{\pi} \frac{(r)(r - \frac{R^2}{r})e^{i\theta}}{(Re^{i\varphi} - re^{i\theta})(Re^{i\varphi} - \frac{R^2}{r}e^{-i\theta})} iRe^{i\varphi} f(Re^{i\varphi}) d\varphi$$

= *

(over)

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$$x = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(r^2 - R^2) e^{i(\theta + \varphi)} (R f(Re^{i\varphi})) d\varphi}{(Re^{i\varphi} - re^{i\theta}) (rRe^{i\varphi} - R^2 e^{i\theta})}$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(r^2 - R^2) e^{i(\theta + \varphi)} f(Re^{i\varphi})}{(Re^{i\varphi} - re^{i\theta}) (re^{i\varphi} - Re^{i\theta})} d\varphi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(r^2 - R^2) f(Re^{i\varphi})}{(Re^{i\varphi} - re^{i\theta}) (re^{-i\theta} - Re^{-i\varphi})} d\varphi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(R^2 - r^2) f(Re^{i\varphi})}{(Re^{i\varphi} - re^{i\theta}) (Re^{-i\varphi} - re^{-i\theta})} d\varphi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(R^2 - r^2) f(Re^{i\varphi})}{R^2 + r^2 - (Rre^{i(\varphi - \theta)} + Rre^{-i(\varphi - \theta)})} d\varphi$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(R^2 - r^2) f(Re^{i\varphi})}{R^2 + r^2 - 2Rr \cos(\varphi - \theta)} d\varphi$$

= Poisson's Formula

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$$b) f(re^{i\theta}) = u(r, \theta) + i v(r, \theta)$$

$$\text{So } f(z)$$

$$u(r, \theta) + i v(r, \theta)$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(R^2 - r^2)}{R^2 + r^2 - 2Rr \cos(\varphi - \theta)} \left(u(R, \varphi) + i v(R, \varphi) \right) d\varphi$$

$$\Rightarrow (*) u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{\dots} u(R, \varphi) d\varphi$$

$$v(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{\dots} v(R, \varphi) d\varphi$$

But recall, $f(z)$ analytic ($f = u + iv$)

$$\Rightarrow \nabla^2 u = 0 \quad \& \quad \nabla^2 v = 0$$

Let $R = 1$ & $r < 1$ and use (*)

$$u(r, \theta) = \frac{1}{2\pi} \int \frac{1 - r^2}{1^2 + r^2 - 2r \cos(\varphi - \theta)} u(1, \varphi) \quad \text{solves}$$

$$\nabla^2 u = 0$$