Cauchy's Integral Formulae

Cauchy's integral formulae are the central pillars of complex analysis. Its basic form shows a differentiable complex function inside a closed curve is completely determined by its values on the curve itself.

THEOREM 1. (B-CIF) Suppose $f : S \to \mathbb{C}$ is continuously* differentiable on an open $S \subseteq \mathbb{C}$, and suppose a closed simple-curve Γ and its interior are in S. Then, for any z in Γ's interior, the basic Cauchy integral formula says

$$f(z) = \frac{1}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta,$$

where the curve Γ is traversed in the counterclockwise sense.

The basic formula is derived as follows. Observe that the function

$$h(\zeta) \equiv \frac{f(\zeta)}{\zeta - z} \quad \text{(w.r.t. variable } \zeta \text{ but fixed } z)$$

is continuously differentiable on the <u>punctured</u> domain $\{\zeta \in S : \zeta \neq z\}$. So, with the aid of Cauchy's theorem, we can deform the closed curve Γ to a circle centered at z

$$C_r \equiv \{\zeta = z + re^{i\theta} : -\pi < \theta \leq \pi\}$$

with small enough radius $r > 0$, to find

$$\oint_\Gamma h(\zeta)\, d\zeta = \oint_{C_r} h(\zeta)\, d\zeta = \int_{-\pi}^{\pi} h(z + re^{i\theta})\; i re^{i\theta}\, d\theta = \int_{-\pi}^{\pi} f(z + re^{i\theta})\, i\, d\theta.$$

(It's very important that you fully understand how I used Cauchy's theorem above to justify the step $\oint_\Gamma = \oint_{C_r}$. This so-called trick will be used many times this semester, and I'll be glad to devote as much time as necessary in lecture to explain.) The derivation of Cauchy's basic integral formula is completed by letting $r \to 0$,

$$\oint_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta = \lim_{r\to 0} \int_{-\pi}^{\pi} f(z + re^{i\theta})\, i\, d\theta = i \int_{-\pi}^{\pi} \lim_{r\to 0} \left(f(z + re^{i\theta})\right)\, d\theta = 2\pi i\, f(z).$$

Not only are the values of f at points strictly inside Γ completely determined by its values on the bounding curve Γ, but the point values of all its derivatives are as well. This fundamental fact can be seen by employing a simple induction argument. Assume the nth

* Use Cauchy-Goursat in the derivation to eliminate the 'continuously' assumption.

derivative of f at each point z inside the open region bounded by Γ exists and is given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

This is true from the basic theorem above when $n = 0$, i.e. $f^{(0)} = f$ and $0! = 1$. By the induction hypothesis, assume it's true for arbitrary $n \geq 0$. To see it's true for $n+1$, take z in the open interior bounded by Γ, take $z + \Delta z$ with $|\Delta z| > 0$ sufficiently small, and form the difference

$$f^{(n)}(z + \Delta z) - f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \left(\frac{1}{(\zeta - (z + \Delta z))^{n+1}} - \frac{1}{(\zeta - z)^{n+1}} \right) f(\zeta)\, d\zeta.$$

Combined the two terms inside the big bracket and use the binomial theorem to see

$$\left(\frac{1}{(\zeta - (z + \Delta z))^{n+1}} - \frac{1}{(\zeta - z)^{n+1}} \right) = \Delta z \left(\frac{(n+1)}{(\zeta - z)(\zeta - (z + \Delta z))^{n+1}} + O(\Delta z) \right),$$

where since $\zeta \in \Gamma$ the term $O(\Delta z) \to 0$ as $\Delta z \to 0$. Therefore,

$$f^{(n+1)}(z) = \lim_{\Delta z \to 0} \frac{f^{(n)}(z + \Delta z) - f^{(n)}(z)}{\Delta z} = \frac{n!}{2\pi i}(n+1) \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+2}} d\zeta,$$

which completes the induction argument.

Theorem 2. (G-CIF) Under the same assumptions stated in the previous theorem, we have the nth ($n = 0, 1, 2, \ldots$) derivative of f exists at any point z inside the open region bounded by Γ, and the derivative's value is given by

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta.$$

This is the general form of the Cauchy integral formula.

As Cauchy's formulae make evident, differentiable complex functions are very special. They are more than just once differentiable; they automatically have continuous derivatives of all orders. This statement cannot be made in general about differentiable real functions. For this reason, the following standard mathematical definitions, whose implications are widely known, are provided below

($*$) An *analytic function* is a complex function that is differentiable everywhere on a given domain.

($*$) An *entire function* is a complex function that is differentiable on the whole complex plane.

Traverse all closed contours in the counterclockwise sense.

1. Evaluate the following integrals on the unit circle $C = \{z : |z| = 1\}$.

(a) $\displaystyle\oint_C \frac{e^z}{z}\,dz.$ (b) $\displaystyle\oint_C \frac{z+2}{z^2}\,dz.$ (c) $\displaystyle\oint_C \frac{e^z}{z^2-2z}\,dz.$ (d) $\displaystyle\oint_C \frac{\sin(z)}{z^2}\,dz.$

2. Let $C_R(z_*) = \{z : |z - z_*| = R\}$ where $R > 0$. That is, the circle centered at z_* with positive radius R. Evaluate the following integrals.

(a) $\displaystyle\oint_{C_4(0)} \frac{z^2+5}{z-3}\,dz.$ (b) $\displaystyle\oint_{C_1(2)} \frac{1}{z^2-4}\,dz.$ (c) $\displaystyle\oint_{C_2(0)} \frac{\sin(z)}{(z-1)^3}\,dz.$

3. Let $C = \{z : |z| = 1\}$ be the unit circle centered at the origin.

(a) Use Cauchy's integral formula to show $\displaystyle\oint_C \left(z + \frac{1}{z}\right)\frac{dz}{z} = 0.$

(b) Use what you got in part (a) to conclude $\displaystyle\int_{-\pi}^{\pi} \cos(\theta)\,d\theta = 0.$

Hint for (b): Parameterize C using $\gamma(\theta) = e^{i\theta}$.

4. Show that for $n \in \mathbb{N}$

$$\int_{-\pi}^{\pi} \cos^{2n}(\theta)\,d\theta = 2\pi\,\frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}.$$

Hint: Consider $\int_{-\pi}^{\pi} \cos^{2n}(\theta)\,d\theta = \oint_C \left[\frac{1}{2}\left(z + \frac{1}{z}\right)\right]^{2n} \frac{dz}{iz}$ with $C = \{z : |z| = 1\}$, and use the binomial theorem together with Cauchy's general integral formula.

When needed, you are free to use the following complex triangle inequality.

$$\left|\int_{t_0}^{t_1} f(\gamma(t))\,\gamma'(t)\,dt\right| \le \int_{t_m}^{t_M} \left|\,f(\gamma(t))\,\gamma'(t)\,\right| dt,$$

where $t_m = \min(t_0, t_1)$, $t_M = \max(t_0, t_1)$.

5. Suppose $f(z)$ is a bounded entire function. (a) Show that $f'(z) = 0$ for every $z \in \mathbb{C}$. Hint: $f'(z) = \frac{1}{2\pi i}\oint_{C_R} f(\zeta)/(\zeta - z)^2\,d\zeta$ where C_R is given by $\{\zeta = z + Re^{i\theta} : -\pi < \theta \le \pi\}$. (b) Conclude $f(z)$ is constant.

6. Here, you'll prove the *fundamental theorem of algebra*. Consider a general degree n $(n \geq 1)$ polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$. (a) Show there is a fixed $R \geq 0$ such that $|p(z)| \geq 1$ for all $|z| >= R$. (b) Conclude there must be at least one $z_* \in \mathbb{C}$ such that $p(z_*) = 0$. Hint for (b): If not, then $f(z) \equiv 1/p(z)$ is bounded and entire.

7. Suppose $f(z)$ is analytic on an open set that contains a closed simple-curve Γ and its interior. Prove that $|f(z)|$ attains its maximum value on Γ. Hint: Suppose $|f(z)|$ takes a strict local maximum at a point z_* in Γ's interior. Use the basic Cauchy formula to show this is impossible.

8. Suppose f is analytic on an open set containing the disk $\{z : |z| \leq R\}$.

(a) Derive Poisson's integral formula.

$$\text{For } 0 \leq r < R: \quad f(re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2R\,r\cos(\theta - \phi) + r^2}\, f(Re^{i\phi})\, d\phi.$$

Hint: Explain why and then use $f(z) = \frac{1}{2\pi i} \oint_{C_R} \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - R^2/\bar{z}} \right) f(\zeta)\, d\zeta$ when $|z| < R$.

(b) Conclude that

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - \phi) + r^2}\, u(1, \phi)\, d\phi$$

solves Laplace's PDE, $\nabla^2 u = 0$, inside the $2d$ unit disk with boundary value $u(1, \theta)$.

I'll conclude by stating and finally completing the proof of Morera's theorem, often referred to as the converse of Cauchy's theorem.

THEOREM 3. (Morera) Let S be an open and connected set. Suppose $f : S \to \mathbb{C}$ is continuous, and for any closed simple-curve Γ in S we have $\oint_\Gamma f(z)\, dz = 0$. Then, f is analytic on S.

The antiderivative theorem, Theorem 3 in your "Complex Line Integrals" homework, says

$$F(z) = \int_{\Gamma_{z_0}^{z}} f(\zeta)\, d\zeta \;\text{ is a well-defined function of } z \qquad \Longrightarrow \qquad F'(z) = f(z).$$

Therefore F is continuously differentiable. Cauchy's general integral formula says F's second derivative must also exist on S. Therefore, f is differentiable on S which implies that f is analytic there.