①

Complex Functions
HW Solutions

1a) $\cosh(z) = \frac{1}{2}(e^{z}+e^{-z})$

$= \frac{1}{2}(e^{x+iy}+e^{-(x+iy)}) = \frac{1}{2}(e^{x}e^{iy}+e^{-x}e^{-iy})$

$= \frac{1}{2}(e^{x}(\cos y + i\sin y)) + e^{-x}(\cos(-y)+i\sin(-y)))$

$= \frac{1}{2}((e^{x}+e^{-x})\cos(y) + i(e^{x}-e^{-x})\sin(y))$

$= \boxed{\cosh(x)\cos(y) + i\sinh(x)\sin(y)}$

1b) $\sinh(z) = \frac{1}{2}(e^{z}-e^{-z})$

$= \frac{1}{2}(e^{x+iy} - e^{-(x+iy)}) = \frac{1}{2}(e^{x}e^{iy}-e^{-x}e^{-iy})$

$= \frac{1}{2}(e^{x}(\cos y + i\sin(y)) - e^{-x}(\cos(y) - i\sin y))$

$= \frac{1}{2}((e^{x}-e^{-x})\cos(y) + i(e^{x}+e^{-x})\sin(y))$

$= \boxed{\sinh(x)\cos(y) + i\cosh(x)\sin(y)}$

②

1c) $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$

$= \frac{1}{2}(e^{ix-y} + e^{-ix}e^{y})$

$= \frac{1}{2}(e^{-y}(\cos x + i\sin x) + e^{y}(\cos x - i\sin x))$

$= \frac{1}{2}((e^{-y} + e^{y})\cos x + i(e^{-y} - e^{y})\sin x)$

$= \boxed{\cosh(y)\cos x - i\sinh(y)\sin x}$

1d) $\sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$

$= \frac{1}{2i}((e^{-y} - e^{y})\cos x + i(e^{-y} + e^{y})\sin x)$

$= -i\frac{1}{2}(e^{-y} - e^{y})\cos x + \frac{1}{2}(e^{-y} + e^{y})\sin x$

$= \boxed{\cosh(y)\sin x + i\sinh(y)\cos x}$

2a) $z-(2-3i)=0$ when $z=2-3i$

So
$f(z)=\dfrac{1}{z-2+3i}$ is NOT defined for $\boxed{z=2-3i\equiv z_0}$

Otherwise

$$f(z)=\frac{1}{z-z_0}=\frac{\overline{z-z_0}}{(z-z_0)(\overline{z-z_0})}$$

$$=\frac{\overline{z-z_0}}{|z-z_0|^2}=\frac{\overline{(x+iy)-(2-3i)}}{(x-2)^2+(y+3)^2}$$

$$=\boxed{\frac{x-2}{(x-2)^2+(y+3)^2}+i\,\frac{y+3}{(x-2)^2+(y+3)^2}}$$

2b) $z^2+1=0$ when $z=\pm i$

So
$f(z)=\dfrac{iz^3+2z}{z^2+1}$ is NOT defined for $z=\pm i$
(i.e. $z^2+1=0$)

(4)

Otherwise

$$f(z) = \frac{iz^3+2z}{z^2+1} = \frac{z(iz^2+2)}{z^2+1} \quad \frac{\overline{z^2+1}}{\overline{z^2+1}}$$

$$= \frac{z(iz^2+2)(\bar{z}^2+1)}{|z^2+1|^2}$$

$$= \frac{z(i|z|^4 + iz^2 + 2\bar{z}^2 + 2)}{|z^2+1|^2} = *$$

Put $z = x+iy$

$$z^2 = x^2-y^2+2ixy$$

So

$$* = \frac{(x+iy)\left(i(x^2+y^2)^2 + i(x^2-y^2+2ixy) + 2(x^2-y^2-2ixy) + 2\right.}{(x^2-y^2+1)^2+(2xy)^2}$$

⑤ oh boy! The hard part

$$\text{Top} = (x+iy)\left(i\left((x^2+y^2)^2 + (x^2-y^2) - 4xy\right)\right.$$
$$\left. - 2xy + 2(x^2-y^2) + 2\right)$$

$$= (x+iy)\left(2\left((x^2-y^2) - xy + 1\right)\right.$$
$$\left. + i\left((x^2+y^2)^2 + (x^2-y^2) - 4xy\right)\right)$$

$$= 2x\left((x^2-y^2) - xy + 1\right)$$
$$- y\left((x^2+y^2)^2 + (x^2-y^2) - 4xy\right)$$
$$+ i\left[2y\left((x^2-y^2) - xy + 1\right)\right.$$
$$\left. + x\left((x^2+y^2)^2 + (x^2-y^2) - 4xy\right)\right]$$

put $\frac{\text{TOP}}{\text{bottom}}$ together on next page

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$\frac{\text{top}}{\text{bot}} = u + iv$ where

$$u(x,y) = \frac{2x\left((x^2-y^2)-xy+1\right) - y\left((x^2+y^2)^2+x^2-y^2-4xy\right)}{(x^2-y^2+1)^2+(2xy)^2}$$

$$v(x,y) = \frac{2y\left((x^2-y^2)-xy+1\right) + x\left((x^2+y^2)^2+x^2-y^2-4xy\right)}{(x^2+y^2+1)^2+(2xy)^2}$$

ugh!

2c) $f(z) = \frac{3z-1}{z^2+z+4}$

$z^2+z+4=0$ when $z = \frac{-1 \pm \sqrt{1-16}}{2}$

$= -\frac{1}{2} \pm \frac{\sqrt{15}}{2} i$

Not defined when

$z = z_0 = -\frac{1}{2} + \frac{\sqrt{15}}{2} i$

or $z_1 = -\frac{1}{2} - \frac{\sqrt{15}}{2} i$

$\neq z_0$

z_1

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otherwise

$$f(z) = \frac{3z-1}{z^2+z+4} = \frac{3z-1}{(z-z_0)(z-z_1)} \cdot \frac{\overline{(z-z_0)(z-z_1)}}{\overline{(z-z_0)(z-z_1)}}$$

$$= \frac{(3z-1)\,\overline{(z-z_0)}\,\overline{(z-z_1)}}{|z-z_0|^2\,|z-z_1|^2}$$

$$\begin{cases} z_1 = x_0 + iy_0 \\ z_1 = x_1 + iy_1 \end{cases}$$

But $= |x - x_0 + i(y-y_0)|^2$

$\times\ |x - x_1 + i(y-y_1)|^2$

$$= \left((x-x_0)^2 + (y-y_0)^2\right)\left((x-x_1)^2 + (y-y_1)^2\right)$$

Top $= (3z-1)(\bar{z}^2 + \bar{z} + 4)$

$= 3\bar{z}|z|^2 + 3|z|^2 + 12z$

$\quad - \bar{z}^2 - \bar{z} - 4$

Let $z = x+iy$

Top $= 3(x-iy)(x^2+y^2) + 3(x^2+y^2) + 12(x+iy)$

$\quad - (x^2-y^2-2ixy) - (x-iy) - 4$ = error

⑧

$$= 3x(x^2+y^2) + 3(x^2+y^2) + 12x$$
$$- (x^2-y^2) - x - y$$
$$+ i\left[-3y(x^2+y^2) + 12y + 2xy + y\right]$$

So

$$\frac{\text{Top}}{\text{bot}} = u + iv$$

$$\boxed{\begin{aligned} u &= \frac{3x(x^2+y^2) + 3(x^2+y^2) + 12x - (x^2-y^2) - x - y}{\left((x-x_0)^2 + (y-y_0)^2\right)\left((x-x_1)^2 + (y-y_1)^2\right)} \\ v &= \frac{-3y(x^2+y^2) + 12y + 2xy + y}{\left((x-x_0)^2 + (y-y_0)^2\right)\left((x-x_1)^2 + (y-y_1)^2\right)} \end{aligned}}$$

where $x_0 = -\frac{1}{2}$ $\quad y_0 = -\frac{\sqrt{15}}{2}$

$x_1 = -\frac{1}{2}$ $\quad y_1 = +\frac{\sqrt{15}}{2}$

⑨

2a) $f(z) = \dfrac{z^2}{(2z^2 - 3z + 1)^2}$

$2z^2 - 3z + 1 = 0$ when $z = \dfrac{3 \pm \sqrt{9 - 4 \cdot 2}}{2 \cdot 2}$

Not defined where $z = 1$ or $z = \frac{1}{2}$

$z = \dfrac{3 \pm 1}{4} = \begin{cases} 1 \\ \text{or} \\ \frac{1}{2} \end{cases}$

$z_0 = 1 \quad z_1 = 1/2$

Otherwise

$$f(z) = \frac{z^2}{(2(z - z_0)(z - z_1))^2}$$

$$= \frac{z^2}{(2(z - z_0)(z - z_1))^2} \quad \frac{\overline{(z - z_0)}^2 \, \overline{(z - z_1)}^2}{\overline{(z - z_0)}^2 \, \overline{(z - z_1)}^2}$$

$= \dfrac{TOP}{BOT}$ where

(Recall $z_0 = 1 \quad z_1 = \frac{1}{2}$

But $|z - z_0|^2 |z - z_1|^2$

$= ((x - 1)^2 + y^2)((x - \frac{1}{2})^2 + y^2)$

(over)

$$\text{Top} = z^2\,\overline{\left(z^2 - \tfrac{3}{2}z + \tfrac{1}{2}\right)}$$

$$= |z|^4 - \tfrac{3}{2}z^2\bar{z} + \tfrac{1}{2}z^2$$

$$= |z|^4 - \tfrac{3}{2}z^2\bar{z} + \tfrac{1}{2}z^2$$

$$= (x^2+y^2)^2 - \tfrac{3}{2}(x+iy)(x^2+y^2) + \tfrac{1}{2}\left((x^2-y^2)+2ixy\right)$$

$$= (x^2+y^2)^2 - \tfrac{3}{2}x(x^2+y^2) + \tfrac{1}{2}(x^2-y^2)$$

$$+ i\left[-\tfrac{3}{2}y(x^2+y^2) + xy\right]$$

So

$$u = \frac{(x^2+y^2)^2 - \frac{3}{2}x(x^2+y^2) + \frac{1}{2}(x^2-y^2)}{\left((x-1)^2+y^2\right)\left((x-\frac{1}{2})^2+y^2\right)}$$

$$v = \frac{-3/2\,y(x^2+y^2) + xy}{\left((x-1)^2+y^2\right)\left((x-1/2)^2+y^2\right)}$$

2e) $f(z) = \dfrac{\sin(z)}{z-\pi}$

Not defined when $z = \pi$

Otherwise

$$f(z) = \frac{\sin(z)}{z-\pi} = \frac{\sin(z)}{z-\pi} \frac{\overline{z-\pi}}{\overline{z-\pi}} = \frac{TOP}{BOT}$$

$z = x+iy$

$$BOT = |z-\pi|^2 = (x-\pi)^2 + y^2$$

$$Top = \frac{e^{iz} - e^{-iz}}{2i} (\overline{z-\pi})$$ use Id

$$= \left(\cosh(y)\sin x + i \sinh(y)\cos(x)\right)\left((x-\pi) - iy\right)$$

$$= (x-\pi)(\cosh(y)\sin x) + y(\sinh(y)\cos x)$$

$$+ i\left[-y\cosh(y)\sin x + (x-\pi)\sinh(y)\cos x\right]$$

(over)

So

$$\boxed{u = \frac{(x-\pi)\cosh y \sin x + y \sinh y \cos x}{(x-\pi)^2 + y^2} \qquad v = \frac{-y \cosh y \sin x + (x-\pi)\sinh y \cos x}{(x-\pi)^2 + y^2}}$$

2f) $f(z) = \dfrac{1-\cos(z)}{z}$ Not defined when $z=0$

Otherwise

$$f(z) = \frac{1-\cos(z)}{z}\,\frac{\bar{z}}{\bar{z}} = \text{TOP}/\text{BOT}$$

use 1c

$$\text{Top} = \left(1-(\cos x \cosh y - i \sin x \sinh y)\right)(x-iy)$$

$$= (1-\cos x \cosh y)\,x + \sin x \sinh y\, y$$

$$+ i\left[-(1-\cos x \cosh y)\,y + \sin x \sinh y\, x\right]$$

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So

$$\boxed{u = \frac{(1-\cos x\cosh y)x + (\sin x\sinh y)y}{x^2+y^2} \qquad v = \frac{-(1-\cos x\cosh y)y + (\sin x\sinh y)x}{x^2+y^2}}$$

3a) $|z|^2 = x^2+y^2$ clearly ≥ 0

Suppose $x^2+y^2=0 \Rightarrow x=0$ & $y=0$

3b) $|z_1 z_2|^2 = z_1 z_2 \overline{z_1 z_2}$

$= z_1\overline{z_1}\, z_2\overline{z_2} = |z_1|^2|z_2|^2$

So $|z_1 z_2| = |z_1||z_2|$

3c) $|z_1+z_2|^2 = (z_1+z_2)(\overline{z_1+z_2})$

$= |z_1|^2 + z_1\overline{z_2} + z_2\overline{z_1} + |z_2|^2$

$= |z_1|^2 + 2\,\mathrm{Re}(z_1\overline{z_2}) + |z_2|^2$

(over)

But given on page 2

$$\mathrm{Re}(z_1\bar{z}_2) \le |z_1\bar{z}_2| = |z_1||z_2|$$

So

$$|z_1+z_2|^2 \le |z_1|^2 + 2|z_1||z_2| + |z_2|^2$$
$$= \left(|z_1|+|z_2|\right)^2$$

$$\Rightarrow |z_1+z_2| \le |z_1|+|z_2|$$

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4) Suppose $f(z)$ continuous at $z_0 = x_0 + iy_0$

So $\forall \varepsilon > 0 \ \exists \delta > 0$ such that

$|f(z) - f(z_0)| < \varepsilon$ whenever $|z - z_0| < \delta$

But This says (From page 2)

$|u(x,y) - u(x_0, y_0)| < \varepsilon$ whenever

and $\sqrt{(x-x_0)^2+(y-y_0)^2} < \delta$

$|v(x,y) - v(x_0, y_0)| < \varepsilon$

$\Rightarrow$ Both $u(x,y)$ & $v(x,y)$ are continuous at $(x,y) = (x_0, y_0)$.

(over)

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Now suppose both $u(x,y)$ and $v(x,y)$ are continuous at $(x,y)=(x_0,y_0)$ and let $\varepsilon>0$. Then

① $\exists\, \delta_u>0$ such that

$$|u(x,y)-u(x_0,y_0)|<\frac{\varepsilon}{\sqrt{2}} \text{ whenever } \sqrt{(x-x_0)^2+(y-y_0)^2}<\delta_u$$

② and $\exists\, \delta_v>0$ such that

$$|v(x,y)-v(x_0,y_0)|<\frac{\varepsilon}{\sqrt{2}} \text{ whenever } \sqrt{(x-x_0)^2+(y-y_0)^2}<\delta_v$$

So for $|z-z_0|<\min(\delta_u,\delta_v)\equiv\delta$

we have

$$\begin{aligned}|f(z)-f(z_0)|^2 &= |u(x,y)-u(x_0,y_0)|^2 \\ &\quad + |v(x,y)-v(x_0,y_0)|^2 \\ &< \frac{\varepsilon^2}{2}+\frac{\varepsilon^2}{2}=\varepsilon^2\end{aligned}$$

$\Rightarrow$ $|f(z)-f(z_0)|<\varepsilon$ whenever $|z-z_0|<\delta$

so $f(z)$ is continuous at z_0.

5a) From Calc I we know $r(x) \equiv e^x$ is continuous at any $x_0 \in \mathbb{R}$, so $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$|x - x_0| < \delta \Rightarrow |e^x - e^{x_0}| < \varepsilon$. This says

Since $|x - x_0| \leq |z - z_0|$

$|z - z_0| < \delta \Rightarrow |x - x_0| < \delta$

$\Rightarrow |e^{Re(z)} - e^{Re(z_0)}| < \varepsilon$

$\therefore \boxed{f_1(z)} \equiv e^{Re(z)}$ is continuous at any $z_0 \in \mathbb{C}$

Also $r(y) \equiv \cos(y)$ is continuous at any $y_0 \in \mathbb{R}$. So $\forall \varepsilon > 0\ \exists \delta > 0$ such that

$|y - y_0| < \delta \Rightarrow |\cos(y) - \cos(y_0)| < \varepsilon$

This says

Since $|y - y_0| \leq |z - z_0|$

$|z - z_0| < \delta \Rightarrow |y - y_0| < \delta$

$\Rightarrow |\cos(Im(z)) - \cos(Im(z_0))| < \varepsilon$

$\therefore \boxed{f_2(z)} \equiv \cos(Im(z))$ is continuous at any $z_0 \in \mathbb{C}$.

(over)

Also $r(y) = \sin(y)$ is continuous at any $y_0 \in \mathbb{R}$. So for any $\varepsilon > 0$ $\exists \delta > 0$ such that $|y - y_0| < \delta \Rightarrow |\sin(y) - \sin(y_0)| < \varepsilon$

So $|z - z_0| < \delta \Rightarrow |y - y_0| < \delta$

$$\Rightarrow |\sin(Im(z)) - \sin(Im(z_0))| < \varepsilon$$

$\therefore$ $\boxed{f_3(z)} \equiv \sin(Im(z))$ is continuous at any $z_0 \in \mathbb{C}$.

Finally $e^z = e^{Re(z)} \cos(Im(z)) + i e^{Re(z)} \sin(Im(z))$

$$= f_1(z) f_2(z) + i f_1(z) f_3(z)$$

by Thrm 1 (i) & (ii) and what is shown above

Says e^z is continuous at any $z_0 \in \mathbb{C}$.

5b) $e^{iz} = e^{ix - y}$

$$= e^{-y} (\cos(x) + i \sin(x))$$

Just as above

$f_1(z) = e^{-Im(z)}$ is continuous

at any $z_0 \in \mathbb{C}$

$f_2(z) = \cos(Re(z))$ is continuous

at any $z_0 \in \mathbb{C}$

$f_3(z) = \sin(Re(z))$ is continuous

at any $z_0 \in \mathbb{C}$

So $e^{iz} = f_1(z)\, f_2(z) + i\, f_1(z) f_3(z)$

by Thrm 1 and above says

e^{iz} is continuous at any $z_0 \in \mathbb{C}$.

5c) $\cosh(z) = \frac{1}{2}(e^{z} + e^{-z})$

But already shown in (a) that

e^{z} is continuous.

$e^{-z} = e^{-x}(\cos(y) - i\sin(y))$

and clearly e^z is <u>never</u> 0.

So $e^{-z} = \frac{1}{e^z}$ is continuous by

Theorem 1 (iii) at any $z_0 \in \mathbb{C}$.

So by Thm 1 (i) & (ii)

$\cosh(z) = \frac{1}{2}(e^z + e^{-z})$ is continuous

at any $z_0 \in \mathbb{C}$.

5d) $\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz})$

By (b) e^{iz} is continuous at any $z_0 \in \mathbb{C}$,

$e^{-iz} = e^{-y}(\cos x + i \sin x)$ is

<u>never</u> 0, i.e. $\cos(x)$ and $\sin(x)$ are never simultaneously zero.

$\therefore$ Thm 1 (iii) says e^{-iz} is continuous

at every $z_0 \in \mathbb{C}$.

So Thm 1 (i) & (ii) says $\cos(z)$ is continuous at every $z_0 \in \mathbb{C}$.

5e) $f(z) = z\cosh(z) + \cos(z)$

Obviously $f_1(z) = z$ is continuous since

$$|f_1(z) - f_1(z_0)| = |z - z_0| \quad (\text{i.e. } \delta = \varepsilon)$$

So by (c) and (d) and Thm 1 (i), (ii)

$z\cosh(z) + \cos(z)$ is continuous at any $z_0 \in \mathbb{C}$.

5f) $f(z) = e^z \cos(z)$

continuous by (a) at all $z_0 \in \mathbb{C}$ — continuous by (d) at all $z_0 \in \mathbb{C}$.

So Thm 1 (ii) says this function is continuous at all $z_0 \in \mathbb{C}$.

6a) $f(z) = 3z^3 + 9z$

Thrm 2 power rule

① $\frac{d}{dz} z^3 = 3z^2$; ② $\frac{d}{dz} z = 1$

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prod rule $\frac{d}{dz}(3z^3) = 3\frac{dz^3}{dz}$ $\frac{d}{dz}(9z) = 9$

$$\frac{d}{dz}(3z^3+9z)$$

$$= \frac{d}{dz}(3z^3) + \frac{d}{dz}(9z)$$ Sum rule

So yes $\boxed{\text{diff'b everywhere}}$ and

$$\boxed{\frac{d}{dz}(3z^3+9z) = 3 \cdot 3z^2 + 9 = 9z^2+9}$$

6b) $f(z) = (z+2)^{15}$

$$\frac{d}{dz}(z+2) = \frac{dz}{dz} + \frac{d2}{dz} = 1$$ Sum and power rule

$$\frac{d}{dw}w^{15} = 15w^{14}$$ power rule

So yes $\boxed{\text{diff'bl everywhere}}$ and

$$\boxed{\frac{d}{dz}(z+2)^{15} = 15(z+2)^{14}}$$

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6 c) $f(z) = \dfrac{9z+2}{z^2+1} \equiv \dfrac{t(z)}{b(z)}$

$b(z) \neq 0$ $\forall z \neq \pm i$ and top and bottom are diff'bl

$\dfrac{dt(z)}{dz} = \dfrac{d}{dz}(9z+2) = \dfrac{d}{dz}9z + \dfrac{d}{dz}2$ Sum prod and power

$= 9$

$\dfrac{db(z)}{dz} = \dfrac{d}{dz}(z^2+1) = \dfrac{d}{dz}z^2 + \dfrac{d}{dz}1 = 2z$ Sum and power rule

So f is diff'b everywhere and

$\dfrac{d}{dz}\left(\dfrac{t(z)}{b(z)}\right) = t(z)\dfrac{d}{dz}\left(b(z)^{-1}\right) + \dfrac{dt(z)}{dz}(b(z))^{-1}$ prod rule

$= t(z)\left(-(b(z))^{-2}\dfrac{db(z)}{dz}\right) = \dfrac{dt(z)}{dz}\dfrac{1}{b(z)}$

Chain & Power rule

$= (9z+2)\left(-\dfrac{2z}{(z^2+1)^2}\right) + 9\dfrac{1}{(z^2+1)}$ (over).

(23)

So $\dfrac{9z+2}{z^2+1}$ diff bl $\forall z \neq \pm i$.

and

$$\boxed{\frac{d}{dz}\left(\frac{9z+2}{z^2+1}\right) = \frac{(z^2+1)9-(9z+2)2z}{(z^2+1)^2}}$$

6d) $f(z) = z + \dfrac{1}{z}$ $\boxed{\text{NOT defined at } z=0 \text{ and } \therefore \text{ NOT diff bl There}}$

power rule $\dfrac{dz}{dz}=1$, $\dfrac{d}{dz}\dfrac{1}{z} = -\dfrac{1}{z^2}$ except at $z=0$

So $\boxed{\dfrac{d}{dz}\left(z+\dfrac{1}{z}\right) = 1-\dfrac{1}{z^2}}$ except NOT at $z=0$

7a) $f(z) = e^{2z}$

$\dfrac{d}{dw}e^w = e^w$ in notes. for $\forall w \in \mathbb{C}$.

$\dfrac{d\, e^{2z}}{dz} = e^{2z}\dfrac{d\,2z}{dz}$ Chain rule

$= 2e^{2z}$

7b) $f(z) = e^{z} \sin(z)$

$$\sin(z) = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$$

both diff for all z.

$$\frac{d}{dz} e^{iz} = \frac{d}{dw} e^{w} \frac{dw}{dz} \qquad w = iz \qquad \text{Chain rule}$$

$$= e^{iz} i$$

$$\frac{d}{dz} e^{-iz} = \frac{d}{dw} e^{w} \frac{dw}{dz} \qquad w = -iz \qquad \text{Chain rule}$$

$$= e^{-iz}(-i)$$

So

$$\frac{d}{dz} \sin(z) = \frac{1}{2i}\left(ie^{iz} - \left(-ie^{-iz}\right)\right) \qquad \text{Sum and prod rule}$$

$$= \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

$$\boxed{\frac{d}{dz}\left(e^{z} \sin(z)\right) = e^{z} \cos(z) + e^{z} \sin(z)}$$

Prod rule

(24) — woops. 2 page 24s.

7c) $f(z) = \cosh z = \frac{1}{2}(e^{z} + e^{-z})$ for all $z \in \mathbb{C}$

$\frac{d}{dz} e^{z} = e^{z}$ and $\frac{d}{dz} e^{-z} = -e^{-z}$

Notes — Chain rule & notes

So

$\frac{d}{dz} \frac{1}{2}(e^{z} + e^{-z}) = \frac{1}{2}(e^{z} - e^{-z})$ Sum, prod rule and above

so $\boxed{\frac{d}{dz} \cosh(z) = \sinh(z)}$

All these work just like you learned in Calc I

7d) $\frac{d}{dz} \sin^2(z)$

$\frac{d}{dz} \sin(z) = \cos(z)$ for every z

done in part b

$\frac{d}{dz}(\sin^2(z)) = \frac{d}{dw} w^2 \frac{dw}{dz}$ $w = \sin(z)$

$= 2w \frac{dw}{dz} = 2\sin(z)\cos(z)$

power rule

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8a) $f(z) = \frac{\sin z}{z} = \sin(z) \frac{1}{z}$

$\frac{d}{dz} \sin(z) = \cos z$ Just like 7b

$\boxed{\forall z \neq 0}$

power rule

$\frac{d}{dz} \frac{1}{z} = \frac{d}{dz} z^{-1} = -z^{-2} \frac{dz}{dz} = -\frac{1}{z^2}$

So $\forall z \neq 0$

Prod rule

$\frac{d}{dz} \frac{\sin(z)}{z} = \sin(z)\left(-\frac{1}{z^2}\right) + \cos(z) \frac{1}{z}$

$= \frac{z \cos z - \sin(z)}{z^2}$

8b) $f(z) = \frac{e^z}{z^2 - 2z + 1} = \frac{e^z}{(z-1)^2}$ Not defined or diffbl at $z = 1$

otherwise

$\frac{d}{dz}(z-1) = \frac{d}{dz} z - \frac{d}{dz} 1 = 1 - 0$ Sum rule and power rule

$\frac{d}{dz}(w)^{-2} = -2w^{-3}$ power rule

as long as $w \neq 0$

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$\frac{d}{dz}(z-1)^{-2} = \frac{-2}{(z-1)^3}$ ok as long as $z \neq 1$.

$\frac{d}{dz}\left(e^z (z-1)^{-2}\right)$ $\qquad$ $\frac{de^z}{dz} = e^z$ Notes

$= e^z \frac{d}{dz}(z-1)^{-2} + e^z (z-1)^{-2}$ prod. rule

$= e^z\left(-2(z-1)^{-3}\right) + e^z (z-1)^{-2}$

From above

$e^z \frac{(z-1)^2}{(z-1)^4} - \frac{2(z-1)}{(z-1)^4} e^z$

$= \boxed{\frac{e^z (z-1)^2 - 2(z-1)e^z}{(z-1)^4}}$

8c) $f(z) = \frac{1}{\sin(z)}$

$\sin(z) = 0 \iff e^{-y}(\cos(x) + i\sin(x))$

$= e^{y}(\cos(x) - i\sin(x))$

(Over)

$\Leftrightarrow \quad e^{-y}\cos x = e^{y}\cos x$

and

$e^{-y}\sin x = -e^{y}\sin x$

Do in cases

$\sin x \neq 0 \Rightarrow e^{-y} = -e^{y} \Rightarrow$ Never

$\sin x = 0 \Rightarrow \cos x \neq 0 \Rightarrow e^{-y} = e^{y} \Rightarrow y = 0$

$\Rightarrow x = n\pi, n \in \mathbb{Z}$

So $\sin(z) = 0 \Leftrightarrow z = n\pi, \; n \in \mathbb{Z}$

By The power rule, chain rule & before

$$\frac{d}{dz}(\sin(z))^{-1} = -\sin(z)^{-2}\frac{d}{dz}\sin(z)$$

$$= -\frac{\cos(z)}{\sin^2(z)}$$

For $z \neq n\pi$ with $n \in \mathbb{Z}$.

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8 d) $f(z) = \dfrac{\sin(z)}{\cos(z)}$

$$\cos(z) = 0 \iff e^{-y}(\cos x + i\sin(x)) = -e^{y}(\cos x - i\sin(x))$$

$$\iff e^{-y}\cos(x) = -e^{y}\cos(x)$$

and $e^{-y}\sin(x) = +e^{y}\sin(x)$

Cases

Case I: $\cos(x) \neq 0 \Rightarrow e^{-y} = -e^{y}$ Never

Case II: $\cos(x) = 0 \Rightarrow \sin(x) \neq 0 \Rightarrow e^{-y} = e^{y}$

$\Rightarrow y = 0$

$\cos(x) = 0$

$\Rightarrow x = \frac{\pi}{2} + n\pi \quad n \in \mathbb{Z}$

By product, power & chain rules

for $z \neq \frac{\pi}{2} + n\pi$ with $n \in \mathbb{Z}$

$$\frac{d}{dz}\frac{\sin(z)}{\cos(z)} = \sin(z)\left(-(\cos(z))^{-2}\right)\frac{d}{dz}\cos z + \frac{d}{dz}\sin(z)\,\frac{1}{\cos(z)}$$

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Part also (haven't done this yet)

$$\frac{d}{dz}\cos(z) = \frac{d}{dz}\left(\frac{e^{iz}+e^{-iz}}{2}\right)$$ use sum rule and stuff from prob 7

$$= \frac{ie^{iz}}{2} - \frac{ie^{-iz}}{2} = \frac{-1}{2i}\left(e^{iz}-e^{-iz}\right)$$

$$= -\sin(z)$$

So for $z \neq \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$

$$\frac{d}{dz}\frac{\sin z}{\cos z}$$

$$= \frac{\sin(z)\sin(z)}{\cos^2(z)} + \frac{\cos^2(z)}{\cos^2(z)}$$

$$= \frac{\sin^2(z)+\cos^2(z)}{\cos^2(z)} \overset{?}{=} \boxed{\frac{1}{\cos^2(z)}}$$

But is $\sin^2(z)+\cos^2(z) = 1$???

Let's check

$$\left(\frac{e^{iz}-e^{-iz}}{2i}\right)^2 + \left(\frac{e^{iz}+e^{-iz}}{2}\right)^2$$

$$= -\frac{1}{2}\left(\frac{e^{2iz}-2+e^{-2iz}}{4}\right) + \frac{1}{2}\left(\frac{e^{2iz}+2+e^{-2iz}}{4}\right) = \frac{4}{4} = 1$$ ✓

9 Recall C-R $u_x = v_y$
$v_x = -u_y$

9a) $(u_x)^2 + (v_x)^2$
$= (v_y)^2 + (-u_y)^2 = (u_y)^2 + (v_y)^2$ ✓

9b) $u_x u_y + v_x v_y$
$= v_y(-v_x) + v_x v_y$
$= -v_x v_y + v_x v_y = 0$ ✓

9c) Assume these are twice partial der and mixed partials are equal

$u_{xx} + u_{yy}$
$= v_{yx} - v_{xy}$
$= 0$

$u_x = v_y$ $\quad u_y = -v_x$
$u_{xx} = v_{yx}$ $\quad u_{yy} = -v_{xy}$

9d) $v_{xx} + v_{yy}$
$= -u_{yx} + u_{xy}$
$= 0$

$v_x = -u_y$ $\quad v_y = u_x$
$v_{xx} = -u_{yx}$ $\quad v_{yy} = u_{xy}$

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10 a) $f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$

(*) $\frac{\partial}{\partial r} f(re^{i\theta}) = f'(z)e^{i\theta} = u_r + i v_r$

$\frac{\partial}{\partial \theta} f(re^{i\theta}) = f'(z)\,ire^{i\theta} = u_\theta + i v_\theta$

$= (f'(z)e^{i\theta})\,ir = u_\theta + i v_\theta$

$r \neq 0 \quad (u_r + i v_r) = \frac{1}{ir}(u_\theta + i v_\theta)$

equate real and imag parts $\Rightarrow$ $\boxed{u_r = \frac{1}{r} v_\theta \qquad v_r = -\frac{1}{r} u_\theta}$

10 b) $f(re^{i\theta}) \equiv \log(r) + i\theta$

Does $\frac{\partial}{\partial r}\log(r) \overset{?}{=} \frac{1}{r}\frac{\partial}{\partial \theta}\theta$

$\frac{1}{r} = \frac{1}{r}$ ✓

$\frac{\partial}{\partial r}\theta \overset{?}{=} -\frac{1}{r}\frac{\partial}{\partial \theta}\log(r), \quad 0 = 0$ ✓

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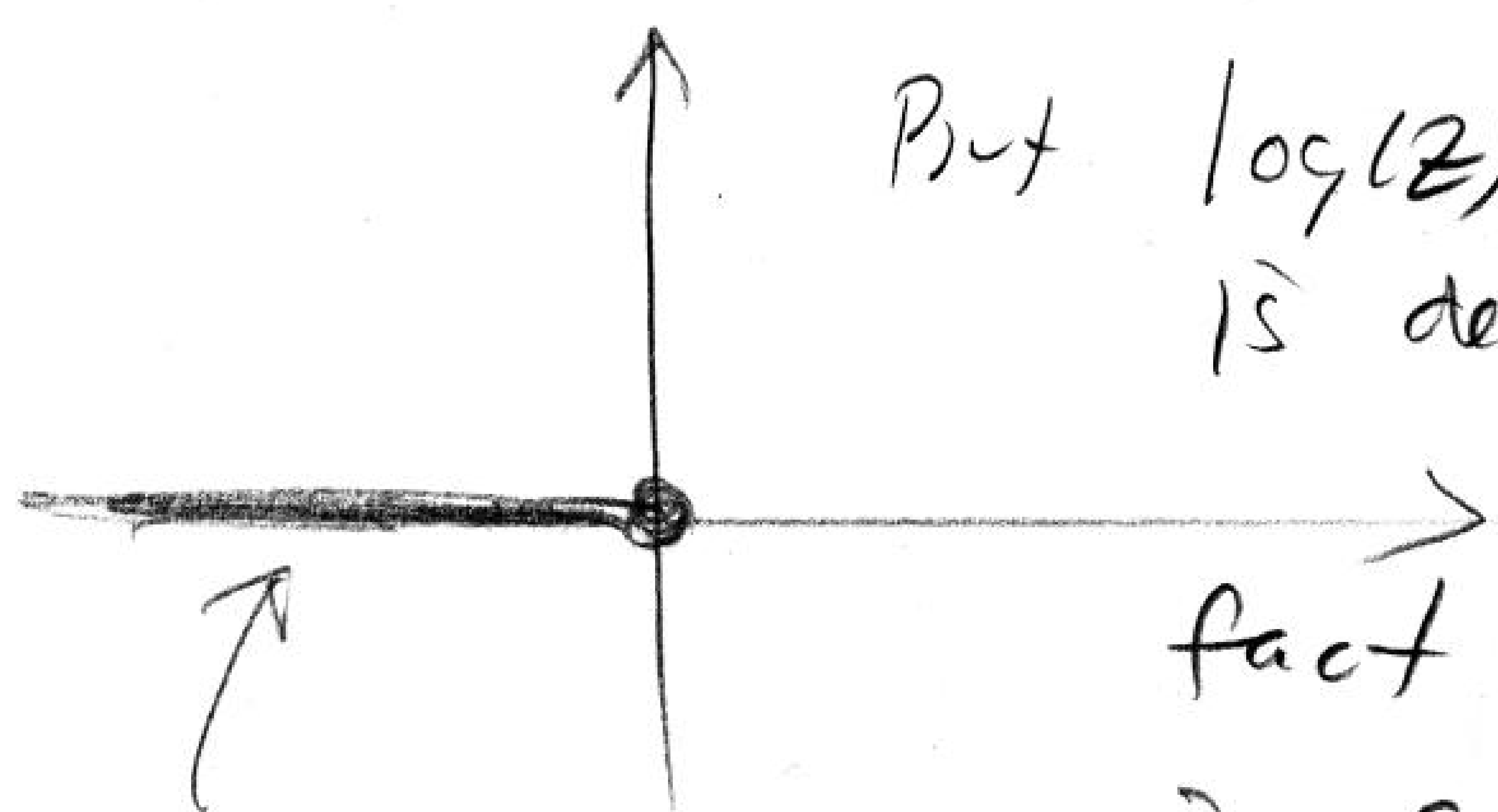
But $\log(r) + i\theta$

is NOT continuous at $r = 0$

or along ray $\theta = \pi, -\pi$



But $\log(z) = \log(r) + i\theta$
is defined and in
fact diff'bl for all
z except along
non positive real axis

From (*) on previous page

$$f'(z) = e^{-i\theta}(u_r + i v_r)$$
$$= e^{-i\theta}\left(\frac{1}{r} + i\,0\right)$$
$$= \frac{1}{re^{i\theta}} = \frac{1}{z} = \frac{d}{dz}\log(z)$$

As expected.