Complex Functions of a Complex Variable

Let $D \subseteq \mathbb{C}$. A complex function of a complex variable, denoted by

$$f : D \to \mathbb{C},$$

is a mapping that assigns to each $z \in D$ a unique $w \in \mathbb{C}$, where $w \equiv f(z)$. The set D is called the *domain* of f. The *range* of f is the set $\{w = f(z) : z \in D\} \subseteq \mathbb{C}$. In standard form we write

$$f(z) = f(x + iy) = u(x, y) + i\, v(x, y),$$

where $u(x, y) \in \mathbb{R}$ and $v(x, y) \in \mathbb{R}$ are called the real and imaginary parts of $f(z)$. Throughout these notes, I will refer to complex functions of a complex variable simply as *complex functions*.

You'll want to visualize complex functions as Calculus III style functions which take $\mathbb{R}^2$ into $\mathbb{R}^2$. For example

$$f(z) \equiv z^2 + 1 = (x + iy)^2 + 1 = (x^2 - y^2 + 1) + i\, 2xy,$$
$$f(z) \equiv e^z = e^{x+iy} = e^x e^{iy} = e^x \cos(y) + i\, e^x \sin(y).$$

Both of these example functions are well defined for any $z \in \mathbb{C}$ and can therefore be regarded as $f : \mathbb{C} \to \mathbb{C}$. On the other hand,

$$f(z) \equiv \frac{1}{z} = \frac{\bar{z}}{z\bar{z}} = \frac{x}{x^2 + y^2} + i\, \frac{-y}{x^2 + y^2}$$

as written, is not defined when $z = 0$. So as written this defines $f : \mathbb{C} \setminus \{0\} \to \mathbb{C}$.

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1. Consider the following complex functions. Write each as $f(z) = u(x, y) + iv(x, y)$.

(a) $f(z) = \cosh(z)$. (c) $f(z) = \cos(z)$.

(b) $f(z) = \sinh(z)$. (d) $f(z) = \sin(z)$.

Selected answers: (a) $\cosh(x + iy) = \cosh(x)\cos(y) + i \sinh(x)\sin(y)$,
(c) $\cos(x + iy) = \cos(x)\cosh(y) - i \sin(x)\sinh(y)$.

2. Consider the following complex functions. Write each as $f(z) = u(x, y) + iv(x, y)$ and determine where, as written, they are not defined.

(a) $f(z) = \dfrac{1}{z - 2 + 3i}$. (c) $f(z) = \dfrac{3z - 1}{z^2 + z + 4}$. (e) $f(z) = \dfrac{\sin(z)}{z - \pi}$.

(b) $f(z) = \dfrac{iz^3 + 2z}{z^2 + 1}$. (d) $f(z) = z^2(2z^2 - 3z + 1)^{-2}$. (f) $f(z) = \dfrac{1 - \cos(z)}{z}$.

Above and throughout, the notation $|z|$ surrounding a complex number z denotes its complex modulus. Note that the modulus satisfies the following.

$$\begin{aligned} &|z| \geq 0 \quad \text{and} \quad |z| = 0 \iff z = 0, \\ &|z_1 z_2| = |z_1|\,|z_2| \quad \text{and} \quad |z_1 + z_2| \leq |z_1| + |z_2|. \end{aligned}$$

You will be asked to show these facts are true. The notation $|x|$ surrounding a real number x denotes its absolute value. The following are easy to see and may prove useful.

$$\mathrm{Re}(z) \leq |\,\mathrm{Re}(z)\,| \leq |z| \quad \text{and} \quad \mathrm{Im}(z) \leq |\,\mathrm{Im}(z)\,| \leq |z|.$$

Consider a function $f : D \to \mathbb{C}$. For ease of presentation, we assume every point in its domain, $D \subseteq \mathbb{C}$, is an interior point. When $z_0 \in D$, the notation

$$L = \lim_{z \to z_0} f(z)$$

is used to signify the following logical statement.

> There is a number $L \in \mathbb{C}$ such that $\forall \epsilon > 0$, $\exists \delta > 0$ where
> we have $|L - f(z)| < \epsilon$ whenever $z \in \{z : 0 < |z - z_0| < \delta\} \subseteq D$.

The goal is to identify the limit L and then find the δ for an arbitrarily given ϵ. Writing $L = \lim_{z \to z_0} f(z)$ tacitly implies the limit exists. However, the limit may or may not exist. When it does, it is unique. The function f is said to be *continuous* at $z_0 \in D$ when

$$f(z_0) = \lim_{z \to z_0} f(z).$$

Here's an example, $f(z) = (3 + 2i)z$ is continuous at any $z_0 \in \mathbb{C}$. To see this, let $\epsilon > 0$ be arbitrary and check that by taking $0 < \delta \leq \epsilon/\sqrt{13}$ we have that

$$|(3 + 2i)z - (3 + 2i)z_0| = |3 + 2i|\,|z - z_0| = \sqrt{13}\,|z - z_0| < \epsilon$$

whenever $0 < |z - z_0| < \delta$. That is, given an $\epsilon > 0$ there is an appropriate $\delta > 0$. Next, consider a second example, $f(z) = z^2$ is continuous at any $z_0 \in \mathbb{C}$.

$$|z^2 - z_0^2| = |(z + z_0)(z - z_0)| \leq (|z| + |z_0|)\,|z - z_0|.$$

First, take $|z - z_0| \leq 1$ which implies $|z| \leq |z_0| + 1$. So, for any $\epsilon > 0$, we find that

$$|z^2 - z_0^2| \leq (1 + 2|z_0|)\,|z - z_0| < \epsilon$$

by taking $1 \geq \delta > 0$ small enough so that $|z - z_0| < \delta \leq \epsilon/(1 + 2|z_0|)$.

THEOREM 1. Suppose that $f : D_f \to \mathbb{C}$ and $g : D_g \to \mathbb{C}$. If both are continuous at $z_0 \in D_f \cap D_g$, then the following functions are also continuous at z_0.

(i) $h(z) \equiv f(z) + g(z)$, (ii) $h(z) \equiv f(z)\, g(z)$,

(iii) $h(z) \equiv f(z)/g(z)$ when $g(z_0) \neq 0$.

If on the other hand, g is continuous at $z_0 \in D_g$ and f is continuous at $w_0 = g(z_0) \in D_f$, then the composition function

(iv) $h(z) \equiv f(g(z))$

is continuous at z_0.

The proofs of these facts are essentially identical to what you were shown for real functions in Calculus I; i.e. the trick is to add and subtract or multiply and divide and then for a given $\epsilon > 0$ appropriately choose δ based on $\delta_f > 0$ and $\delta_g > 0$.

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3. Derive the following.

(a) $|z| \geq 0$ and $|z| = 0 \iff z = 0$. Hint: $|z|^2 = x^2 + y^2$.

(b) $|z_1 z_2| = |z_1|\,|z_2|$. Hint: $|z_1 z_2|^2 = z_1 z_2\, \overline{z_1 z_2}$.

(c) $|z_1 + z_2| \leq |z_1| + |z_2|$. Hint: $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1 \bar{z}_2)$.

4. Suppose $f(z) = u(x, y) + iv(x, y)$. Prove $f(z)$ is continuous at $z_0 = x_0 + iy_0$ if and only if both its real part $u(x, y)$ and imaginary part $v(x, y)$ are continuous at (x_0, y_0).

5. Use the results of the previous exercise as well as Theorem 1 above, together with your knowledge from Calculus I concerning continuity and local boundedness of the real functions e^x, $\sin(x)$ and $\cos(x)$ to conclude the following complex functions are continuous at any $z_0 \in \mathbb{C}$.

(a) $f(z) = e^z$. (c) $f(z) = \cosh(z)$. (e) $f(z) = z\cosh(z) + \cos(z)$.

(b) $f(z) = e^{iz}$. (d) $f(z) = \cos(z)$. (f) $f(z) = e^z \cos(z)$.

A complex function $f : D \to \mathbb{C}$ is *differentiable* at a point $z \in D$ if

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}.$$

Above $f'(z)$ denotes f's derivative at z. If this limit does not exist, at that z the derivative of f does not exist. I will use the notation

$$\frac{d}{dz} f(z) \equiv f'(z)$$

whenever it's convenient.

THEOREM 2. The familiar rules of differentiation from Calculus I carry over to complex functions. If f and g are both differentiable at z, then the sums and products are differentiable at z and satisfy

(i) $\dfrac{d}{dz}(f(z) + g(z)) = \dfrac{d}{dz} f(z) + \dfrac{d}{dz} g(z)$,

(ii) $\dfrac{d}{dz}(f(z)\, g(z)) = f(z) \left(\dfrac{d}{dz} g(z) \right) + \left(\dfrac{d}{dz} f(z) \right) g(z)$.

If g is differentiable at z and f is differentiable at $w = g(z)$, then the composition is differentiable at z and satisfies

(iii) $\dfrac{d}{dz}(f(g(z)) = f'(g(z)) \dfrac{d}{dz} g(z)$.

(i) is called the *sum rule*, (ii) is called the *product rule* and (iii) is the celebrated *chain rule*. We finally have the *power rules*. For any $n \in \mathbb{Z} \setminus \{0\}$ and $z \neq 0$ when $n \leq -1$

(iv) $\dfrac{d}{dz} z^n = n z^{n-1}$,

and when $f(z) = c$ (i.e. constant)

(v) $\dfrac{d}{dz} c = 0$.

Again, the proofs of facts (i)–(iii) and (v) are essentially identical to what you were shown for real functions in Calculus I. The proof of (iv) for cases $n \geq 1$ uses the Binomial Theorem

$$\begin{aligned} & (z + \Delta z)^n = z^n + n z^{n-1} \Delta z + \cdots + \Delta z^n \\ \Longrightarrow \quad & \frac{1}{\Delta z} \left((z + \Delta z)^n - z^n \right) = n z^{n-1} + \cdots + \Delta z^{n-1} \to n z^{n-1} \text{ as } \Delta z \to 0. \end{aligned}$$

This is valid for any $z \in \mathbb{C}$. For the case $n = -1$ and $z \neq 0$, the difference quotient gives

$$\frac{1}{\Delta z} \left(\frac{1}{z + \Delta z} - \frac{1}{z} \right) = \frac{1}{\Delta z} \left(\frac{z - (z + \Delta z)}{(z + \Delta z)\, z} \right) = - \left(\frac{1}{(z + \Delta z)\, z} \right) \to -z^{-2} \text{ as } \Delta z \to 0.$$

For cases $n \leq -1$, take $z \neq 0$ to write $z^n = \left(z^{-n} \right)^{-1}$ and apply the results of the previous two cases along with the chain rule.

The well-known *quotient rule* is derived by considering $d/dz\,\big(f(z)(g(z))^{-1}\big)$ and applying to it the product rule, the $n=-1$ power rule and the chain rule.

From Theorem 2, we easily see that any complex polynomial is everywhere differentiable. Moreover, the quotient of any two polynomials is differentiable at all points except those (isolated) points where the denominator polynomial is zero.

This is a good place to show the function e^z is differentiable everywhere and its derivative is, as expected, e^z. To this end, let's first consider its difference quotient at $z=0$

$$\frac{e^{0+\Delta z}-e^0}{\Delta z} = \frac{e^{\Delta x}e^{i\Delta y}-1}{\Delta z} = \frac{e^{\Delta x}\left(\cos(\Delta y)+i\,\sin(\Delta y)\right)-1}{\Delta z}.$$

You learned from Taylor's theorem in Calculus II that

$$e^{\Delta x} = 1+\Delta x+O(\Delta x^2),\quad \cos(\Delta y) = 1+O(\Delta y^2),\quad \sin(\Delta y) = \Delta y+O(\Delta y^3).$$

Insert these into the difference quotient above, set $\Delta z = re^{i\theta}\neq 0$ and collect terms to find

$$\frac{e^{0+\Delta z}-e^0}{\Delta z} = 1+O(|\Delta z|)\to 1 \ \text{ as } \Delta z\to 0.$$

Now for arbitrary $z\in\mathbb{C}$ observe

$$\frac{e^{z+\Delta z}-e^z}{\Delta z} = e^z\left(\frac{e^{0+\Delta z}-e^0}{\Delta z}\right)\to e^z \ \text{ as } \Delta z\to 0 \quad\Longrightarrow\quad \frac{d}{dz}e^z = e^z.$$

Differentiable complex functions are **very**, **very** special. For example, the function

$$u(x,y)+i\,v(x,y)\equiv(x+y)+i(x+2y)$$

can be written as a function of z by setting $x=\frac{1}{2}(z+\bar{z})$ and $y=\frac{1}{2}(z-\bar{z})/i$,

$$\begin{aligned} u(x,y)+i\,v(x,y) &= \tfrac{1}{2}\big((z+\bar{z})-i(z-\bar{z})\big)+\tfrac{1}{2}i\big((z+\bar{z})-2i(z-\bar{z})\big)\\ &= \frac{3}{2}\,z-\frac{1-2i}{2}\,\bar{z}\equiv f(z). \end{aligned}$$

However, this function is not differentiable anywhere. To see this, it suffices to show that $c(z)\equiv\bar{z}$ is not differentiable anywhere. Let $\Delta z = re^{i\theta}\neq 0$ to find

$$\frac{c(z+\Delta z)-c(z)}{\Delta z} = \frac{\overline{z+\Delta z}-\bar{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{e^{-i\theta}}{e^{i\theta}} = e^{-2i\theta}$$

and observe the difference quotient does not have a limit as $\Delta z\to 0$.

A <u>necessary</u> condition concerning the differentiability of a complex function is easy to derive. Suppose $f(z) = f(x+iy) = u(x,y) + i\,v(x,y)$ is differentiable at z. Then,

$$\begin{aligned}
&\frac{\partial}{\partial x} f(x+iy) = f'(z)\,\frac{\partial}{\partial x}(x+iy) = f'(z) = u_x(x,y) + i\,v_x(x,y) \\
&\frac{\partial}{\partial y} f(x+iy) = f'(z)\,\frac{\partial}{\partial y}(x+iy) = f'(z)\,i = u_y(x,y) + i\,v_y(x,y) \\
\Longrightarrow \qquad &u_x(x,y) + i\,v_x(x,y) = \frac{1}{i}\left(u_y(x,y) + i\,v_y(x,y)\right).
\end{aligned}$$

Equate the real and imaginary parts above to see we must have both

$$\frac{\partial}{\partial x}\,u(x,y) = \frac{\partial}{\partial y}\,v(x,y) \quad \text{and} \quad \frac{\partial}{\partial x}\,v(x,y) = -\frac{\partial}{\partial y}\,u(x,y).$$

This is the renowned *Cauchy-Riemann* condition. When z is written in polar coordinates, $z = re^{i\theta}$, and $f(z) = u(r,\theta) + i\,v(r,\theta)$, you will derive as an exercise the polar form of the Cauchy-Riemann condition

$$\frac{\partial}{\partial r}\,u(r,\theta) = \frac{1}{r}\,\frac{\partial}{\partial \theta}\,v(r,\theta) \quad \text{and} \quad \frac{\partial}{\partial r}\,v(r,\theta) = -\frac{1}{r}\,\frac{\partial}{\partial \theta}\,u(r,\theta).$$

Here are two Cauchy-Riemann examples. For the first example

$$f(z) = (x+y) + i(x+2y) \implies \begin{array}{ll} u_x = 1, & u_y = 1, \\ v_x = 1, & v_y = 2, \end{array}$$

and so this complex function can not be differentiable anywhere. This confirms what was explicitly shown above. For the second

$$f(z) = r\left(\cos(\theta) + i\,\sin(\theta)\right) \implies \begin{array}{ll} u_r = \cos(\theta), & u_\theta = -r\sin(\theta), \\ v_r = \sin(\theta), & v_\theta = r\cos(\theta), \end{array}$$

and this complex function <u>may</u> be everywhere differentiable. In fact, since this $f(z) = z$, it clearly is differentiable for every $z \in \mathbb{C}$.

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6. Determine where $f(z)$ is differentiable and at those points compute its derivative.

   (a) $f(z) = 3z^3 + 9z$. (c) $f(z) = \dfrac{9z+2}{z^2+1}$.

   (b) $f(z) = (z+2)^{15}$. (d) $f(z) = z + \dfrac{1}{z}$.

7. Do the same as asked in the previous exercise.

   (a) $f(z) = e^{2z}$. (c) $f(z) = \cosh(z)$.

   (b) $f(z) = e^z \sin(z)$. (d) $f(z) = \sin^2(z)$.

8. Do the same as asked in the previous exercise.

(a) $f(z) = \dfrac{\sin z}{z}$. (c) $f(z) = \dfrac{1}{\sin(z)}$.

(b) $f(z) = \dfrac{e^z}{z^2 - 2z + 1}$. (d) $f(z) = \dfrac{\sin(z)}{\cos(z)}$.

Hint (c) and (d): Show $\sin(z)$ and $\cos(z)$ can only be zero along the real axis.

9. Suppose $f(z) = u(x,y) + i\,v(x,y)$ is differentiable. Use the Cauchy-Riemann condition to show the following are true.

(a) $(u_x)^2 + (v_x)^2 = (u_y)^2 + (v_y)^2$ (c) $u_{xx} + u_{yy} = 0$

(b) $u_x u_y + v_x v_y = 0$ (d) $v_{xx} + v_{yy} = 0$

Note that (c) and (d) show the real and imaginary parts of a differentiable complex function are what are called *harmonic* functions.

10. (a) Derive the polar form of the Cauchy-Riemann conditions.
Hint: Consider $f(z) = f(re^{i\theta}) = u(r,\theta) + i\,v(r,\theta)$ then use the $\partial/\partial r$ and $\partial/\partial\theta$ chain rule.
10. (b) Consider $f(re^{i\theta}) \equiv \log(r) + i\theta$ where $r \geq 0$ and $-\pi < \theta \leq \pi$. Determine where this function is not defined. Determine where this function is not continuous. Can this be differentiable otherwise?

It is possible for the Cauchy-Riemann condition to be true at a point but the complex function is not differentiable there. For example, consider

$$f(z) \equiv \begin{cases} \bar{z}^2/z & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \quad \Longrightarrow \quad f(x + i\,0) = x + i\,0 \;\text{ and }\; f(0 + i\,y) = 0 + i\,y.$$

By evaluating the partial derivative difference quotients at $(0,0)$, you'll find $u_x(0,0) = 1$, $v_x(0,0) = 0$, $u_y(0,0) = 0$ and $v_y(0,0) = 1$, and so the Cauchy-Riemann condition is met. However, setting $\Delta z = re^{i\theta} \neq 0$ we see that

$$\frac{f(0 + \Delta z) - f(0)}{\Delta z} = \frac{r^2 e^{-2i\theta}/re^{i\theta} - 0}{re^{i\theta}} = e^{-4i\theta},$$

which does not have a limit as $\Delta z \to 0$.

If we add one further requirement concerning the real functions $u(x,y)$ and $v(x,y)$ (which by the way are not entirely necessary) we can show the following.

Suppose the partial derivatives of u and v exist <u>and</u> are continuous functions at a point $(x_0, y_0) \in \mathbb{R}^2$. Then, if the Cauchy-Riemann condition is satisfied at (x_0, y_0), the complex function $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at $z_0 = x_0 + iy_0$.

To prove this, first use your Calculus III extension of Taylor's theorem

$$u(x_0+\Delta x, y_0+\Delta y) = u(x_0,y_0) + u_x(x_0,y_0)\,\Delta x + u_y(x_0,y_0)\,\Delta y + e_u\sqrt{\Delta x^2+\Delta y^2},$$
$$v(x_0+\Delta x, y_0+\Delta y) = v(x_0,y_0) + v_x(x_0,y_0)\,\Delta x + v_y(x_0,y_0)\,\Delta y + e_v\sqrt{\Delta x^2+\Delta y^2},$$

where, given that u_x, u_y, v_x and v_y are continuous real valued functions at (x_0, y_0), the error terms, e_u and e_v, both tend to zero as $\sqrt{\Delta x^2+\Delta y^2} \to 0$. Insert the expansions for u and v into f's difference quotient (with $\Delta z = \Delta x + i\,\Delta y$)

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}$$
$$= \frac{1}{\Delta z}\Big(\big(u_x(x_0,y_0)\,\Delta x + u_y(x_0,y_0)\,\Delta y\big) + i\big(v_x(x_0,y_0)\,\Delta x + v_y(x_0,y_0)\,\Delta y\big)\Big) + e,$$

where here note $e \equiv (e_u + ie_v)\dfrac{\sqrt{\Delta x^2+\Delta y^2}}{\Delta z} \to 0$ as $\Delta z \to 0$.

Substitute partial derivatives according to Cauchy-Riemann, i.e. $v_y(x_0,y_0) = u_x(x_0,y_0)$ and $u_y(x_0,y_0) = -v_x(x_0,y_0)$, into the bracketed expression to find

$$\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z} = u_x(x_0,y_0) + i\,v_x(x_0,y_0) + e$$

Finally, as already noted, the rightmost term above tends to zero as $\Delta z \to 0$.