

①

Complex Line Integrals

$$1) \quad \tilde{\Gamma}_{z_0}^{z_1} = \left\{ (1-4s^2)z_0 + 4s^2z_1 : 0 \leq s \leq 1/2 \right\}$$

(Note: $\tilde{\Gamma}_{z_0}^{z_1}$ is same set as $\Gamma_{z_0}^{z_1}$)

$$a) \quad \int_{\tilde{\Gamma}_{z_0}^{z_1}} z \, dz = \int_0^{1/2} ((1-4s^2)z_0 + 4s^2z_1) (-8s z_0 + 8s z_1) \, ds$$

$$= (z_1 - z_0) \int_0^{1/2} ((1-4s^2)z_0 + 4s^2z_1) 8s \, ds = *$$

Substitute $4s^2 = t \Rightarrow 8s \, ds = dt$
 $s=0 \Rightarrow t=0$
 $s=1/2 \Rightarrow t=1$

So

$$* = (z_1 - z_0) \int_0^1 ((1-t)z_0 + tz_1) \, dt$$

as before in notes

$$= (z_1 - z_0) \left(\frac{1}{2} z_0 + \frac{1}{2} z_1 \right) = \boxed{\frac{1}{2} z_1^2 - \frac{1}{2} z_0^2}$$

$$b) \quad \int_{\tilde{\Gamma}_{z_0}^{z_1}} z^2 \, dz$$

$$= (z_1 - z_0) \int_0^{1/2} ((1-4s^2)z_0 + 4s^2z_1)^2 8s \, ds = *$$

(over)

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Make some substitution

$$* = (z_1 - z_0) \int_0^1 ((1-t)z_0 + tz_1)^2 dt$$

$$= (z_1 - z_0) \left[z_0^2 \int_0^1 (1-t)^2 dt + 2z_0 z_1 \int_0^1 (1-t)t dt + z_1^2 \int_0^1 t^2 dt \right]$$

$$= (z_1 - z_0) \left[\frac{1}{3} z_0^2 + 2\left(\frac{1}{6}\right) z_0 z_1 + \frac{1}{3} z_1^2 \right]$$

$$= \frac{1}{3} \left[\cancel{z_0^2 z_1} + \cancel{z_0 z_1^2} + z_1^3 - z_0^3 - \cancel{z_0^2 z_1} - \cancel{z_0 z_1^2} \right]$$

$$= \boxed{\frac{1}{3} z_1^3 - \frac{1}{3} z_0^3}$$

These reparam's must agree with what I got in notes.

B.C. a reparam is just a change of "t" variables

③

$$2 a) \quad \Gamma_{z_0}^{z_1} = \left\{ \gamma(t) = (1-t)z_1 + tz_0 : 0 \leq t \leq 1 \right\}$$

$$\begin{aligned} \int_{\Gamma_{z_0}^{z_1}} \bar{z} dz &= \int_0^1 \overline{((1-t)z_1 + tz_0)} (z_1 - z_0) dt \\ &= (z_1 - z_0) \bar{z}_1 \int_0^1 (1-t) dt + (z_1 - z_0) \bar{z}_0 \int_0^1 t dt \\ &= \frac{1}{2} (z_1 - z_0) \bar{z}_1 + \frac{1}{2} (z_1 - z_0) \bar{z}_0 \\ &= \left[\frac{1}{2} \left[|z_1|^2 - (z_0 \bar{z}_1 + \bar{z}_0 z_1) - |z_0|^2 \right] \right] \end{aligned}$$

$$b) \quad \tilde{\Gamma}_{z_0}^{z_1} = \left\{ \gamma(t) = (1-4s^2)z_0 + 4s^2z_1 : 0 \leq s \leq \frac{1}{2} \right\}$$

(Note $\Gamma_{z_0}^{z_1}$ is same set as $\tilde{\Gamma}_{z_0}^{z_1}$)

$$\begin{aligned} \int_{\tilde{\Gamma}_{z_0}^{z_1}} \bar{z} dz &= \int_0^{1/2} \overline{((1-4s^2)z_0 + 4s^2z_1)} 8s (z_1 - z_0) ds \\ &= (z_1 - z_0) \bar{z}_1 \int_0^{1/2} (1-4s^2) 8s ds \\ &\quad + (z_1 - z_0) \bar{z}_0 \int_0^{1/2} 4s^2 8s ds \\ &= (\text{over}) * \end{aligned}$$

④ Substitute

$$4s^2 = t \Rightarrow 8s ds = dt$$

$$s=0 \Rightarrow t=0$$

$$s=1/2 \Rightarrow t=1$$

So * becomes

$$* = (z_1 - z_0) \bar{z}_1 \int_0^1 (1-t) dt$$

$$+ (z_1 - z_0) \bar{z}_0 \int_0^1 t dt$$

$$= \frac{1}{2} (z_1 - z_0) (\bar{z}_1 + \bar{z}_0)$$

$$= \left[\frac{1}{2} [|z_1|^2 - (z_0 \bar{z}_1 - \bar{z}_0 z_1) - |z_0|^2] \right]$$

Same as in part (a).

These examples demonstrate a general fact.

Let $\tilde{\gamma}(s) \equiv \gamma(t(s))$ be a reparameterization of a curve. (ener)

⑤ As I hope you remember, change of variables in the chain rule in action,

$$\begin{aligned}\gamma'(t) dt &= \frac{d\gamma(t)}{dt} dt = \frac{d\gamma(t(s))}{ds} \frac{ds}{dt} dt \\ &= \tilde{\gamma}'(s) ds\end{aligned}$$

So it's always true that

$$\begin{aligned}\int_{t_0}^{t_1} f(\gamma(t)) \gamma'(t) dt & \quad \begin{aligned} \gamma(t_0) &= z_0 \quad \dots \\ \tilde{\gamma}(s_0) &= z_0 \end{aligned} \\ &= \int_{s_0}^{s_1} f(\tilde{\gamma}(s)) \tilde{\gamma}'(s) ds, \quad \begin{aligned} &\text{or} \\ &\leftarrow \begin{cases} t_0 = t(s_0) \\ t_1 = t(s_1) \end{cases} \end{aligned}\end{aligned}$$

3) Here we have two different curves that connect

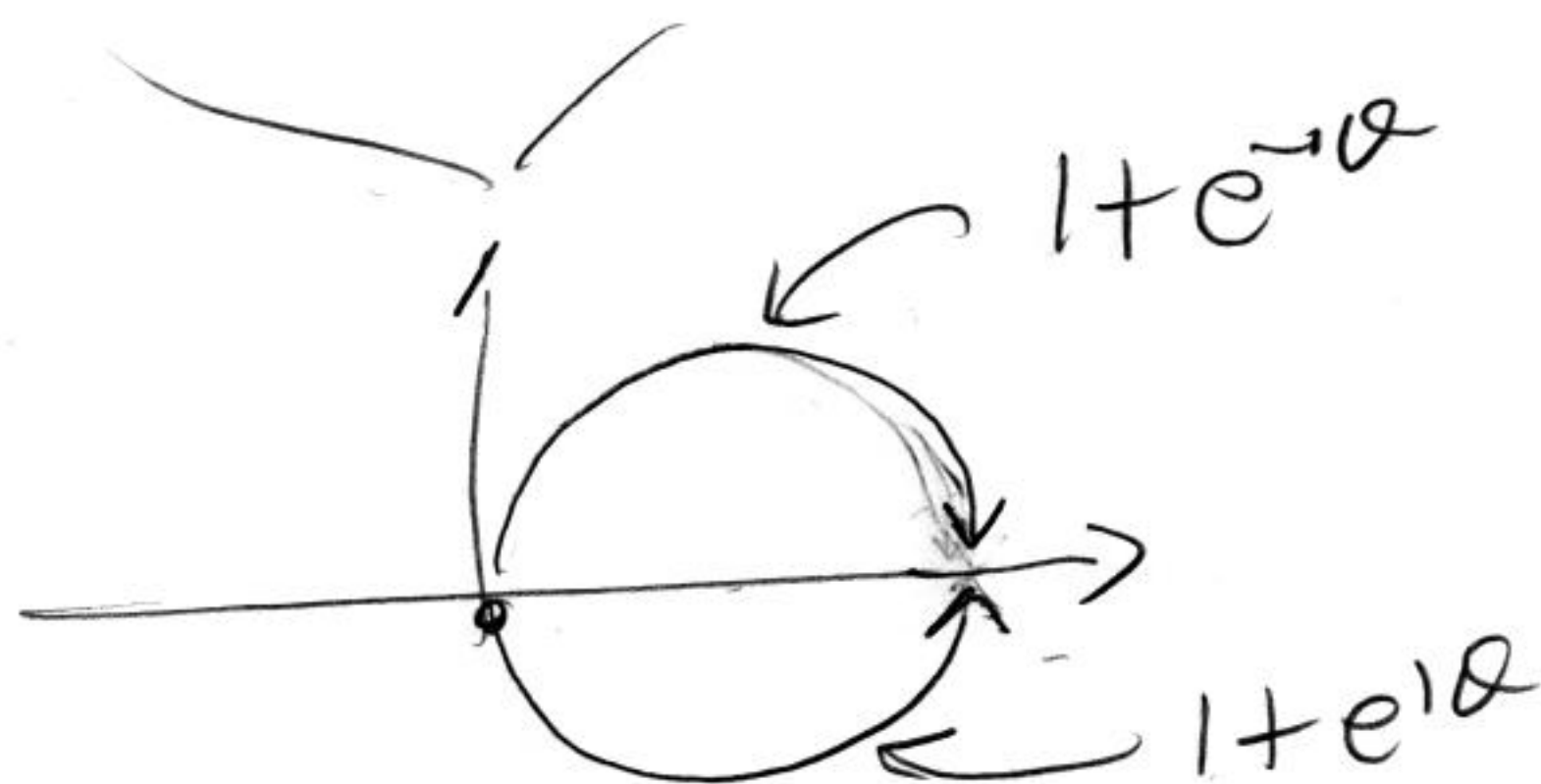
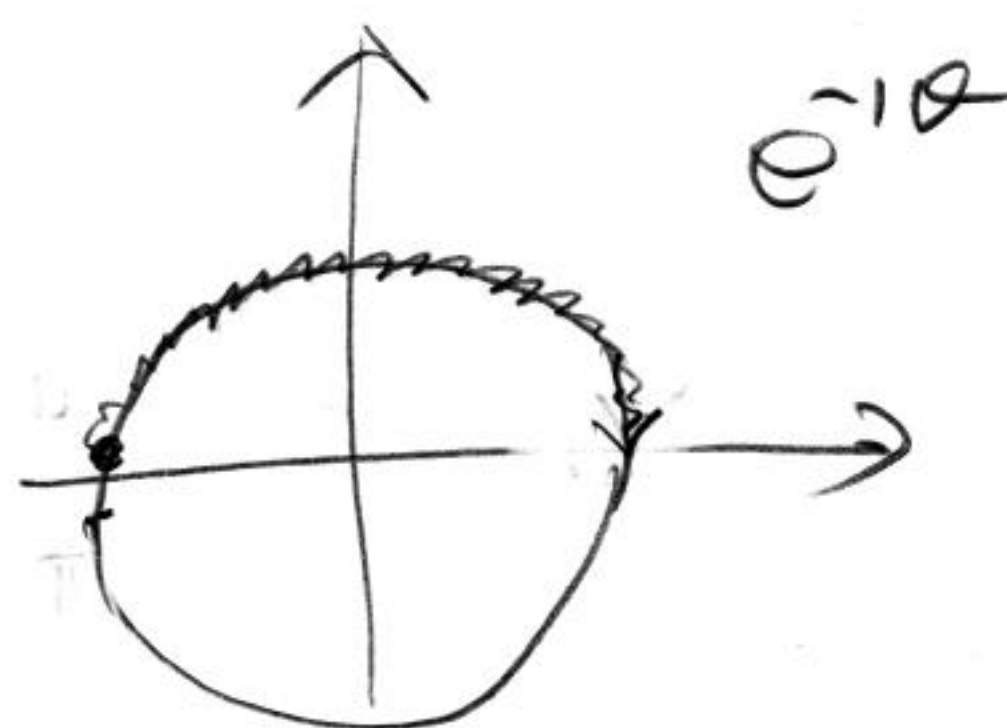
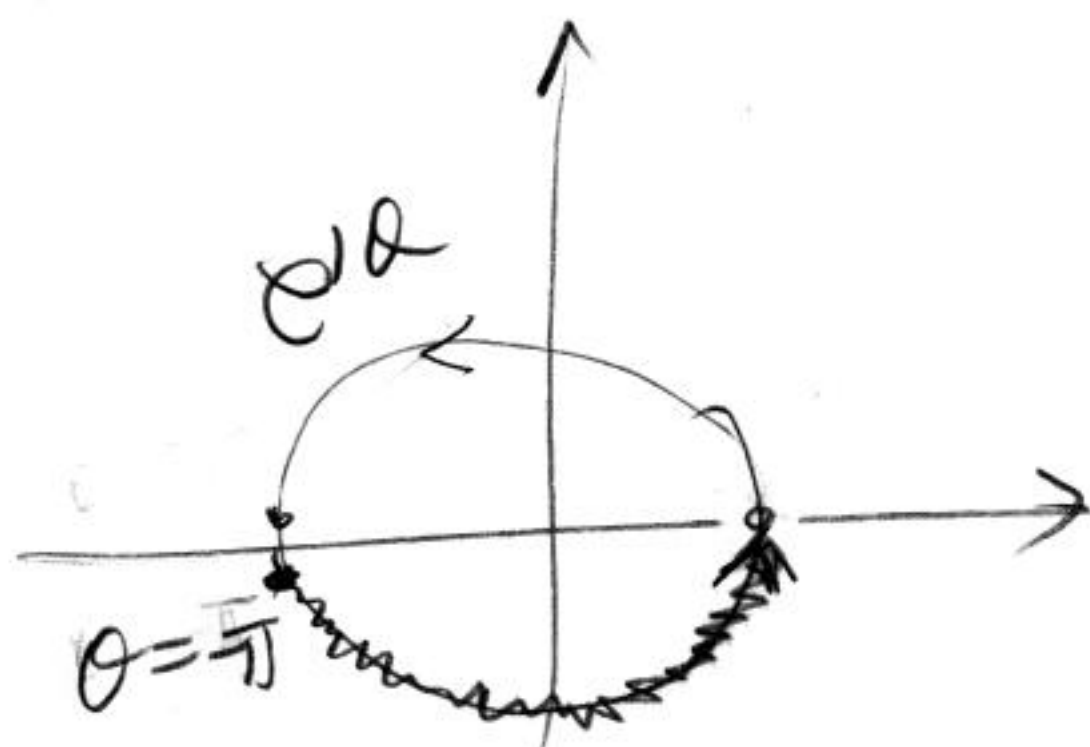
$$z_0 = 0 \quad \text{to} \quad z_1 = 2$$

6)

$$\Gamma_0^2 = \{1 + e^{i\theta} : \theta \in [-\pi, 0]\}$$

and

$$\tilde{\Gamma}_0^2 = \{1 + e^{-i\theta} : \theta \in [-\pi, 0]\}$$



compute the following by using the given parametrizations

$$a) \int_{\Gamma_0^2} z dz = \int_{-\pi}^0 (1 + e^{i\theta}) i e^{i\theta} d\theta$$

$$= i \int_{-\pi}^0 (e^{i\theta} + e^{2i\theta}) d\theta = i \left(\frac{e^{i\theta}}{i} + \frac{e^{2i\theta}}{2i} \right) \Big|_{-\pi}^0$$

$$\textcircled{4} = (e^0 - e^{-\pi i}) + \left(\frac{e^0 - e^{-2\pi i}}{2}\right) = \boxed{2}$$

$$b) \int_{\tilde{\Gamma}_0^2} z dz = \int_{-\pi}^0 (1 + e^{-i\alpha}) - i e^{-i\alpha} d\alpha$$

$$= -i \int_{-\pi}^0 (e^{-i\alpha} + e^{-2i\alpha}) d\alpha = -i \left(\frac{e^{-i\alpha}}{-i} + \frac{e^{-2i\alpha}}{-2i} \right) \Big|_{-\pi}^0$$

$$= -i \left(\frac{e^0 - e^{\pi i}}{-i} \right) + \left(\frac{e^0 - e^{2\pi i}}{2} \right) = \boxed{2}$$

Observe That $z = \frac{1}{2} \frac{d}{dz} z^2$ so

$$\int_{\Gamma_0^2} z dz = \frac{1}{2} z^2 - 0 = 2$$

Different curves
Same endpoints
Same integral.

and

$$\int_{\tilde{\Gamma}_0^2} z dz = \frac{1}{2} z^2 - 0 = 2$$

⑧

$$\begin{aligned}
 c) \int_{\Gamma_0^2} \bar{z} dz &= \int_{-\pi}^0 \overline{(1+e^{i\theta})} i e^{i\theta} d\theta \\
 &= \int_{-\pi}^0 (1+e^{-i\theta}) i e^{i\theta} d\theta = i \int_{-\pi}^0 (e^{i\theta} + 1) d\theta \\
 &= i \left(\frac{e^{i\theta}}{i} + \theta \right) \Big|_{-\pi}^0 = \boxed{2 + \pi i}
 \end{aligned}$$

$$\begin{aligned}
 d) \int_{\tilde{\Gamma}_0^2} \bar{z} dz &= \int_{-\pi}^0 \overline{(1+e^{-i\theta})} -i e^{-i\theta} d\theta \\
 &= \int_{-\pi}^0 (1+e^{i\theta}) (-i e^{-i\theta}) d\theta = -i \int_{-\pi}^0 (e^{-i\theta} + 1) d\theta \\
 &= -i \left(\frac{e^{-i\theta}}{-i} + \theta \right) \Big|_{-\pi}^0 = \boxed{2 - \pi i}
 \end{aligned}$$

$\bar{z}$ does not have a complex antiderivative so you shouldn't expect these two integrals to agree.

⑨

4) Each integrand here has a complex antiderivative.

$$a) \int_{\pi z_*}^0 (z^2 + 3z) dz$$

$$= \int_{\pi z_*}^0 \frac{d}{dz} \left(\frac{z^3}{3} + \frac{3z^2}{2} \right) dz$$

$$= \left[\frac{1}{3} z_*^3 + \frac{3}{2} z_*^2 - 0 \right]$$

$$b) \int_{\pi z_*}^0 e^{iz} dz$$

$$= \int_{\pi z_*}^0 \frac{d}{dz} \left(\frac{1}{i} e^{iz} \right) dz$$

$$= \frac{1}{i} e^{iz_*} - \frac{1}{i} = \boxed{i(1 - e^{iz_*})}$$

$$c) \int_{\Gamma_{z_*}} \cos(z) dz$$

$$= \int_{\Gamma_{z_*}} \frac{d}{dz} \sin(z) dz = \boxed{\sin(z_*) - 0}$$

$$d) \int_{\Gamma_{z_*}} \sinh(2z) dz$$

$$= \int_{\Gamma_{z_*}} \frac{d}{dz} \left(\frac{1}{2} \cosh(2z) \right) dz$$

$$= \left[\frac{1}{2} \cosh(2z_*) - \frac{1}{2} \right]$$

5) If f & g are diff'bl on S

Then so is $h = fg$.

From prod rule

$$\frac{dh}{dz} = f \frac{dg}{dz} + \frac{df}{dz} g$$

(over)

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From FTC if $\Gamma_{z_0}^{z_1} \subseteq S$

$$h(z_1) - h(z_0)$$

$$= \int_{\Gamma_{z_0}^{z_1}} \frac{dh}{dz} dz$$

$$\text{So } = \int_{\Gamma_{z_0}^{z_1}} (f g' + f' g) dz$$

$$= \int_{\Gamma_{z_0}^{z_1}} f g' dz + \int_{\Gamma_{z_0}^{z_1}} f' g dz$$

$$\text{So } \int_{\Gamma_{z_0}^{z_1}} f g' dz = h(z_1) - h(z_0) - \int_{\Gamma_{z_0}^{z_1}} f' g dz$$

$$= f(z_1) g(z_1) - f(z_0) g(z_0)$$

$$- \int_{\Gamma_{z_0}^{z_1}} f' g dz //$$

⑫

6. $\Gamma_0^{z_*}$ is a simple-curve joining 0 to z_* .

a) $\int_{\Gamma_0^{z_*}} f g' dz$

$f(z) = z$
 $g'(z) = e^z$
 $\Rightarrow g(z) = e^z$

$= z e^z \Big|_0^{z_*}$

$-\int_{\Gamma_0^{z_*}} 1 e^z dz = z_* e^{z_*} - e^z \Big|_0^{z_*}$

$= \left[z_* e^{z_*} - (e^{z_*} - 1) \right]$

b) $\int_{\Gamma_0^{z_*}} (z+2) e^{2z} dz$

$f(z) = z+2$
 $g'(z) = e^{2z}$
 $\Rightarrow g(z) = \frac{1}{2} e^{2z}$

$= (z+2) \frac{1}{2} e^{2z} \Big|_0^{z_*}$

$-\int_{\Gamma_0^{z_*}} 1 \frac{1}{2} e^{2z} dz = \left(\frac{z_*+2}{2} e^{2z_*} - 1 \right) - \frac{1}{4} e^{2z} \Big|_0^{z_*}$
 $= \left[\left(\frac{z_*+2}{2} e^{2z_*} - 0 \right) - \frac{1}{4} (e^{2z_*} - 1) \right]$

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$$c) \int_{\Gamma_{0, z_*}^{z_*}} z (z+1)^{100} dz$$

$$= \frac{z(z+1)^{101}}{101} \Big|_0^{z_*}$$

$$- \int_{\Gamma_0^{z_*}} \frac{1}{101} (z+1)^{101} dz$$

$$= \frac{z_* (z_*+1)^{101}}{101} - 0 - \int_{\Gamma_0^{z_*}} \frac{1}{101} (z+1)^{101} dz$$

$$= \frac{z_* (z_*+1)^{101}}{101} - \frac{1}{101} \frac{(z+1)^{102}}{102} \Big|_0^{z_*}$$

$$= \left[\frac{z_* (z_*+1)^{101}}{101} - \frac{1}{101 \cdot 102} ((z_*+1)^{102} - 1) \right]$$

$$d) \int_{\Gamma_0^{z_*}} \sinh(z) \cosh(z) dz$$

$$= 0 \text{ ver.}$$

$$f(z) = z$$

$$g'(z) = (z+1)^{100}$$

$$\Rightarrow g(z) = \frac{1}{101} (z+1)^{101}$$

$$f(z) = \sinh(z)$$

$$g'(z) = \cosh(z)$$

$$\Rightarrow g(z) = \cosh(z)$$

⑭

$$* = \sinh(z) \cosh(z) \Big|_0^{z_*}$$

$$- \int_0^{z_*} \cosh^2(z) dz$$

But $\cosh^2(z) - \sinh^2(z) = 1$

Because $\left(\frac{e^z + e^{-z}}{2}\right)^2 - \left(\frac{e^z - e^{-z}}{2}\right)^2$

$$= \frac{1}{4} (e^{2z} + 2 + e^{-2z}) - (e^{2z} - 2 + e^{-2z})$$

$$= \frac{1}{4} (4) = 1$$

So use this

$$* = \sinh(z) \cosh(z) - \int_0^{z_*} (1 + \sinh^2(z)) dz$$

$$\Rightarrow 2 \int_0^{z_*} \sinh^2(z) dz = \sinh(z) \cosh(z) - z$$

$$\int_0^{z_*} \sinh^2(z) dz = \left[\frac{1}{2} \sinh(z) \cosh(z) - \frac{1}{2} z \right]_0^{z_*}$$

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$$7.) \oint_{\Gamma} z^{-n} dz \quad n=2, 3, \dots$$

$$\Gamma = \{e^{it} : -\pi < t \leq \pi\}$$

$$= \int_{-\pi}^{\pi} e^{-ni\theta} i e^{i\theta} d\theta$$

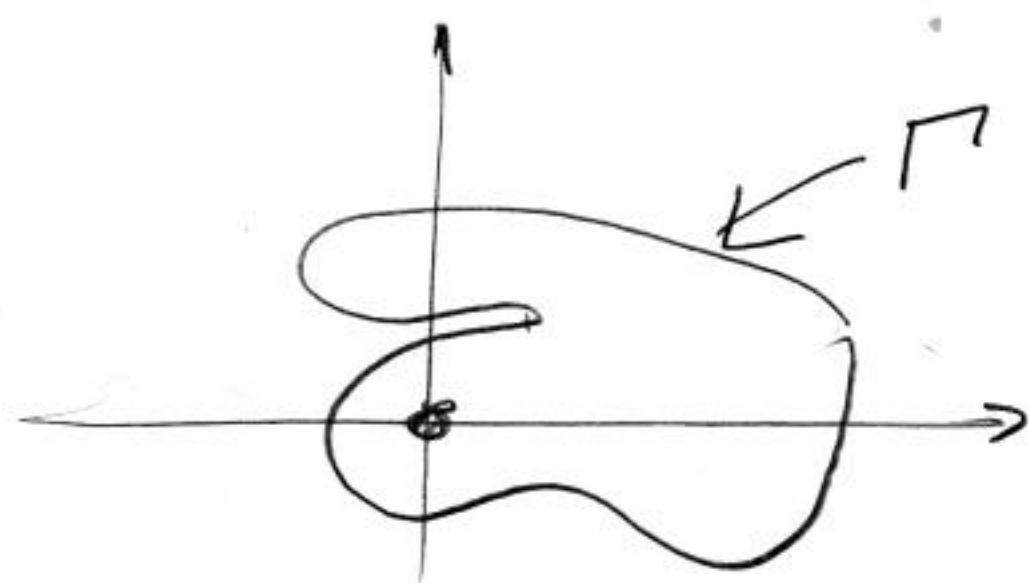
$$= i \int_{-\pi}^{\pi} e^{(1-n)i\theta} d\theta = i \left. \frac{e^{(1-n)i\theta}}{(1-n)i} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{1-n} \left(e^{(1-n)\pi i} - e^{-(1-n)\pi i} \right)$$

$$= \frac{1}{1-n} \left((-1)^{1-n} - (-1)^{1-n} \right) = 0 \quad \text{when } n=2, 3, \dots$$

8) Note that $\frac{1}{z}$ is diff'ble for

all z EXCEPT at $z=0$

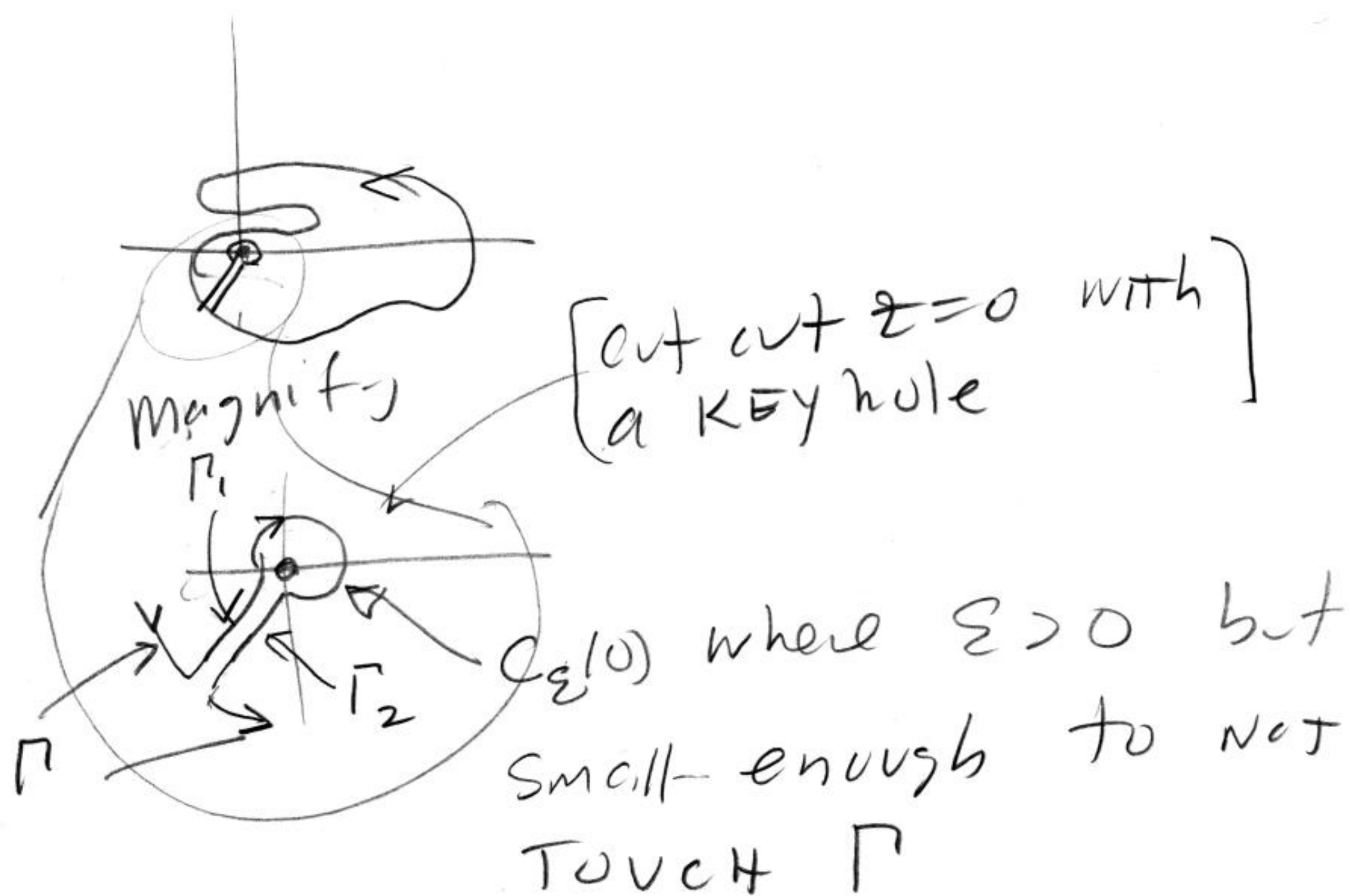


where $z=0$ is
INSIDE Γ .

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Consider Γ modified as follows



$$\tilde{\Gamma} = \Gamma \cup \Gamma_1 \cup C_\epsilon(0) \cup \Gamma_2$$

Note That inside $\tilde{\Gamma}$ $\frac{1}{z}$ is diff'bl for all z inside.

$\therefore$ Cauchy's Thm says

$$\oint_{\tilde{\Gamma}} \frac{1}{z} dz = 0$$

$$= \oint_{\Gamma} \frac{1}{z} dz - \oint_{C_\epsilon(0)} \frac{1}{z} dz + \int_{\Gamma_1} \frac{1}{z} dz - \int_{\Gamma_2} \frac{1}{z} dz$$

going around clockwise

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up direction to Γ_1

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$$\int_{\Gamma_1} \frac{1}{z} dz - \int_{\Gamma_2} \frac{1}{z} dz = 0$$

These cancel each other,



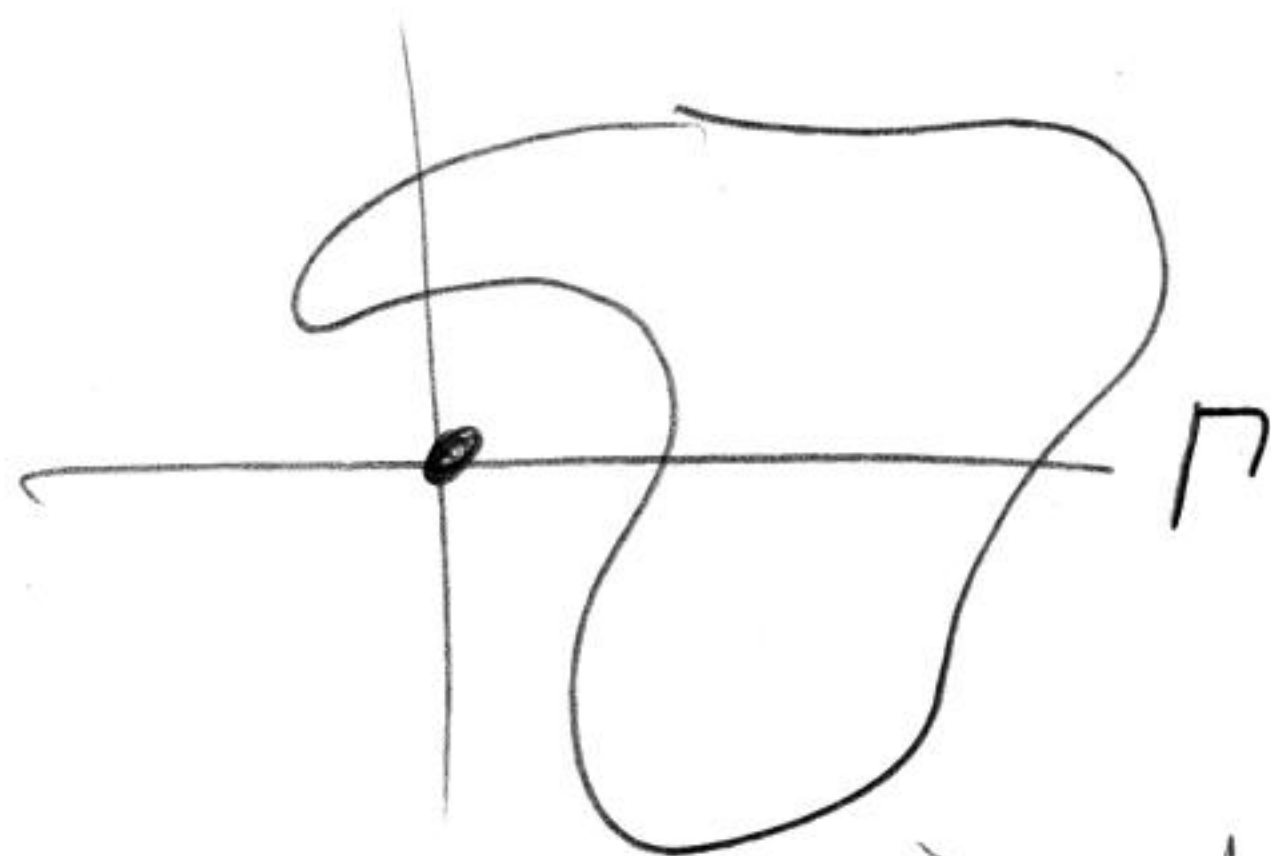
$$\oint_{C_{\varepsilon}(0)} \frac{1}{z} dz = \int_{\theta_*}^{\theta_* + 2\pi} \frac{1}{\varepsilon e^{i\theta}} i \varepsilon e^{i\theta} d\theta$$

$$= i \int_{\theta_*}^{\theta_* + 2\pi} d\theta = 2\pi i$$

$$\therefore \oint_{\Gamma} \frac{1}{z} dz = \cancel{\int_{\Gamma_1}} - \cancel{\int_{\Gamma_2}} + \underline{\underline{2\pi i}}$$

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b) Suppose $z=0$ outside Γ .



Inside Γ $\frac{1}{z}$ is diff'ble

so again using Cauchy's Thm

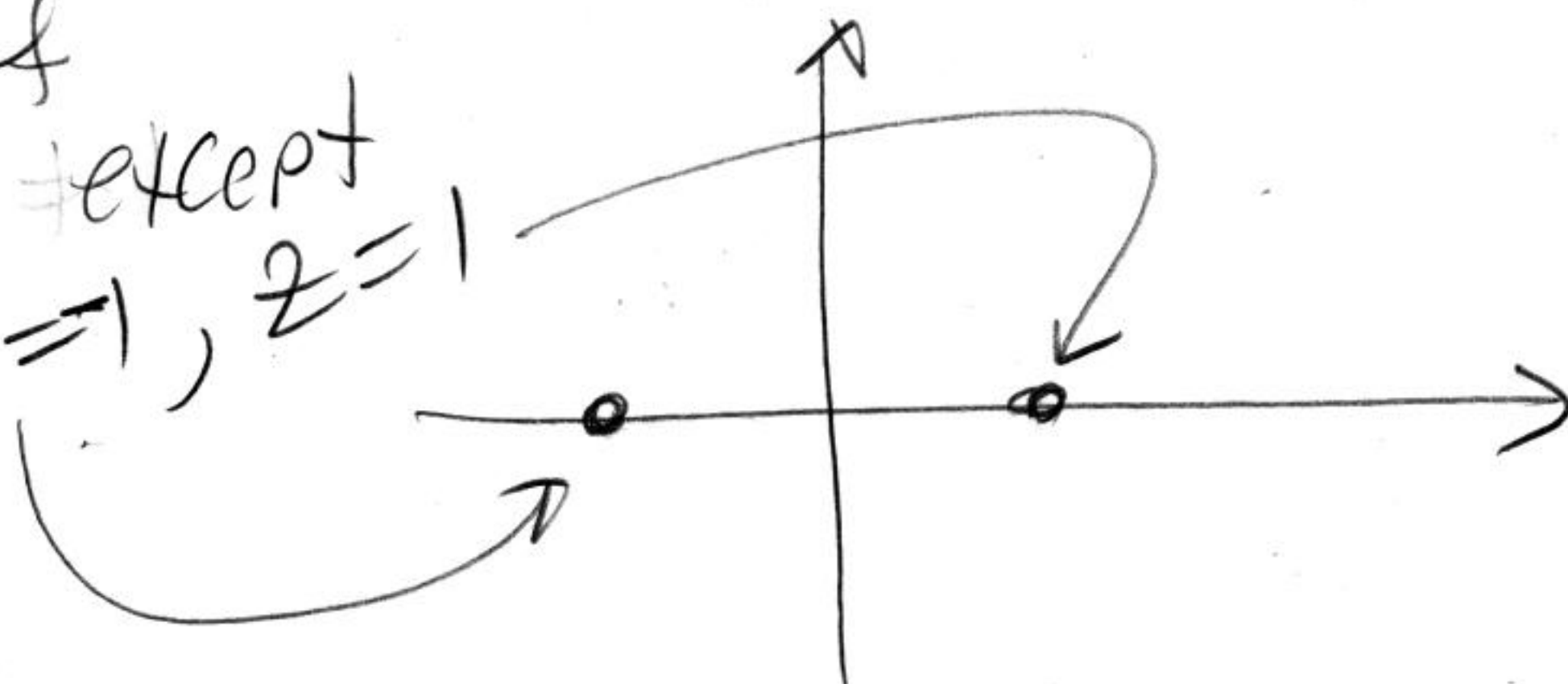
$$\oint_{\Gamma} \frac{1}{z} dz = 0$$

9) Write

Partial
Fraction

$$\frac{1}{z^2 - 1} = \frac{1}{2} \frac{1}{z-1} - \frac{1}{2} \frac{1}{z+1}$$

diff
except
 $z=1, z=-1$

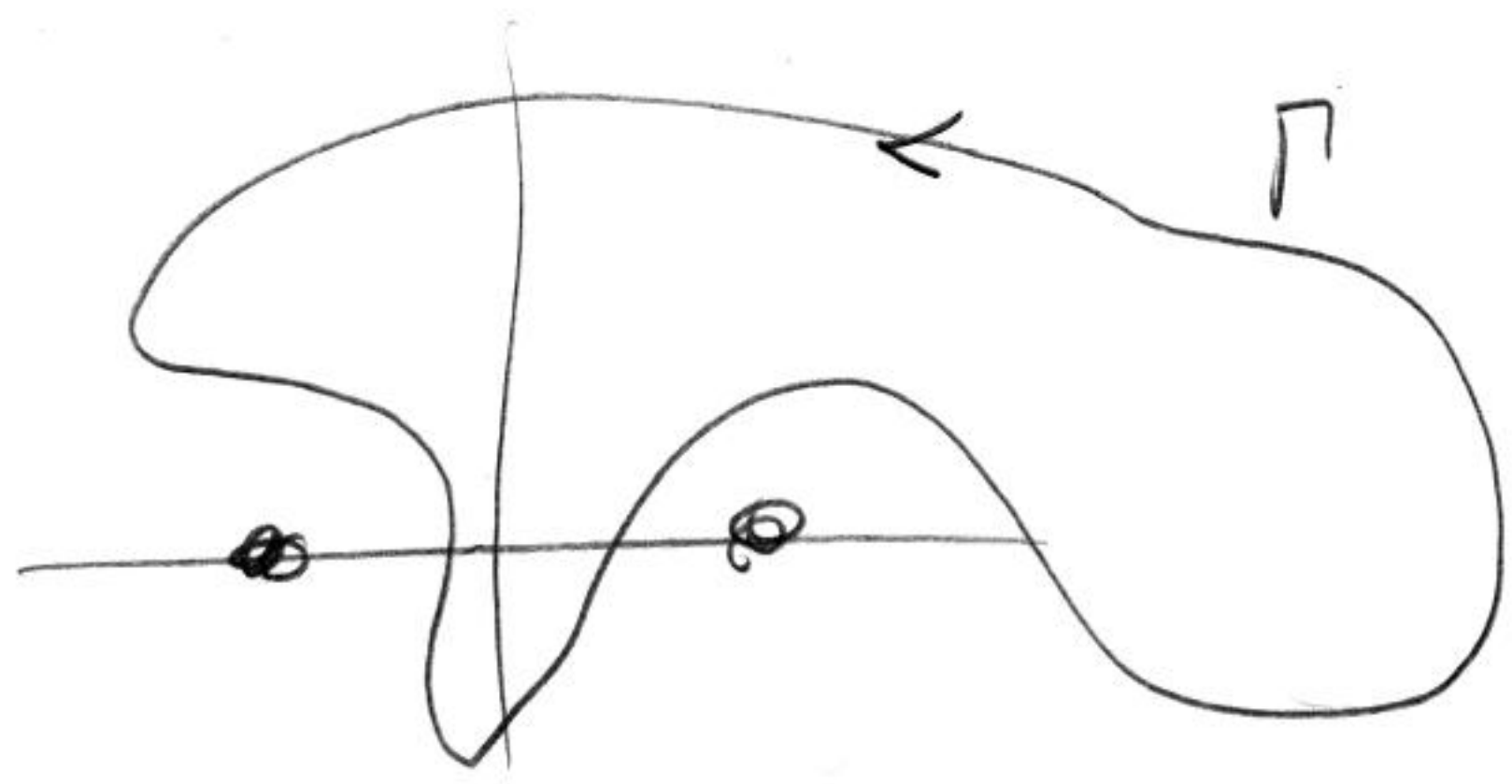


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Suppose Γ looks like This

Case 1
Neither



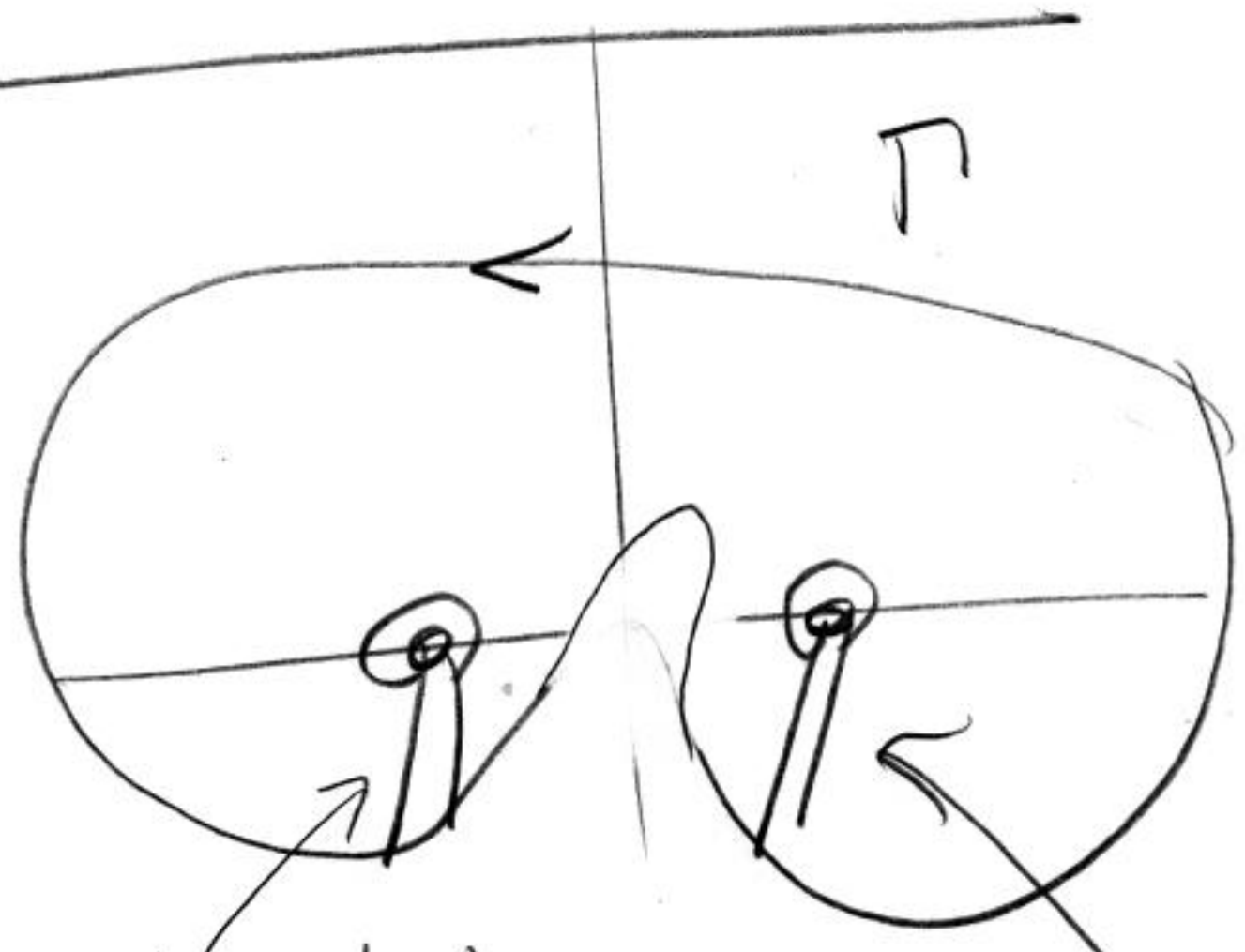
Then $\frac{1}{z^2-1}$ is diffbl for all z

inside Γ .

So Cauchy says.

$$\oint_{\Gamma} \frac{1}{z^2-1} dz = 0$$

Case 2
both



Keyhole
cut

Key hole
cut

$$\tilde{\Gamma} = \Gamma \cup \Gamma_{\text{Keyhole 1}} \cup \Gamma_{\text{Keyhole 2}}$$

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$$\oint_{\tilde{\Gamma}} \frac{1}{z^2-1} dz = 0$$

Since by Cauchy's Thm
 $\frac{1}{z^2-1}$ is diffbl inside $\tilde{\Gamma}$,

Break $\tilde{\Gamma}$ up into pieces

$$= \oint_{\Gamma} \frac{1}{z^2-1} dz + \int_{\Gamma_{KH1}} \frac{1}{z^2-1} dz + \int_{\Gamma_{KH2}} \frac{1}{z^2-1} dz$$

Put $\int_{\Gamma_{KH1}} \frac{1}{z^2-1} dz =$

orientation

$\oint_{C_\varepsilon(-1)}$

These cancel out

$$= - \int_{C_\varepsilon(-1)} \left(\frac{1}{2} \frac{1}{z-1} - \frac{1}{2} \frac{1}{z+1} \right) dz$$

partial frac

$$= -\frac{1}{2} \int_{C_\varepsilon(-1)} \frac{1}{z-1} dz + \frac{1}{2} \int_{C_\varepsilon(-1)} \frac{1}{z+1} dz$$

b.c. $\frac{1}{z-1}$ is diffbl inside $C_\varepsilon(-1)$

parameterize

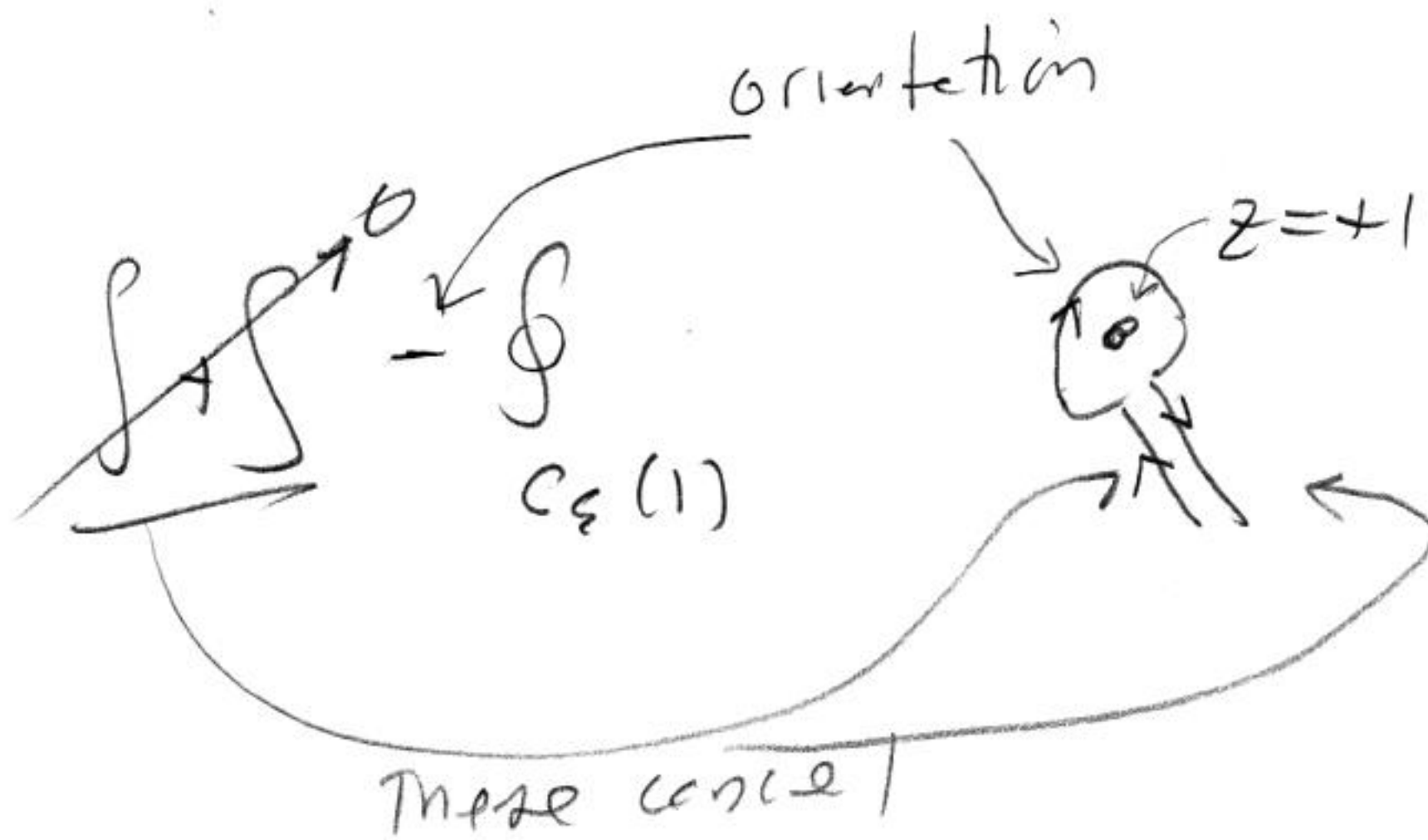
$$z+1 = + \varepsilon e^{i\theta}$$

$$= \frac{1}{2} \int_{-\pi}^{\pi} \frac{1}{\varepsilon e^{i\theta}} i \varepsilon e^{i\theta} d\theta = \pi i$$

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But

$$\int_{\Gamma_{K+2}} \frac{1}{z^2-1} dz =$$



$$= - \oint_{C_\epsilon(1)} \left(\frac{1}{2} \frac{1}{z-1} - \frac{1}{2} \frac{1}{z+1} \right) dz$$

partial frac

$$= -\frac{1}{2} \oint_{C_\epsilon(1)} \frac{1}{z-1} dz + \frac{1}{2} \oint_{C_\epsilon(1)} \frac{1}{z+1} dz$$

parameterize

$$z-1 = \epsilon e^{i\theta}$$

$$= -\frac{1}{2} \int_{-\pi}^{\pi} \frac{1}{\epsilon e^{i\theta}} i \epsilon e^{i\theta} d\theta$$

OK $\frac{1}{z+1}$ is analytic b.c. $z+1$ inside $C_\epsilon(1)$

$$= -\pi i$$

So Case 2 when both -1 & $+1$ inside

$$0 = \oint_{\Gamma} \frac{1}{z^2-1} dz + \pi i - \pi i$$

$$\Rightarrow \left| \oint_{\Gamma} \frac{1}{z^2-1} dz = 0 \right|$$

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Case 3 only -1 inside, $+1$ outside

$$0 = \oint_{\Gamma} \frac{1}{z^2-1} dz + \int_{\Gamma_{KH1}} \frac{1}{z^2-1} dz$$

$$= \oint_{\Gamma} \frac{1}{z^2-1} dz + \pi i$$

↖ Bottom of page 20

So

$$\left\{ \oint_{\Gamma} \frac{1}{z^2-1} dz = -\pi i \right.$$

Case 4 only $+1$ inside, -1 outside

$$0 = \oint_{\Gamma} \frac{1}{z^2-1} dz + \int_{\Gamma_{KH2}} \frac{1}{z^2-1} dz$$

$$= \oint_{\Gamma} \frac{1}{z^2-1} dz - \pi i$$

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So

$$\left\{ \oint_{\Gamma} \frac{1}{z^2-1} dz = \pi i \right.$$