Complex Line Integrals

Consider a smooth simple-curve $\Gamma_{z_0}^{z_1}$ defined by a one-to-one and continuously differentiable function $\gamma : [t_0, t_1] \to \mathbb{C}$, where $z_0 = \gamma(t_0)$ and $z_1 = \gamma(t_1)$. See your "Complex Point Sets and Curves" notes. The complex line integral of $f(z)$ on $\Gamma_{z_0}^{z_1}$ is given by

$$\int_{\Gamma_{z_0}^{z_1}} f(z)\, dz \equiv \int_{t_0}^{t_1} f(\gamma(t))\, \gamma'(t)\, dt.$$

Provided that $f(z)$ is a continuous complex function on an open set S and $\Gamma_{z_0}^{z_1} \subseteq S$, it is straightforward to interpret this complex line integral as the limit of Riemann sums. Moreover, the value of the line integral depends only on the point set $\Gamma_{z_0}^{z_1}$ itself and not on any particular reparameterization of the function that defines it.

The following examples may prove useful.

First, let $f(z) = z$ and suppose $\gamma(t) = (1-t)\, z_0 + t\, z_1$ for $0 \le t \le 1$ gives the curve $\Gamma_{z_0}^{z_1}$. So

$$\int_{\Gamma_{z_0}^{z_1}} z\, dz = \int_0^1 \Big((1-t)\, z_0 + t\, z_1\Big)(z_1 - z_0)\, dt = \tfrac{1}{2}\, z_1^2 - \tfrac{1}{2}\, z_0^2.$$

Second, let $f(z) = z^2$ on the same curve.

$$\int_{\Gamma_{z_0}^{z_1}} z^2\, dz = \int_0^1 \Big((1-t)\, z_0 + t\, z_1\Big)^2 (z_1 - z_0)\, dt = \tfrac{1}{3}\, z_1^3 - \tfrac{1}{3}\, z_0^3.$$

And here's a third example that may later prove surprising. Let $f(z) = 1/z$ and $\gamma(t) = e^{it}$ for $-\pi < t \le \pi$ giving the closed circular curve Γ_{-1}^{-1}. Notice this closed curve is traversed from $z_0 = -1 + 0\, i$ around the origin in the right-hand sense to $z_1 = -1 + 0\, i$.

$$\oint_{\Gamma_{-1}^{-1}} (1/z)\, dz = \int_{-\pi}^{\pi} \Big(1/e^{it}\Big)\, ie^{it}\, dt = 2\pi i.$$

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1. Let $\Gamma_{z_0}^{z_1}$ be the straight line segment joining z_0 to z_1, but now defined by

$$\widetilde{\gamma}(s) = (1 - 4s^2)\, z_0 + 4s^2\, z_1 \text{ where } 0 \le s \le 1/2 \quad \Longrightarrow \quad \Gamma_{z_0}^{z_1} = \{\widetilde{\gamma}(s) : s \in [0, 1/2]\}.$$

Notice $\widetilde{\gamma}$ here is simply a reparameterization of γ given in the two earlier examples. Use this parameterization to compute the following.

(a) $\displaystyle\int_{\Gamma_{z_0}^{z_1}} z\, dz.$ (b) $\displaystyle\int_{\Gamma_{z_0}^{z_1}} z^2\, dz.$

2. Use both parameterizations describing the straight line segment joining z_0 to z_1, i.e. the one used in the first two examples and the other from the previous exercise, to compute the line integral $\int_{\Gamma_{z_0}^{z_1}} \bar{z}\, dz$. Both parameterizations must yield the same result.

3. Consider two curves:

$$\Gamma_0^2 \text{ is defined by } \{1+e^{i\theta} : \theta \in [-\pi, 0]\}, \quad \widetilde{\Gamma}_0^2 \text{ is defined by } \{1+e^{-i\theta} : \theta \in [-\pi, 0]\}.$$

Both connect $z = 0$ to $z = 2$ but are different curves. Γ_0^2 is a semicircle in the lower half-plane while $\widetilde{\Gamma}_0^2$ is a semicircle in the upper half-plane. Compute the following.

$$\text{(a)} \int_{\Gamma_0^2} z\, dz. \qquad \text{(b)} \int_{\widetilde{\Gamma}_0^2} z\, dz. \qquad \text{(c)} \int_{\Gamma_0^2} \bar{z}\, dz. \qquad \text{(d)} \int_{\widetilde{\Gamma}_0^2} \bar{z}\, dz.$$

My answers: (a) and (b) 2, (c) $2+\pi\, i$, (d) $2-\pi\, i$.

Theorem 1. (FTC) This is a complex version of the *Fundamental Theorem of Calculus.* Suppose $f(z)$ is differentiable on an open set $S \subseteq \mathbb{C}$ and a given smooth simple-curve satisfies $\Gamma_{z_0}^{z_1} \subseteq S$. Then

$$\int_{\Gamma_{z_0}^{z_1}} f'(z)\, dz = f(z_1) - f(z_0).$$

This can be derived as follows. Suppose $\Gamma_{z_0}^{z_1} = \{\gamma(t) : t \in [t_0, t_1]\}$. Insert the curve's parameterization to write

$$\int_{\Gamma_{z_0}^{z_1}} f'(z)\, dz = \int_{t_0}^{t_1} f'(\gamma(t))\, \gamma'(t)\, dt = \int_{t_0}^{t_1} \frac{d}{dt} f(\gamma(t))\, dt,$$

where the rightmost identity above comes from the version of the chain rule given in your "Complex Point Sets" notes. Finally, the Calculus I version of the FTC yields

$$\int_{t_0}^{t_1} \frac{d}{dt} f(\gamma(t))\, dt = f(\gamma(t_1)) - f(\gamma(t_0)) = f(z_1) - f(z_0),$$

which is what we wanted to show.

An alternate form the FTC reads as follows. Suppose $F(z)$ is an *antiderivative* of $f(z)$. That is, $F(z)$ is differentiable on an open set S and $F'(z) = f(z)$. Then, on a given smooth simple-curve $\Gamma_{z_0}^{z_1} \subseteq S$ we have

$$\int_{\Gamma_{z_0}^{z_1}} f(z)\, dz = F(z_1) - F(z_0).$$

4. Let $\Gamma_0^{z_*}$ denote any smooth simple-curve from $z = 0$ to $z = z_*$. Use the FTC (the antiderivative form) to evaluate the following.

(a) $\displaystyle\int_{\Gamma_0^{z_*}} (z^2 + 3z)\, dz$. (c) $\displaystyle\int_{\Gamma_0^{z_*}} \cos(z)\, dz$.

(b) $\displaystyle\int_{\Gamma_0^{z_*}} e^{iz}\, dz$. (d) $\displaystyle\int_{\Gamma_0^{z_*}} \sinh(2z)\, dz$.

5. Suppose $f(z)$ and $g(z)$ are differentiable on an open set S. For any smooth simple-curve $\Gamma_{z_0}^{z_1} \subseteq S$, derive the integration by parts formula

$$\int_{\Gamma_{z_0}^{z_1}} f(z)\, g'(z)\, dz = f(z_1)\, g(z_1) - f(z_0)\, g(z_0) - \int_{\Gamma_{z_0}^{z_1}} f'(z)\, g(z)\, dz.$$

6. Let $\Gamma_0^{z_*}$ denote any smooth simple-curve joining $z = 0$ to $z = z_*$. Use integration by parts to evaluate the following.

(a) $\displaystyle\int_{\Gamma_0^{z_*}} z\, e^{z}\, dz$. (c) $\displaystyle\int_{\Gamma_0^{z_*}} z\,(z+1)^{100}\, dz$.

(b) $\displaystyle\int_{\Gamma_0^{z_*}} (z+2)\, e^{2z}\, dz$. (d) $\displaystyle\int_{\Gamma_0^{z_*}} \sinh^2(z)\, dz$.

Hint for (d). Verify and use $\cosh^2(z) - \sinh^2(z) = 1$.

Notice that up to now, I've restricted my attention to "smooth" simple-curves. A *piecewise smooth simple-curve* (often called a contour or path) is a continuous, nonintersecting curve made by combining a finite number of smooth segments. To extend the complex line integral from a smooth to a piecewise smooth curve Γ, simply write

$$\Gamma = \bigcup_{k=1}^{n} \Gamma_k \quad \Longrightarrow \quad \int_{\Gamma} f(z)\, dz = \sum_{k=1}^{n} \int_{\Gamma_k} f(z)\, dz,$$

and then evaluate each "smooth-curve" integral separately. Even for smooth curves, however, the line integral can be problematic to define. For example, the following function $\gamma : [0,1] \to \mathbb{C}$ given by

$$\gamma(t) = \begin{cases} t + i\, t \sin(1/t) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

is continuous on $[0,1]$ and differentiable of any order on $(0,1)$. So this function defines a smooth simple-curve $\Gamma_0^{1+i\sin(1)}$. But, a fun exercise will show you this curve has infinite arc length. I plan to avoid considering such pathologies during the entirety of our class.

Recall that Green's theorem from Calculus III has the following statement. Let C be a positively oriented, piecewise smooth, simple closed curve in the plane, and let R be the region enclosed by C. Suppose $p(x, y)$ and $q(x, y)$ are real functions with continuous partial derivatives on an open set S containing R. Then, the following identity holds.

$$\oint_C p\,dx + q\,dy = \iint_R \Big(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y}\Big)\,dx\,dy.$$

C is positively oriented means it goes around R in the counterclockwise (right-hand) sense. C is simple and closed means it never intersects itself except at its starting and ending points.

Let $f = u + i\,v$ be a continuously differentiable complex function on $S \supseteq R$ (i.e. u and v have continuous partial derivatives which pointwise on S satisfy the Cauchy-Riemann condition). Use Green's theorem to determine

$$\begin{aligned}\oint_C f(z)\,dz = \oint_C \Big(u + i\,v\Big)\,(dx + i\,dy) &= \Big(\oint_C u\,dx - v\,dy\Big) + i\,\Big(\oint_C v\,dx + u\,dy\Big)\\ &= -\iint_R \Big(v_x + u_y\Big)\,dxdy + i\iint_R \Big(u_x - v_y\Big)\,dxdy.\end{aligned}$$

Cauchy-Riemann says $v_x + u_y = 0$ at every point in R, so the first integral on the right side above is zero. Cauchy-Riemann also says $u_x - v_y = 0$ in R, so the second integral on the right side is zero. This is Cauchy's original derivation of the following.

Theorem 2. (Cauchy) Let Γ denote a piecewise smooth closed simple-curve, and suppose $f(z)$ is continuously differentiable on a open set that contains the region enclosed by Γ. Then

$$\oint_\Gamma f(z)\,dz = 0.$$

We will use this important result many times. In 1900, Édouard Goursat relaxed Cauchy's original assumption that f must be continuously differentiable on S to simply differentiable on S. His stronger result is often called the Cauchy-Goursat theorem.

The so-called converse of Cauchy's theorem, often called Morera's theorem, is also true. Its statement and proof will be delayed to a future homework. A key ingredient in its proof, however, is the following.

Theorem 3. (Antiderivative) Suppose a complex function f is (only) continuous on an open connected set S, and also suppose $\oint_\Gamma f(z)\,dz = 0$ on any closed simple-curve $\Gamma \subseteq S$. Then, there is an antiderivative F such that $F'(z) = f(z)$ for any $z \in S$.

To see this is true, fix $z_0 \in S$ and let $z \in S$ be arbitrary, and let $\Gamma_{z_0}^{z}$ denote a simple-curve contained in S joining z_0 to z. The critical assumption that $\oint_\Gamma f(\zeta)\,d\zeta = 0$ on any closed simple-curve Γ in S gives us the fact that

$$F(z) = \int_{\Gamma_{z_0}^{z}} f(\zeta)\,d\zeta$$

is a well-defined function of z. That is, its value does not depend on which simple-curve (of many possible) is used to join z_0 to z. This is called *path independence.* I will explain in class why here this fact is true. Since $z \in S$ and S is open, $z + \Delta z \in S$ for all $|\Delta z| \neq 0$ small enough. Let $\Gamma_{z}^{z+\Delta z}$ denote the particular *straight-line* curve, parameterized by $\gamma(t) = z + t\,\Delta z$ for $t \in [0,1]$, joining z to $z + \Delta z$, and write

$$\begin{aligned} F(z+\Delta z) &= \int_{\Gamma_{z_0}^{z} \cup \Gamma_{z}^{z+\Delta z}} f(\zeta)\,d\zeta = \int_{\Gamma_{z_0}^{z}} f(\zeta)\,d\zeta + \int_{\Gamma_{z}^{z+\Delta z}} f(\zeta)\,d\zeta \\ &= F(z) + \int_{\Gamma_{z}^{z+\Delta z}} f(\zeta)\,d\zeta = F(z) + \int_0^1 f(z + t\,\Delta z)\,\Delta z\,dt. \end{aligned}$$

Therefore,

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} = \int_0^1 f(z + t\,\Delta z)\,dt.$$

Finally, since f is assumed to be continuous in S, we have

$$\lim_{\Delta z \to 0} \int_0^1 f(z + t\,\Delta z)\,dt = f(z) \quad \Longrightarrow \quad \lim_{\Delta z \to 0} \frac{F(z+\Delta z) - F(z)}{\Delta z} = f(z).$$

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7. Let Γ be given by the closed circular curve $\{e^{it} : -\pi < t \leq \pi\}$. Use this explicit parameterization to determine $\oint_\Gamma z^{-n}\,dz = 0$ for any integer $n >= 2$.

8. Recall from an earlier example it was shown that $\oint_\Gamma z^{-1}\,dz = 2\pi i$ on the particular circular curve given in the previous exercise. Here, let Γ denote an arbitrary closed simple-curve that does not go through $z = 0$. Use this and some theorem quoting to conclude the following.

(a) $\displaystyle\oint_\Gamma z^{-1} dz = 2\pi i$ when $z = 0$ is strictly inside Γ.

(b) $\displaystyle\oint_\Gamma z^{-1} dz = 0$ when $z = 0$ is strictly outside Γ.

9. Consider the integral $\oint_\Gamma 1/(z^2 - 1)\,dz$ where Γ is a close simple-curve that does not go through either $z = -1$ or $z = +1$. Use the results of the previous exercise, and some theorem quoting, to determine the value of this integral depending on Γ. That is, when Γ surrounds $z = -1$, or $z = +1$, or both, or neither.
