Complex Numbers

1a) F's mult ident is unique

Suppose there are two, say $\tilde{1}$ and $\tilde{\tilde{1}}$.

By 3b $\quad \tilde{\tilde{1}}\,\tilde{1} = \tilde{\tilde{1}}$ and $\tilde{1}\,\tilde{\tilde{1}} = \tilde{1}$.

By 2b $\quad \tilde{\tilde{1}}\,\tilde{1} = \tilde{1}\,\tilde{\tilde{1}} \Rightarrow \tilde{1} = \tilde{\tilde{1}}$.

b) The additive inverse of a is unique.

By 3a $\quad a+\tilde{a} = 0$ and $a + \tilde{\tilde{a}} = 0$

So $\quad a+\tilde{a} = a + \tilde{\tilde{a}}$

Add $\tilde{a}$ to both sides

$$\tilde{a} + a + \tilde{a} = \tilde{a} + a + \tilde{\tilde{a}}$$

But $\tilde{a} + a = 0$ so

$$0 + \tilde{a} = 0 + \tilde{\tilde{a}}$$

and since by 2a and 3a

$$0 + \tilde{a} = \tilde{a} + 0 = \tilde{a} \Rightarrow \tilde{a} = \tilde{\tilde{a}}.$$

$$0 + \tilde{\tilde{a}} + \tilde{\tilde{a}} + 0 = \tilde{\tilde{a}}$$

c) $a \neq 0$ then a's mult inverse is unique.

By 5 $\quad a \cdot \tilde{a} = 1$ and $a \cdot \tilde{\tilde{a}} = 1$.

mult both sides by $\tilde{\tilde{a}}$ to get

$$(a \cdot \tilde{a})\tilde{\tilde{a}} = (a \cdot \tilde{\tilde{a}})\tilde{\tilde{a}}$$

By 2b and 1b

$$(a \cdot \tilde{a})\tilde{\tilde{a}} = (\tilde{a} \cdot a)\tilde{\tilde{a}} = \tilde{a}$$

(over)

and $(a\cdot\tilde{\hat{a}})\,\hat{\tilde{a}} = (\tilde{\hat{a}}\cdot a)\,\hat{\tilde{a}} = \tilde{\hat{a}}$

So $\hat{\tilde{a}} = \tilde{\hat{a}}$.

d) $(-1) \equiv$ add inverse of mult ident

Then $(-1)a$ is add inverse of a,

$$1 + (-1) = 0$$

so $(1+(-1))a = 0\cdot a = 0$ from Notes

$$\begin{aligned} &\ \ \ \, \Vert \\ &1a + (-1)a \quad \text{by 6 and 3} \\ &\ \ \ \, \Vert \\ &a + (-1)a \quad \text{by 2b and 3} \end{aligned}$$

So $a + (-1)a = 0 \Rightarrow (-1)a$ is a's add idnt.

e) Suppose $1 = 0$ and the field has more than one number.

From notes $a\cdot 0 = 0$ always.

But if 0 also mult ident then

$$a\cdot 0 = a$$

This says $a = 0$ so field has only one number, i.e. $F = \{0\}$.

So if field has more than one number we must have $1 \neq 0$.

2a) Add & mult on $\mathbb{C}$ are assoc & comm is easy to show. Just let

$z_1 = x_1 + iy_1$

$z_2 = x_2 + iy_2$ and use the rules given

$z_3 = x_3 + iy_3$

b) Dist prop also obvious, use z_1 z_2 & z_3 and the given rules.

c) $\mathbb{C}$'s add idnt

let $0 = \alpha + i\beta$ and $z = x + iy$ arb.

$$z + 0 = x + iy + \alpha + i\beta = x + iy = z$$

Need $x + \alpha = x$ and $y + \beta = y$

$\Rightarrow \alpha = 0 \quad \beta = 0$

$0 = 0 + i0.$

d) Let $1 = \alpha + i\beta$ and $z = x + iy$

$$z \cdot 1 = (x+iy)(\alpha + i\beta)$$
$$= x\alpha - y\beta + i(x\beta + y\alpha) = x + iy$$
$$= z$$

Need $x\alpha - y\beta = x, \quad x\beta + y\alpha = y$

(over)

WLOG take $z \neq 0$ since $0 \cdot$ anything $= 0$.

So we have 2 eqs in 2 unknowns

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{x^2+y^2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{x^2+y^2}\begin{pmatrix} x^2+y^2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

so $\alpha = 1$
$\beta = 0$

$\therefore \boxed{1 = 1 + 0i}$

e) $z + \tilde{z} = 0$, $(x+iy) + (\alpha + i\beta) = 0 + i0$

$(x+\alpha) = 0 \quad (y+\beta) = 0$

$\alpha = -x \quad \beta = -y$

$\boxed{\tilde{z} = (-x) + i(-y)}$

f) $z \neq 0$ $\quad (x+iy)(\alpha + i\beta) = 1 + 0i$

$$\parallel$$

$x\alpha - y\beta + i(x\beta + y\alpha)$

$x\alpha - y\beta = 1$ & $x\beta + y\alpha = 0$

So again, 2 eqs in 2 unknowns

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \to \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{x^2+y^2}\begin{pmatrix} x & y \\ -y & x \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\boxed{\tilde{z} = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)} \Leftarrow \quad = \begin{pmatrix} x/(x^2+y^2) \\ -y/(x^2+y^2) \end{pmatrix}$$

3a) $z = \dfrac{3}{i} = \dfrac{3i}{i^2} = \boxed{-3i}$

b) $z = i(2-i) = 2i - i^2 = \boxed{1+2i}$

c) $z = (1+i)(1-i) = 1+1 = \boxed{2}$

d) $z = (1+i)^2 = (1+i)(1+i) = 1-1+2i$
$= \boxed{2i}$

e) $z = \dfrac{1+i}{1-i} = \dfrac{1+i}{1-i}\dfrac{(1+i)}{(1+i)}$

$= \dfrac{1-1+2i}{1+1} = \boxed{i}$

f) $z = \dfrac{(1+i)^2}{(1-i)^2} = \left(\dfrac{1+i}{1-i}\right)^2 = (i)^2 = \boxed{-1}$

4a) $\overline{\dfrac{1}{z}} = \overline{\dfrac{x-iy}{x^2+y^2}} = \dfrac{x}{x^2+y^2} + i\dfrac{y}{x^2+y^2}$

BTW $z \neq 0$

$\dfrac{1}{\bar{z}} = \dfrac{1}{x-iy} = \dfrac{x+iy}{x^2+y^2} = \dfrac{x}{x^2+y^2} + i\dfrac{y}{x^2+y^2}$

⑥

b) $\overline{z_1+z_2} = \overline{x_1+iy_1+x_2+iy_2}$

$= x_1+x_2-i(y_1+y_2)$

$\overline{z_1}+\overline{z_2} = (x_1-iy_1)+(x_2-iy_2)$

$= x_1+x_2-i(y_1+y_2)$ ✓

c) $\overline{z_1 z_2} = \overline{(x_1+iy_1)(x_2+iy_2)}$

$= \overline{(x_1x_2-y_1y_2)+i(x_1y_2+x_2y_1)}$

$= (x_1x_2-y_1y_2)-i(x_1y_2+x_2y_1)$

$\overline{z_1}\,\overline{z_2} = (x_1-iy_1)(x_2-iy_2)$

$= x_1x_2-y_1y_2-i(x_1y_2+x_2y_1)$ ✓

d) $\overline{z_1 z_2^n} = \overline{z_1 z_2 \cdots z_2}$

$= \overline{z_1}\,\overline{z_2}\cdots\overline{z_2} = \overline{z_1}(\overline{z_2})^n$ ✓

7

5a) $\overline{\dfrac{1}{1+2i}} = \overline{\dfrac{1-2i}{5}} = \boxed{\dfrac{1}{5}} + \dfrac{2}{5}i$

$\dfrac{1}{\overline{1+2i}} = \dfrac{1}{1-2i} = \dfrac{1+2i}{5} = \boxed{\dfrac{1}{5}} + \dfrac{2}{5}i$

b) $\overline{(2+i)+(3-2i)} = \overline{5-i} = \boxed{5+i}$

$\overline{2+i} + \overline{(3-2i)} = 2-i+3+2i = \boxed{5+i}$

c) $\overline{(1+2i)(1-i)} = \overline{1+2+i} = \boxed{3-i}$

$\overline{1+2i}\;\overline{1-i} = (1-2i)(1+i) = 1+2-i$
$= \boxed{3-i}$

d) $\overline{(1-2i)(1+i)^2} = \overline{(1-2i)(1-1+2i)}$
$= \overline{2i+4} = \boxed{4-2i}$

$\overline{1-2i}\,(\overline{1+i})^2 = (1+2i)(1-i)^2$
$= (1+2i)(1-i)(1-i)$
$= (1+2i)(-2i)$
$= -2i+4 = \boxed{4-2i}$

8

6a) $|2e^{i\theta}| = |2\cos\theta + 2i\sin\theta|$

$= \sqrt{4\cos^2\theta + 4\sin^2\theta} = \sqrt{4} = \boxed{2}$

b) $\overline{e^{i\theta}} = \overline{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$

$= \cos(-\theta) + i\sin(-\theta)$

$= \boxed{e^{-i\theta}}$

7a) $z = -1 = 1(\cos(\pi) + i\sin(\pi))$

$= \boxed{e^{\pi i}}$

b) $z = -2i = 2(-i) = 2(\cos(-\pi/2) + i\sin(-\pi/2))$

$= \boxed{2e^{-\pi/2\, i}}$

c) $z = 1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$

$= \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4)) = \boxed{\sqrt{2}e^{i\frac{\pi}{4}}}$

d) $z = 2-2i = \sqrt{4+4}\left(\frac{2}{\sqrt{8}} - \frac{2}{\sqrt{8}}i\right)$

$\sqrt{8}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i\right)$ (over)

⑨

$$\sqrt{8}\left(\cos(-\pi/4)+i\sin(-\pi/4)\right)=\boxed{\sqrt{8}\,e^{-\frac{\pi}{4}i}}$$

8a) $z=2e^{\pi i}=2(\cos(\pi)+i\sin(\pi))=\boxed{-2}$

b) $z=e^{\frac{3\pi}{4}i}=\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)$

$\frac{3\pi}{4}$

$$=\boxed{-\frac{\sqrt{2}}{2}}+i\,\frac{\sqrt{2}}{2}$$

c) $z=3e^{\pi i}=3(\cos(\pi)+i\sin(\pi))$

$$=\boxed{-3}$$

d) $z=4e^{-\frac{\pi}{2}i}=4(\cos(-\pi/2)+i\sin(-\pi/2))$

$$=4(0-i)=\boxed{-4i}$$

9a) $z^5=1=e^{2n\pi i}$ $\quad n=0,\pm 1,\ldots$

$\parallel$

$|z|^5 e^{i5\arg(z)}$

$\Rightarrow |z|=1$ & $\arg(z)=\frac{2n\pi}{5}$ (over) so $\arg(z) \in (-\pi, \pi]$

$\Rightarrow n=0,\ n=\pm 1,\ n=\pm 2$

$$\boxed{\begin{array}{ll} z_1 = 1 \cdot e^0 & \\ z_2 = 1\, e^{+\frac{\pi}{5}i} & z_3 = 1\, e^{-\frac{\pi}{5}i} \\ z_4 = 1\, e^{\frac{2\pi}{5}i} & z_5 = e^{-\frac{2\pi}{5}i} \end{array}}$$

b) $z^6 = -1 = e^{\pi i + 2n\pi i}$

$\quad \| $

$|z|^6 e^{i\,6\arg(z)}$

$\Rightarrow |z| = 1$ & $\arg(z) = \frac{\pi}{6} + \frac{2n\pi}{6}$

$= \frac{\pi}{6} + \frac{n\pi}{3}$

So since $\arg(z) \in (-\pi, \pi]$

$\Rightarrow n=0,\ n=\pm 1\ \ n=\pm 2,\ n=-3$

so
$$\left[\begin{array}{ll} z_1 = e^{\frac{\pi}{6}i} & \\ z_2 = e^{(\frac{\pi}{6}+\frac{\pi}{3})i} & z_3 = e^{(\frac{\pi}{6}-\frac{\pi}{3})i} \\ z_4 = e^{(\frac{\pi}{6}+\frac{2\pi}{3})i} & z_5 = e^{(\frac{\pi}{6}-\frac{2}{3}\pi)i} \\ z_6 = e^{(\frac{\pi}{6}-\pi)i} & \end{array}\right]$$

11

c) $z^2 = i = e^{\frac{\pi}{2}i + 2n\pi i}$

$\quad \| $

$|z|^2 e^{i\,2\arg(z)}$

$\Rightarrow |z| = 1$ & $\arg(z) = \frac{\pi}{4}i + n\pi i$

So $\arg(z) \in (-\pi, \pi]$

$\Rightarrow n = 0 \quad n = -1$

So $\left[z_1 = e^{\frac{\pi}{4}i} \quad z_2 = e^{(\frac{\pi}{4} - \pi)i} \right]$

d) $z^3 = -i = e^{(-\frac{\pi}{2}i + 2n\pi i)}$

$\quad \|$

$|z|^3 e^{3\arg(z)i}$

$\Rightarrow |z| = 1$ & $\arg(z) = -\frac{\pi}{6} + \frac{2}{3}n\pi$

So since $\arg(z) \in (-\pi, \pi]$

$\rightarrow n = 0, \quad n = -1, \quad n = 1$

$\left[z_1 = e^{-\frac{\pi}{6}i} \quad z_2 = e^{(-\frac{\pi}{6} + \frac{2}{3}\pi)i} \quad z_3 = e^{(-\frac{\pi}{6} - \frac{2}{3}\pi)i} \right]$

(12)

e) $z^3 = 8\sqrt{2}(1+i)$

$= 8 \cdot 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i\right)$

$= 16\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

$= 16 e^{i\frac{\pi}{4} + 2n\pi i}$

$\parallel$

$|z|^3 e^{i3\arg(z)}$

$\Rightarrow |z| = \sqrt[3]{16}$ & $\arg(z) = \frac{\pi}{12} + \frac{2}{3}n\pi$

Since $\arg(z) \in (-\pi, \pi]$

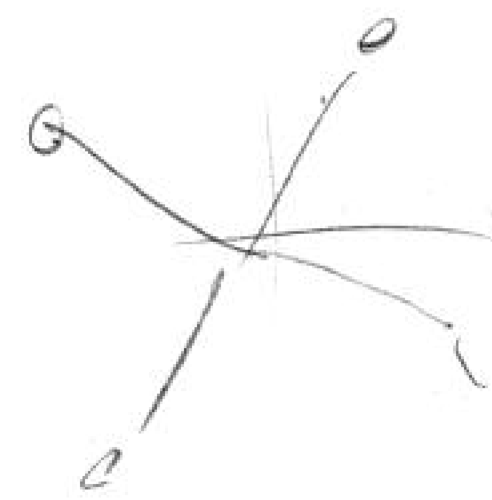
$n=0 \quad n=-1 \quad n=1$

$$\left\{\begin{aligned} z_1 &= \sqrt[3]{16}\, e^{\frac{\pi i}{12}} \\ z_2 &= \sqrt[3]{16}\, e^{\left(\frac{\pi}{12} + \frac{2}{3}\pi\right) i} \\ z_3 &= \sqrt[3]{16}\, e^{\left(\frac{\pi}{12} - \frac{2}{3}\pi\right) i} \end{aligned}\right.$$

(13)

f) $z^4 = \sqrt{2}(1-i)$

$= 2\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} i\right)$



$= 2\left(e^{\left(-\frac{\pi}{4}+2n\pi\right) i}\right)$

$|z|^4 e^{i4\arg(z)}$

$\Rightarrow |z| = \sqrt[4]{2} \quad \arg(z) = \frac{-\pi}{16} + \frac{1}{2}n\pi$

Since $\arg(z) \in (-\pi, \pi]$

$n=0 \quad n=1, \ n=2 \quad n=-1$

$$\begin{aligned}
z_1 &= \sqrt[4]{2}\, e^{-\frac{\pi}{16} i} \\
z_2 &= \sqrt[4]{2}\, e^{\left(-\frac{\pi}{16}+\frac{\pi}{2}\right) i} \\
z_3 &= \sqrt[4]{2}\, e^{\left(-\frac{\pi}{16}+\pi\right) i} \\
z_4 &= \sqrt[4]{2}\, e^{\left(-\frac{\pi}{16}-\frac{\pi}{2}\right) i}
\end{aligned}$$

10) Taylor's Thrm gives

for $|z|<1$ $\quad \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$

This is the geometric series.

Let $z = re^{i\theta}$

$$\frac{1}{1-re^{i\theta}} = \sum_{n=0}^{\infty} r^n e^{in\theta}$$

$$= \sum_{n=0}^{\infty} r^n \cos(n\theta) + i \sum_{n=0}^{\infty} r^n \sin(n\theta)$$

$$\frac{1}{(1-re^{i\theta})} \frac{(1-re^{-i\theta})}{(1-re^{-i\theta})}$$

$$= \frac{1-re^{-i\theta}}{1+r^2-(re^{i\theta}+re^{-i\theta})}$$

But from Euler

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

So

$$\frac{1}{1-re^{i\theta}} = \frac{1-re^{-i\theta}}{1+r^2-2r\cos\theta}$$

$$= \frac{1-r\cos(-\theta)}{1+r^2-2r\cos\theta} - i\,\frac{r\sin(-\theta)}{1+r^2-2r\cos\theta}$$

$$= \frac{1-r\cos\theta}{1+r^2-2r\cos\theta} + i\,\frac{r\sin\theta}{1+r^2-2r\cos\theta}$$

$$= \sum_{n=0}^{\infty} r^n e^{in\theta}$$

$$= \sum_{n=0}^{\infty} r^n \cos(n\theta) + i\sum_{n=0}^{\infty} r^n \sin(n\theta)$$

Equate real & imag parts to get both formulae.