Complex Numbers

In mathematics, a *field* F is a set of *numbers* on which two binary operations, addition $(+)$ and multiplication $(\cdot)$, are defined. A binary operation on F is a mapping $F \times F \to F$ which sends each ordered pair of elements of F to a uniquely determined element of F. These operations are required to satisfy the following field properties. Let a, b and c denote arbitrary numbers in F.

(1) Addition and multiplication are associative:

$$a + (b + c) = (a + b) + c, \text{ and } a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

(2) Addition and multiplication are commutative:

$$a + b = b + a, \text{ and } a \cdot b = b \cdot a.$$

(3) The field has an additive and a multiplicative identity:

$$\exists\, 0 \in F \text{ such that } a + 0 = a, \text{ and } \exists\, 1 \in F \text{ such that } a \cdot 1 = a.$$

It is standard to denote the additive identity by 0 and multiplicative identity by 1.

(4) Has additive inverses:

$$\forall\, a \in F,\ \exists\, (-a) \in F \text{ such that } a + (-a) = 0.$$

Here, the notation $(-a)$ is used to denote the additive inverse of a.

(5) Has multiplicative inverses:

$$\forall\, a \in F \setminus \{0\},\ \exists\, a^{-1} \in F \text{ such that } a \cdot a^{-1} = 1.$$

Here, the notation a^{-1} denotes the multiplicative inverse of numbers $a \neq 0$.

(6) Distributivity:

$$a \cdot (b + c) = (a \cdot b) + (a \cdot c).$$

Several facts not explicitly stated above can be deduced from the given field properties. For example, a field's additive identity is necessarily unique. To see this is true, suppose it has two, say 0_1 and 0_2.

From (3a) $\quad 0_2 + 0_1 = 0_2$ and $0_1 + 0_2 = 0_1$.
From (2a) $\quad 0_1 + 0_2 = 0_2 + 0_1$.
So $\quad 0_1 = 0_2$.

It must also be true that for any $a \in F$ we have $0 \cdot a = 0$.

From (3a)	$0 + 0 = 0.$
So	$a \cdot (0 + 0) = a \cdot 0.$
From (6)	$a \cdot (0 + 0) = a \cdot 0 + a \cdot 0.$
So	$a \cdot 0 + a \cdot 0 = a \cdot 0.$
Add the additive inverse of $a \cdot 0$ to both sides.	
So	$a \cdot 0 = 0.$
From (2a)	$0 \cdot a = a \cdot 0 = 0.$

When $1 \neq 0$, do you see why 0 can never have a multiplicative inverse?

Several other properties can be deduced from the particular field properties listed above.

1. Let F denote a field. Prove the following. (a) F's multiplicative identity must be unique. (b) Each number's additive inverse must be unique. (c) Each nonzero number's multiplicative inverse must be unique. (d) If (-1) denotes the additive inverse of the multiplicative identity, then $(-1) \cdot a$ is the additive inverse of a. (e) If the field has more than one number, the additive identity and the multiplicative identity are necessarily distinct.

We are all familiar with the field of real numbers $\mathbb{R}$. The *standard form* of a *complex number* is

$$z = x + iy, \text{ where } x \in \mathbb{R} \text{ and } y \in \mathbb{R}.$$

x is called the *real part* of z and is denoted by $\mathrm{Re}(z)$. y is called the *imaginary part* of z and is denoted by $\mathrm{Im}(z)$. The placeholder i is called the *imaginary unit.* Notice that a complex number can be viewed as a two-dimensional vector $(x, y) \in \mathbb{R}^2$. Two complex numbers, z_1 and z_2, are equal if and only if both $\mathrm{Re}(z_1) = \mathrm{Re}(z_2)$ and $\mathrm{Im}(z_1) = \mathrm{Im}(z_2)$.

Consider two complex numbers, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Complex *addition* is defined by

$$(A) \qquad z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 \equiv (x_1 + x_2) + i(y_1 + y_2).$$

Complex addition can be viewed as the usual notion of vector addition on $\mathbb{R}^2$. Complex *multiplication* is defined by letting $i^2 = -1$ and multiplying out

$$(M) \qquad z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) \equiv (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

The set of complex numbers together with operations (A) and (M) is denoted by $\mathbb{C}$.

The complex *conjugate* of $z = x + iy$ is denoted by $\bar{z}$ and is given by

$$(C) \qquad \bar{z} = x - iy.$$

When z is viewed as a vector $(x, y) \in \mathbb{R}^2$, its conjugate is the vector's reflection across the x-axis. Observe that given $z = x + iy$ we have

$$\tfrac{1}{2}(z + \bar{z}) = x, \;\; \tfrac{1}{2}(z - \bar{z}) = iy \text{ and } z\bar{z} = \bar{z}z = x^2 + y^2.$$

The *modulus* of $z = x + iy$, denoted by $|z|$, is given by

$$|z| \equiv \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

and can be viewed as the Euclidean length of the two-dimensional vector (x, y).

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2. Here, you'll verify that $\mathbb{C}$ is a field. (a) Verify that addition and multiplication are associative and commutative. (b) Verify the distributive property. (c) Determine $\mathbb{C}$'s additive identity. (d) Determine $\mathbb{C}$'s multiplicative identity. (e) For $z = x + iy$, determine its additive inverse. (f) For $z = x + iy \neq 0 + i\,0$, determine its multiplicative inverse. Answers: (c) $0 + i\,0$, (d) $1 + i\,0$, (e) $(-x) + (-y)\,i$, (f) $(x/(x^2 + y^2)) + i\,(-y/(x^2 + y^2))$.

3. Write the following complex numbers z in standard $x + iy$ form.

(a) $z = \dfrac{3}{i}$ (c) $z = (1 + i)(1 - i)$ (e) $z = \dfrac{1 + i}{1 - i}$

(b) $z = i(2 - i)$ (d) $z = (1 + i)^2$ (f) $z = \dfrac{(1 + i)^2}{(1 - i)^2}$

4. Establish the following.

(a) $\overline{(1/z)} = 1/\bar{z}$ (c) $\overline{z_1 z_2} = \bar{z}_1\, \bar{z}_2$

(b) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (d) $\overline{z_1 z_2^n} = \bar{z}_1 (\bar{z}_2)^n$, ($n$ is an integer)

5. Write separately the left and right side of following in standard form to corroborate the previous exercise's general results.

(a) $\overline{(1/(1 + 2i))} = 1/(1 - 2i)$.

(b) $\overline{(2 + i) + (3 - 2i)} = (2 - i) + (3 + 2i)$.

(c) $\overline{(1 + 2i)(1 - i)} = (1 - 2i)(1 + i)$.

(d) $\overline{(1 - 2i)(1 + i)^2} = (1 + 2i)(1 - i)^2$.

Historically, complex numbers arose out of attempts to find roots of and to eventually factor polynomial equations

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0.$$

Even when p's coefficients, $a_n, \ldots, a_0$, are real numbers, its roots may not be real. For example, we find from the quadratic formula that

$$z^2 - 2z + 2 = 0 \quad \Rightarrow \quad z = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i.$$

The fact that this example's roots appear in complex conjugate pairs is not a fluke coming from the quadratic formula. In general, when a polynomial has real coefficients, its nonreal roots always appear in conjugate pairs. To see this, consider a general nth degree polynomial with real coefficients, say $p(z)$, and suppose $p(z_*) = 0$. Use the results of exercise 4 above together with the facts that $\overline{a_k} = a_k$ for each k to see

$$\begin{aligned}
0 = \overline{0} = \overline{p(z_*)} &= \overline{a_n z_*^n + a_{n-1} z_*^{n-1} + \cdots + a_0} \\
&= \overline{a_n z_*^n} + \overline{a_{n-1} z_*^{n-1}} + \cdots + \overline{a_0} = a_n \bar{z}_*^n + a_{n-1} \bar{z}_*^{n-1} + \cdots + a_0 \\
&= p(\bar{z}_*).
\end{aligned}$$

Therefore, the complex roots of a polynomial with real coefficients are either real or when nonreal appear in conjugate pairs.

In 1748 Leonhard Euler first presented his celebrated formula

$$e^{ix} = \cos(x) + i \sin(x), \text{ for any } x \in \mathbb{R}.$$

Recall the following Taylor series expansions you studied in Calculus II:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}, \quad \sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Each series has an infinite radius of convergence. From these, Euler's formula can be derived as follows. Partition the index set into sets of even and odd indices

$$\{0, 1, 2, 3, 4, 5, \ldots\} = \{0, 2, 4, \ldots\} \cup \{1, 3, 5, \ldots\}$$

in order to write

$$e^{ix} = \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \left(\sum_{k=0}^{\infty} \frac{(ix)^{2k}}{(2k)!} \right) + \left(\sum_{k=0}^{\infty} \frac{(ix)^{2k+1}}{(2k+1)!} \right) = (I) + (II).$$

Since $i^{2k} = (-1)^k$ and $i^{2k+1} = i\,(-1)^k$, conclude the following

$$(I) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = \cos(x), \quad (II) = \sum_{k=0}^{\infty} i\,(-1)^k \frac{x^{2k+1}}{(2k+1)!} = i \sin(x).$$

Recall that a complex number $z = x + iy$ can be viewed as a vector $(x, y) \in \mathbb{R}^2$. In Calculus III, you learned to write such vectors in *polar coordinates*

$$x = r\cos(\theta), \; y = r\sin(\theta), \quad \text{where } r = \sqrt{x^2 + y^2} \text{ and } \theta = \operatorname{atan2}(y, x).$$

Note that the function $\operatorname{atan2}(y, x) \in (-\pi, \pi]$ calculates the four-quadrant inverse tangent of a point (x, y). It returns the angle θ between the positive x-axis and the ray to the point. It is standard notation in complex analysis to write $\theta = \mathrm{atan2}(y, x) \equiv \arg(z)$. So this together with Euler's formula allows us to write z's polar form as

$$z = |z|\, e^{i\arg(z)}, \quad \text{where } |z| = r \text{ and } \arg(z) = \theta \in (-\pi, \pi].$$

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6. For $\theta \in \mathbb{R}$ establish the following.

   (a) $|2e^{i\theta}| = 2$. (a) $\overline{e^{i\theta}} = e^{-i\theta}$.

7. Write the following standard form complex numbers in polar form.

   (a) $z = -1$. (b) $z = -2i$. (c) $z = 1 + i$. (d) $z = 2 - 2i$.

8. Write the following polar form complex numbers in standard form.

   (a) $z = 2e^{\pi i}$. (b) $z = e^{3\pi\, i/4}$. (c) $z = 3e^{\pi i}$. (d) $z = 4e^{-\pi\, i/2}$.

Let's use the polar representation for z to find all solutions to the equation $z^4 = 16$.

$$z^4 = \left(|z|e^{i\arg(z)}\right)^4 = |z|^4\, e^{i4\arg(z)} = 16 = 16\, e^{i2n\pi} \text{ for arbitrary integer } n.$$

$|z|^4 = 16$ implies $|z| = 2$ and $4\arg(z) = 2n\pi$ implies $\arg(z) = \frac{n}{2}\pi$. Since $\arg(z) \in (-\pi, \pi]$ we have $\arg(z)$ equals $-\frac{1}{2}\pi$, or $\frac{0}{2}\pi$, or $\frac{1}{2}\pi$, or $\frac{2}{2}\pi$. Using $z = |z|\, e^{i\arg(z)}$ yields four solutions

$$z_1 = -2i,\ z_2 = 2,\ z_3 = 2i,\ z_4 = -2.$$

A second example may prove helpful. Find all solutions to $z^2 = 1+i$. In polar coordinates write $1 + i = \sqrt{2}\, e^{i\pi/4} = \sqrt{2}e^{i(\pi/4+2n\pi)}$ for arbitrary integer n. So

$$z^2 = \left(|z|e^{i\arg(z)}\right)^2 = |z|^2\, e^{2i\arg(z)} = \sqrt{2}e^{i(\pi/4+2n\pi)},$$

which implies $|z| = \sqrt[4]{2}$ and $\arg(z) = \pi/8 + n\pi$. Since $\arg(z) \in (-\pi, \pi]$, $\arg(z) = \pi/8 + n\pi$ with $n = 0$ or $n = -1$. So the two solutions are

$$z_1 = \sqrt[4]{2}\, e^{\pi i/8}, \quad z_2 = \sqrt[4]{2}\, e^{\pi i/8 - \pi i} = -z_1.$$

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9. Find all solutions to the following.

   (a) $z^5 = 1$ (c) $z^2 = i$ (e) $z^3 = 8\sqrt{2}\,(1 + i)$

   (b) $z^6 = -1$ (d) $z^3 = -i$ (f) $z^4 = \sqrt{2}\,(1 - i)$

10. For any $|r| < 1$ and any θ derive

$$\text{(a)} \quad 1 + \sum_{n=1}^{\infty} r^n \cos(n\theta) = \frac{1 - r\cos\theta}{1 + r^2 - 2r\cos\theta}, \qquad \text{(b)} \quad \sum_{n=1}^{\infty} r^n \sin(n\theta) = \frac{r\sin\theta}{1 + r^2 - 2r\cos\theta}.$$

Hint: When $|z| < 1$ the geometric series gives $\sum_{n=0}^{\infty} z^n = 1/(1-z)$.
