Complex Point Sets, Complex Curves and Other Sundries

In the following, let $S \subseteq \mathbb{C}$, $U \subseteq \mathbb{C}$ and $V \subseteq \mathbb{C}$.

$(*)$ A point $z_* \in S$ is called an *interior point* of S when

$$B_\delta(z_*) \equiv \{z \in \mathbb{C} : |z - z_*| < \delta\} \subseteq S \quad \text{for some } \delta > 0.$$

The notation $B_\delta(z_*)$ represents what is called an *open ball* of radius δ centered at z_*.

$(*)$ A set S is *open* when each of its points is interior to itself.

$(*)$ Two sets U and V are *disjoint* when $U \cap V$ is empty.

$(*)$ A set S is *disconnected* if there are two open disjoint sets U and V such that $S \cap U$ and $S \cap V$ are nonempty and $S = (S \cap U) \cup (S \cap V)$.

$(*)$ A set is *connected* when it is not disconnected.

Basically, a set is disconnected when it can be separated into points inside two disjoint open subsets. A set is connected when it can not be separated in such a manner.

Let I denote a nonempty interval of real numbers, and consider functions of the form

$$\gamma : I \to \mathbb{C}, \text{ where } \gamma(t) = x(t) + i\,y(t), \text{ and } x : I \to \mathbb{R}, \quad y : I \to \mathbb{R}.$$

In most cases here, I will assumed the interval $I = [t_0, t_1]$ with $t_0 < t_1$.

$(*)$ A *curve* is the range, $\Gamma = \{z = \gamma(t) : t \in I\}$, of a continuous function $\gamma : I \to \mathbb{C}$.

In a very general setting, the continuous image of a connected set is connected. (I'll prove this fact in class if you ask me to.) Therefore, a curve is composed of a connected, locally one dimensional set of points. A function is said to be *one-to-one* when every output value of the function comes from a unique input. That is, a function γ is one-to-one says $\gamma(t) = \gamma(t_*) \implies t = t_*$.

$(*)$ A *simple-curve*, Γ, is the range of a continuous, one-to-one function $\gamma : I \to \mathbb{C}$.

I will use the notation

$$\Gamma_{z_0}^{z_1} = \{z = \gamma(t) : z \in [t_0, t_1]\}$$

to signify a simple-curve which goes from $z_0 = \gamma(t_0)$ to $z_1 = \gamma(t_1)$. Note that a simple-curve does not intersect itself.

$(*)$ A *smooth simple-curve*, Γ, is given by a continuous, one-to-one function $\gamma : I \to \mathbb{C}$ that has a continuous, nonzero derivative $\gamma'(t) \neq 0$ at every interior $t \in I$.

Recall $\gamma(t) = x(t) + i\,y(t)$ where $x(t)$ and $y(t)$ are Calculus I type real valued functions of t. So γ' is given by $\gamma'(t) = x'(t) + i\,y'(t)$ and $\gamma' \neq 0$ says $|\gamma'| = \sqrt{(x')^2 + (y')^2} \neq 0$. Therefore, γ generates a point set Γ which is guaranteed to have a well defined and continuously varying tangent.

$(*)$ A *closed simple-curve* is a simple-curve that closes on itself; that is $\gamma(t_1) = \gamma(t_0)$.

The Jordan curve theorem states that any non-intersecting, closed continuous loop in the plane (a Jordan curve) divides the plane into exactly two connected components: a bounded interior region and an unbounded exterior region, with the curve serving as the common boundary for both.

$(*)$ A set S is *path-connected* when there is a simple-curve $\Gamma_{z_0}^{z_1} \subseteq S$ connecting any two distinct points $z_0 \in S$ and $z_1 \in S$.

A path-connected set is always connected. While many connected sets (e.g. open connected sets) are also path-connected, there are examples of connected sets which are not path-connected.

Here are a few examples.

$S_1 = \{z \in \mathbb{C} : \mathrm{Re}(z) \neq 0\}$ is open but not connected.

$S_2 = \{z \in \mathbb{C} : \mathrm{Re}(z) > 0\}$ is open and path-connected.

$S_3 = \{z \in \mathbb{C} : \mathrm{Re}(z) \geq 0\}$ is not open but is path-connected.

$S_4 = \{z = t + i\,\sin(1/t) : t > 0\}$ is path-connected.

$S_5 = \{z = t + i\,\sin(1/t) : t \neq 0\}$ is not connected.

$S_6 = \{z = i\,t : -1 \leq t \leq 1\} \cup S_5$ is connected but not path-connected.

In your Calculus III class you already saw the next (and last) set type I'll define here. I personally consider the type name to be somewhat misleading, but be aware, the name is very commonly used in mathematics.

$(*)$ A *simply-connected* set $S \subseteq \mathbb{C}$ is a path-connected set where every <u>closed</u> simple-curve inside S can be continuously shrunk to a single point.

Loosely speaking, a simply-connected set has no holes inside itself. For example, the set $\{z \in \mathbb{C} : z \neq 0\}$ is path-connected but is not simply-connected.

Sets of complex numbers (or multi-d sets in general) are much more complicated (and therefore more interesting) compared to sets of real numbers. Check out, for example, this link: https://homepages.math.uic.edu/~culler/chaos/.

I will close by giving you a slight variant of the chain rule. Suppose $\gamma : I \to \mathbb{C}$ is differentiable at an interior $t_* \in I$. Also, suppose a complex function $f(w)$ is differentiable at $w_* = \gamma(t_*)$. Then, the composition $h(t) \equiv f(\gamma(t))$ is differentiable at $t = t_*$, and its derivative there is given by

$$h'(t_*) = f'(\gamma(t_*))\,\gamma'(t_*).$$

To see this, take $\Delta t \neq 0$ and consider

$$\frac{h(t_* + \Delta t) - h(t_*)}{\Delta t} = \left(\frac{f(\gamma(t_* + \Delta t)) - f(\gamma(t_*))}{\gamma(t_* + \Delta t) - \gamma(t_*)}\right)\left(\frac{\gamma(t_* + \Delta t) - \gamma(t_*)}{\Delta t}\right).$$

i.e. divide and multiply by $\Delta w \equiv \gamma(t_* + \Delta t) - \gamma(t_*)$. Note however, it's possible for Δw in the denominator above to be zero even though Δt is not zero. To get around this difficulty, a standard trick is to let

$$\begin{aligned} &e_{w_*}(w) \equiv \begin{cases} (f(w) - f(w_*))/(w - w_*) - f'(w_*) & \text{if } w \neq w_* \\ 0 & \text{if } w = w_* \end{cases} \\ &\Longrightarrow \quad e_{w_*}(w) \text{ is well defined and } \lim_{w \to w_*} e_{w_*}(w) = 0. \end{aligned}$$

So, taking $w = \gamma(t_* + \Delta t)$ and $w_* = \gamma(t_*)$, we find for $\Delta t \neq 0$

$$\frac{h(t_* + \Delta t) - h(t_*)}{\Delta t} = (f'(w_*) + e_{w_*}(w))\left(\frac{w - w_*}{\Delta t}\right).$$

Since $\Delta t \to 0$ implies $w \to w_*$, we see the first bracketed term above tends to $f'(w_*)$. Also, the second bracketed term tends to $\gamma'(t_*)$. These together give us the desired result.