

The Taylor Series and Analytic Functions via Power Series

The truncated geometric series formula is derived as follows. For $z \neq 1$ we have

$$\begin{aligned}
 S_n &= 1 + z + z^2 + \cdots + z^{n-1} + z^n \\
 zS_n &= z + z^2 + z^3 + \cdots + z^n + z^{n+1} \\
 \implies S_n - zS_n &= 1 - z^{n+1} \implies (1 - z)S_n = 1 - z^{n+1} \\
 \implies S_n &\equiv \sum_{k=0}^n z^k = \frac{1 - z^{n+1}}{1 - z} = \frac{1}{1 - z} - \frac{z^{n+1}}{1 - z}.
 \end{aligned}$$

Suppose f is analytic on an open set $S \subseteq \mathbb{C}$, let $z_0 \in S$ and take $\rho > 0$ so that the region surrounded by the circle $C_\rho(z_0)$ lies in S . For $|z - z_0| < \rho$, the basic Cauchy integral formula gives us

$$\begin{aligned}
 f(z) &= \frac{1}{2\pi i} \oint_{C_\rho(z_0)} \frac{f(\zeta)}{\zeta - z} d\zeta \\
 &= \frac{1}{2\pi i} \oint_{C_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0) - (z - z_0)} d\zeta \\
 &= \frac{1}{2\pi i} \oint_{C_\rho(z_0)} \left(\frac{1}{\zeta - z_0} \right) \left(\frac{1}{1 - (z - z_0)/(\zeta - z_0)} \right) f(\zeta) d\zeta.
 \end{aligned}$$

Write the second bracketed term above as

$$\left(\frac{1}{1 - (z - z_0)/(\zeta - z_0)} \right) \equiv \frac{1}{1 - r} \quad \text{where } r = \frac{z - z_0}{\zeta - z_0} \neq 1,$$

and use the truncated geometric series derived above to rewrite this term as

$$\left(\frac{1}{1 - r} \right) = \sum_{k=0}^n r^k + \frac{r^{n+1}}{1 - r}.$$

Insert this identity into the integral above, and then apply Cauchy's general integral formula to discover the celebrated *Taylor polynomial with remainder*

$$\begin{aligned}
 f(z) &= \sum_{k=0}^n \left(\frac{1}{2\pi i} \oint_{C_\rho(z_0)} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta \right) (z - z_0)^k + R_n \\
 &= \sum_{k=0}^n \left(\frac{1}{k!} f^{(k)}(z_0) \right) (z - z_0)^k + R_n,
 \end{aligned}$$

where the remainder term, R_n , is given explicitly by

$$R_n = \frac{1}{2\pi i} \oint_{C_\rho(z_0)} \left(\frac{1}{\zeta - z} \right) \left(\frac{z - z_0}{\zeta - z_0} \right)^{n+1} f(\zeta) d\zeta.$$

By applying the triangle inequality, the remainder term can be bounded as

$$\begin{aligned} |R_n| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{z_0 + \rho e^{i\theta} - z} \right| \left| \frac{z - z_0}{\rho} \right|^{n+1} |f(z_0 + \rho e^{i\theta})| d\theta \\ &\leq \frac{1}{\rho - |z - z_0|} \left(\frac{|z - z_0|}{\rho} \right)^{n+1} \max_{\theta} |f(z_0 + \rho e^{i\theta})|. \end{aligned}$$

Finally, since $|z - z_0| < \rho$ we always have $\lim_{n \rightarrow \infty} R_n = 0$. We now state what was just shown as a theorem.

THEOREM 1. (Taylor) Suppose $f(z)$ is analytic on an open set S , and $z_0 \in S$. Then, there is a $\rho > 0$ such that whenever z satisfies $|z - z_0| < \rho$ we have

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n.$$

An important remark is in order here. The *radius of convergence* of a Taylor series, say R_c , is the largest ρ possible in the previous theorem. That is, for a given analytic function $f(z)$, its Taylor series around z_0 has $R_c = |z_* - z_0|$ where z_* is the nearest point to z_0 where $f(z_*)$ is either undefined or not analytic. Here're two examples to illustrate this.

Use the geometric series to determine

$$\frac{1}{z^2 + 1} = \frac{1}{1 - (-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = \sum_{n=0}^{\infty} (-1)^n z^{2n}.$$

Obviously this function is undefined when $z = \pm i$ and analytic everywhere else. Therefore, this Taylor series will have radius of convergence $R_c = |\pm i - 0| = 1$.

For $n = 0, 1, 2, \dots$ observe that $(d^n/dz^n) e^z|_{z=0} = 1$. So the Taylor series for e^z around $z_0 = 0$ is

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n.$$

The function e^z is entire, therefore this Taylor series has $R_c = \infty$.

1. Use the fact that for $|z| < 1$ we have $1/(1 - z) = \sum_{n=0}^{\infty} z^n$ to determine the Taylor series for the following functions. Also, state the radius of convergence for each.

$$\begin{array}{ll} \text{(a) } f(z) = \frac{1}{2 - z} \text{ around } z_0 = 0. & \text{(c) } f(z) = \frac{1 + z}{1 - z} \text{ around } z_0 = 0. \\ \text{(b) } f(z) = \frac{1}{2 - z} \text{ around } z_0 = 1. & \text{(d) } f(z) = \frac{2 + z}{2 - z} \text{ around } z_0 = 1. \end{array}$$

2. Use the fact that the Taylor series for the entire function $e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$ to determine the Taylor series for the following functions. Also, conclude each has an infinite radius of convergence.

(a) $f(z) = e^{iz}$ around $z_0 = 0$. (c) $f(z) = ze^z$ around $z_0 = 0$.

(b) $f(z) = e^{iz}$ around $z_0 = \pi$. (d) $f(z) = ze^z$ around $z_0 = 1$.

For part (d) I got $ze^z = e + \sum_{n=1}^{\infty} e \left(\frac{1}{(n-1)!} + \frac{1}{n!} \right) (z-1)^n$.

3. Use your answer in 2(d) above to compute the following.

(a) $\left. \frac{d}{dz}(ze^z) \right|_{z=1}$. (b) $\left. \frac{d^{25}}{dz^{25}}(ze^z) \right|_{z=1}$.

4. Use your results from 2(a) and 2(b) to determine the Taylor series for the following.

(a) $f(z) = \cos(z)$ around $z_0 = 0$. (c) $f(z) = \cos(z)$ around $z_0 = \pi$.

(b) $f(z) = \sin(z)$ around $z_0 = 0$. (d) $f(z) = \sin(z)$ around $z_0 = \pi$.

Often we are given a function that is exclusively defined in terms of a *power series*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

where $a_0, a_1, a_2, \dots$ are given complex numbers. For example, the power series

$$f(z) \equiv \sum_{n=1}^{\infty} \frac{n!}{n^n} z^n$$

locally defines an analytic function on $|z| < e$. (We will show this later.) Another interesting example comes from probability theory. The *error function* can be defined by the power series

$$\operatorname{erf}(z) \equiv \sum_{n=0}^{\infty} \left(\frac{2}{\sqrt{\pi}} \frac{(-1)^n}{(2n+1)n!} \right) z^{2n+1} = \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{5 \cdot 2!} - \frac{z^7}{7 \cdot 3!} + \cdots \right).$$

and this power series (also shown later) defines an entire complex function.

You learned the *root test* and the *ratio test* in Calculus II to test for absolute convergence of the series $\sum_{n=0}^{\infty} b_n$.

Root Test: $\limsup_{n \rightarrow \infty} \sqrt[n]{|b_n|} = \rho$. Ratio Test: $\lim_{n \rightarrow \infty} \frac{|b_{n+1}|}{|b_n|} = \rho$.

When $\rho > 1$, the series is divergent. When $\rho < 1$, the series is absolutely convergent. Either test can be used to determine the radius of convergence of a power series. However, as you probably recall, the ratio test may fail when the limit of the ratio fails to exist. You might also recall that the root test can be difficult to evaluate, especially when factorials are involved. The well-known fact

$$\text{Stirling's Approximation: } n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \text{ as } n \rightarrow \infty,$$

can often be a very handy tool to exploit when applying the root test.

I'm going to state the following fact here just in case it's not known to you already. Suppose $\{|b_n|\}_n$ is a bounded real sequence. If the ratio test limit, $\lim_{n \rightarrow \infty} |b_{n+1}|/|b_n|$, exists, then the root test limit, $\limsup_{n \rightarrow \infty} \sqrt[n]{|b_n|}$, also exists and both give the exact same value. Stated the other way around, however, this fact is not true in general. That is, the root test limsup may exist but the ratio test limit may not. Consequently, the root test will always succeed to determine the radius of convergence for a given power series, whereas the ratio test may fail to do so. Remember this. (See exercise 6 below.)

Let's first compute the radius of convergence for the power series $f(z) = \sum_{n=1}^{\infty} (n!/n^n) z^n$ by employing the root test together with Stirling's approximation.

$$\begin{aligned} \sqrt[n]{|b_n|} &\equiv \sqrt[n]{|(n!/n^n) z^n|} = \sqrt[n]{n!} (|z|/n) \\ &\sim \sqrt[n]{2\pi n} (n/e) (|z|/n) \rightarrow |z|/e \text{ as } n \rightarrow \infty. \end{aligned}$$

Therefore, this power series pointwise defines a function for any z satisfying $|z|/e < 1$. This will imply $R_c = e$.

Let's now consider the $\text{erf}(z)$ power series given above and apply the ratio test.

$$\begin{aligned} \frac{|b_{n+1}|}{|b_n|} &\equiv \left(\frac{1}{(2(n+1)+1)(n+1)!} |z|^{2(n+1)+1} \right) / \left(\frac{1}{(2n+1)n!} |z|^{2n+1} \right) \\ &= \left(\frac{(2n+1)n!}{(2n+3)(n+1)!} \right) |z|^2 \rightarrow (0) |z|^2 \text{ as } n \rightarrow \infty. \end{aligned}$$

Clearly, whatever the value of z is, we have $0 \cdot |z| < 1$. Therefore, the $\text{erf}(z)$ power series pointwise defines a function for any $z \in \mathbb{C}$. Here we will find $R_c = \infty$.

Finally, apply the ratio test to this power series $f(z) = \sum_{n=0}^{\infty} n! z^n$.

$$\frac{|b_{n+1}|}{|b_n|} \equiv \frac{(n+1)! |z|^{n+1}}{n! |z|^n} = (n+1) |z|.$$

Therefore, the series is divergent for any $z \neq 0$, and so the power series $f(z)$ is undefined unless $z = 0$. In such cases, we say the power series has radius of convergence $R_c = 0$.

5. Use both the root **and** ratio tests to determine all open balls $B_\rho(0)$ (if any are possible) on which the following power series define a function of z .

$$(a) \sum_{n=0}^{\infty} 3^n z^{2n}. \quad (b) \sum_{n=1}^{\infty} \frac{1}{n 2^n} z^n. \quad (c) \sum_{n=0}^{\infty} \frac{n!}{(2n)!} z^n. \quad (d) \sum_{n=0}^{\infty} \frac{n^n n!}{(2n)!} z^{2n}.$$

6. Consider the power series

$$\sum_{n=0}^{\infty} a_n z^n \equiv \sum_{n=0}^{\infty} \frac{2 - (-1)^n}{2^n} z^n.$$

(a) Why does the ratio test fail on this example? (b) Use the root test to see that in fact this power series does define a function on $B_\rho(0)$ for any $\rho < 2$.

Hint: $\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n z^n|} \equiv \lim_{n \rightarrow \infty} \sup_{k \geq n} \left(\sqrt[k]{|2 - (-1)^k| \frac{|z|^k}{2^k}} \right) = \lim_{n \rightarrow \infty} \sup_{k \geq n} \sqrt[k]{|2 - (-1)^k|} \frac{|z|}{2}.$

An early form of the root test (employing a simple limit instead of the limsup) was originally introduced by Cauchy in 1821. Jacques Hadamard refined it to its current form in 1892. A few facts concerning the limsup are in order. Let $\{|b_n|\}_{n=0}^{\infty}$ denote a nonnegative sequence of real numbers. The $\limsup_{n \rightarrow \infty} |b_n|$ is defined by

$$\limsup_{n \rightarrow \infty} |b_n| \equiv \lim_{n \rightarrow \infty} \left(\sup \{|b_k| : k \geq n\} \right).$$

When the sequence $\{|b_n|\}_{n=0}^{\infty}$ **is not** bounded, observe that $\sup \{|b_k| : k \geq n\} \equiv \infty$. In this case (for the sake of notational convenience) we write $\limsup_{n \rightarrow \infty} |b_n| = \infty$. On the other hand, when the sequence **is** bounded, observe that $s_n \equiv \sup \{|b_k| : k \geq n\}$ is a bounded nonincreasing sequence of real numbers, and such sequences always have a well-defined limit, say L . In this case we write $\limsup_{n \rightarrow \infty} |b_n| = L$. For later use below, you should also verify the following important fact.

$$\text{If } \lim_{n \rightarrow \infty} |x_n| = |x| \text{ and } \limsup_{n \rightarrow \infty} |b_n| = |b|, \text{ then } \limsup_{n \rightarrow \infty} |x_n b_n| = |x b|.$$

THEOREM 2. (Cauchy-Hadamard) Consider the sequence of *partial sums*

$$f_n(z) \equiv \sum_{k=0}^n a_k (z - z_0)^k, \text{ and let } R_c \text{ denote } R_c \equiv 1 / \limsup_{k \rightarrow \infty} \sqrt[k]{|a_k|}.$$

Given that $R_c > 0$, we have $f_n(z) \rightarrow f(z)$ as $n \rightarrow \infty$ pointwise on the open ball $B_{R_c}(z_0)$. Moreover, the limit function, $f(z)$, is analytic on $B_{R_c}(z_0)$.

The proof I give here for the Cauchy-Hadamard theorem is pretty long. Some parts are routine, similar to what you saw in calculus. Other parts are tricky. Read what's given, but please don't get buried in the details. However, make sure you understand what the statement says! It's a foundational result in complex and real analysis.

The pointwise convergence result is proved just as done in calculus. For $m > n$, write

$$|f_m(z) - f_n(z)| = \left| \sum_{k=n+1}^m a_k (z - z_0)^k \right| \leq \sum_{k=n+1}^m |a_k| |z - z_0|^k.$$

Let $z \in B_{R_c}(z_0)$ where $0 < |z - z_0| \leq r < R_c$. The sequence $\sup_{l \geq k} \left(\sqrt[l]{|a_l|} \right) \rightarrow 1/R_c$ as $k \rightarrow \infty$, so by taking $\epsilon \equiv \frac{1}{2}(1/r - 1/R_c) > 0$ we see there is an N_ϵ such that

$$\sqrt[k]{|a_k|} \leq \sup_{l \geq k} \sqrt[l]{|a_l|} \leq \frac{1}{R_c} + \epsilon = \frac{1}{2} \left(\frac{1}{r} + \frac{1}{R_c} \right) \quad \forall k \geq N_\epsilon.$$

Multiply this by $|z - z_0|$ and raise to the k th power

$$\sqrt[k]{|a_k|} |z - z_0| \leq \left(\frac{R_c + r}{2R_c} \right) \implies |a_k| |z - z_0|^k \leq \left(\frac{R_c + r}{2R_c} \right)^k.$$

Note that $\rho \equiv \frac{1}{2}(R_c + r)/R_c = \frac{1}{2}(1 + r/R_c) < 1$, and then use the geometric series to find

$$|f_m(z) - f_n(z)| \leq \sum_{k=n+1}^m \rho^k \leq \sum_{k=n+1}^{\infty} \rho^k = \frac{\rho^{n+1}}{1 - \rho} \quad \text{for all } m > n \geq N_\epsilon.$$

This inequality shows $\{f_n(z)\}_n$ is a *Cauchy sequence* since $0 \leq \rho < 1$ and $\rho^n \rightarrow 0$ as $n \rightarrow \infty$. The excluded case when $z = z_0$ is trivial because $f_n(z_0) = a_0$ for each n . Therefore, for every $z \in B_{R_c}(z_0)$ there is a function value, denoted by $f(z)$, such that $f_n(z) \rightarrow f(z)$ pointwise as $n \rightarrow \infty$.

To conclude that $f(z)$ must also be analytic, first consider the derivative of the sequence of partial sums

$$f'_n(z) = \sum_{k=1}^n k a_k (z - z_0)^{k-1} = \sum_{k=0}^{n-1} (k+1) a_{k+1} (z - z_0)^k,$$

and note that its radius of convergence is given by $R'_c \equiv 1/\limsup_{k \rightarrow \infty} \sqrt[k]{(k+1)|a_{k+1}|}$. Compute the simple limit

$$\log(\sqrt[k]{(k+1)}) = \frac{\log(k+1)}{k} \rightarrow 0 \implies \sqrt[k]{(k+1)} \rightarrow 1 \quad \text{as } k \rightarrow \infty,$$

to see (using the limit within a limsup fact I stated earlier in this section)

$$\limsup_{k \rightarrow \infty} \sqrt[k]{(k+1)|a_{k+1}|} = \limsup_{k \rightarrow \infty} \sqrt[k]{|a_k|} \implies R'_c = R_c.$$

So, from what was shown in the previous paragraph, there is a function $h(z)$ such that

$$f'_n(z) \rightarrow h(z) \text{ as } n \rightarrow \infty \text{ pointwise on } B_{R_c}(z_0).$$

The second, and final step of this part of the proof, is to show that as $\Delta z \rightarrow 0$

$$\begin{aligned} & \left(\frac{f(z + \Delta z) - f(z)}{\Delta z} - h(z) \right) \\ &= \sum_{k=1}^{\infty} a_k \left(\frac{(z + \Delta z - z_0)^k - (z - z_0)^k}{\Delta z} - k(z - z_0)^{k-1} \right) \rightarrow 0. \end{aligned}$$

The trick used here is to re-express the big bracketed term in the sum above by means of the following extension of the truncated geometric series formula given to you at the top of page 1 of this homework. Recall for $\theta \neq 1$ and $n \geq 1$ we have

$$\begin{aligned} \frac{1 - \theta^n}{1 - \theta} &= \sum_{k=0}^{n-1} \theta^k \\ \implies \frac{1 - \theta^n}{1 - \theta} - n\theta^{n-1} &= \sum_{k=0}^{n-1} (\theta^k - \theta^{n-1}) = (1 - \theta) \sum_{k=0}^{n-2} \theta^k \left(\frac{1 - \theta^{n-1-k}}{1 - \theta} \right) \\ &= (1 - \theta) \sum_{k=0}^{n-2} \left(\sum_{l=0}^{n-2-k} \theta^{k+l} \right) = (1 - \theta) \sum_{l=0}^{n-2} (l+1) \theta^l. \end{aligned}$$

Note that the right hand side of this extended formula is zero when $n = 1$. Suppose $z_1 \neq z_2$, and for the moment assume $z_2 \neq 0$. Set $\theta = z_1/z_2 \neq 1$ and take $n = k$ in the formula above to find

$$\frac{z_2^k - z_1^k}{z_2 - z_1} - k z_1^{k-1} = (z_2 - z_1) \sum_{l=0}^{k-2} (l+1) z_1^l z_2^{k-2-l}.$$

Observe that in the case when $z_2 = 0$, the left hand side above equals $(1 - k) z_1^{k-1}$, and the right side equals $-z_1 (k - 2 + 1) z_1^{k-2}$ (i.e. when $l = k - 2$) and both sides agree. We're now ready to estimate. Let $z_1 = (z - z_0)$ and $z_2 = (z + \Delta z - z_0)$ above, and use the triangle inequality

$$\begin{aligned} \left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - h(z) \right| &\leq \sum_{k=1}^{\infty} |a_k| \left| \frac{(z + \Delta z - z_0)^k - (z - z_0)^k}{\Delta z} - k(z - z_0)^{k-1} \right| \\ &= \sum_{k=2}^{\infty} |a_k| \left| \Delta z \sum_{l=0}^{k-2} (l+1) (z - z_0)^l (z + \Delta z - z_0)^{k-2-l} \right| \\ &\leq |\Delta z| \sum_{k=2}^{\infty} |a_k| \sum_{l=0}^{k-2} (l+1) |z - z_0|^l |z + \Delta z - z_0|^{k-2-l}. \end{aligned}$$

Take $z \in B_{R_c}(z_0)$ and take $|\Delta z| > 0$ small enough so that we also have $z + \Delta z \in B_{R_c}(z_0)$. That is, there is an $0 < r < R_c$ such that $|z - z_0| \leq r$ and $|z + \Delta z - z_0| \leq r$. Therefore

$$\left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - h(z) \right| \leq |\Delta z| \sum_{k=2}^{\infty} \left(\left(\sum_{l=0}^{k-2} (l+1) \right) |a_k| r^{k-2} \right).$$

The innermost sum above is called an *arithmetic progression*. It can be explicitly evaluated as follows.

$$\begin{aligned} \sum_{l=0}^{k-2} (l+1) &\equiv S_{k-2} = 1 + 2 + \cdots + k-2 + k-1, \\ S_{k-2} &= k-1 + k-2 + \cdots + 2 + 1, \\ \implies 2 S_{k-2} &= k(k-1) \implies S_{k-2} = \frac{1}{2} (k-1) k. \end{aligned}$$

Use this and reindex to find

$$\left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - h(z) \right| \leq \frac{1}{2} |\Delta z| \sum_{k=0}^{\infty} (k+1)(k+2) |a_{k+2}| r^k.$$

The root test applied to the sum on the right establishes its boundedness; see M below.

$$\begin{aligned} \limsup_{k \rightarrow \infty} \sqrt[k]{(k+1)(k+2) |a_{k+2}|} &= \limsup_{k \rightarrow \infty} \sqrt[k]{|a_k|} = \frac{1}{R_c}, \\ \implies \left| \frac{f(z + \Delta z) - f(z)}{\Delta z} - h(z) \right| &\leq |\Delta z| M \rightarrow 0 \text{ as } \Delta z \rightarrow 0. \end{aligned}$$

Therefore, for any $z \in B_{R_c}(z_0)$, $f'(z)$ exists and is equal to $h(z)$ (determined by term-by-term differentiation). That's the proof of Cauchy-Hadamard.

Two obvious extensions of the Cauchy-Hadamard theorem should be pointed-out here. A power series $f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$ with positive radius of convergence $R_c > 0$ has derivatives of all orders at points $z \in B_{R_c}(z_0)$, and its n th derivative is given by term-by-term differentiation, i.e. $f^{(n)}(z) = \sum_{k=n}^{\infty} (k!/(k-n)!) a_k (z - z_0)^{k-n}$. Moreover, a power series can be integrated term-by-term any number of times inside its radius of convergence.

Here're two examples illustrating the utility of what I just said.

Determine a closed form expression for the power series $\sum_{n=1}^{\infty} n(n+1) z^{n-1}$ inside its radius of convergence. To do this, recall the geometric series $(1-z)^{-1} = \sum_{n=0}^{\infty} z^n$ which is valid when $|z| < 1$. Differentiate term-by-term twice to find

$$\frac{2}{(1-z)^{-3}} = \frac{d^2}{dz^2} (1-z)^{-1} = \sum_{n=2}^{\infty} n(n-1) z^{n-2} = \sum_{n=1}^{\infty} n(n+1) z^{n-1},$$

and this is valid for $|z| < 1$.

Determine a closed form expression for the power series $\sum_{n=2}^{\infty} 1/(n(n-1)) z^n$ inside its radius of convergence. To do this, integrate the geometric series twice term-by-term inside $|z| < 1$ to find

$$\begin{aligned}(1-z) \log(1-z) + z &= \int_0^z \int_0^{\zeta_1} (1-\zeta_2)^{-1} d\zeta_2 d\zeta_1 \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} z^{n+2} = \sum_{n=2}^{\infty} \frac{1}{n(n-1)} z^n,\end{aligned}$$

and this is also valid for $|z| < 1$.

7. Consider the following power series given by $f(z)$. Determine the radius of convergence by the root test. Then differentiate term by term and identify a closed form expression for the derivative $f'(z)$. Use your closed form expression for $f'(z)$ to determine a closed form expression for $f(z)$.

$$(a) f(z) = \sum_{n=1}^{\infty} \frac{2^n}{n} z^n. \quad (b) f(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)n!} z^{2n+1}.$$

My answers: (a) $f'(z) = 2/(1-2z) \Rightarrow f(z) = -\log(1-2z)$. (b) $f'(z) = e^{-z^2} \Rightarrow f(z) = \int_0^z e^{-\zeta^2} d\zeta$.

8. Consider the following power series given by $f'(z)$. Determine the radius of convergence by the root test. Then integrate term by term and identify a closed form expression for the antiderivative $f(z)$. Use your closed form expression for $f(z)$ to determine a closed form expression for $f'(z)$.

$$(a) f'(z) = \sum_{n=1}^{\infty} n z^{2n-1}. \quad (b) f'(z) = \sum_{n=0}^{\infty} \frac{n+1}{n!} z^n.$$

My answers: (a) $f(z) = \frac{1}{2} z^2 / (1-z^2) \Rightarrow f'(z) = z / (1-z^2)^2$. (b) $f(z) = z e^z \Rightarrow f'(z) = (z+1) e^z$.

9. Determine the radius of convergence for the following sum. Show it has the given value.

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(4n+3)(2n+1)!} z^{4n+3} = \int_0^z \sin(\zeta^2) d\zeta.$$

This special function is called the *Fresnel sine integral*.

10. Determine the radius of convergence for the following sum. Show it has the given value.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} z^n = \int_0^z -\frac{\log(1-\zeta)}{\zeta} d\zeta.$$

This special function is called the *dilogarithm function*.