Math 3364 Exam 2. Sanders Spring 2017

This exam has five problems, and all five will be graded. Use my supplied paper only. Return your solution sheets with the problems in order. Put your name, last name first, and student id number on each solution sheet you turn in. Each problem is worth 20 points with parts equally weighted unless indicated otherwise.

Recall, Cauchy’s integral formula says that when $f$ is analytic on an open and simply connected set $\Omega \subseteq \mathbb{C}$, and $\Gamma$ is a positively oriented simple path in $\Omega$ which surrounds a point $z$, then

$$2\pi i f(z) = \oint_{\Gamma} \frac{f(\zeta)}{\zeta - z} d\zeta,$$

and more generally

$$2\pi i f^{(k)}(z) = k! \oint_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^{k+1}} d\zeta,$$

where $f^{(k)}(z)$ represents the $k$th derivative of $f$ at $z$.

1. Let $\Gamma_R$ denote the circle centered at $z = 0$ with radius $R > 0$. Use the Cauchy Integral Formula to evaluate the following.

(a) $\oint_{\Gamma_R} \frac{\cos z}{z} dz$

(b) $\oint_{\Gamma_R} \frac{\cos z}{z^2 + 1} dz$

On part (b) (only) you need not consider the case when $R = 1$. However, clearly distinguish the two cases when $0 < R < 1$ and when $R > 1$.

2. Let $\Gamma_R$ be as given in the previous problem. Again use the Cauchy Integral Formula to determine the values of the following integrals for $0 < R < 1$ and for $R > 1$.

(a) $\oint_{\Gamma_R} \frac{1}{z^2(z + 1)} dz$

(b) $\oint_{\Gamma_R} \frac{\cos z}{z^2(z + 1)} dz$

3. Determine the radius of convergence for each of the following power series.

(a) $\sum_{k=0}^{\infty} \frac{k}{k+1} z^k$

(b) $\sum_{k=0}^{\infty} \frac{1}{2^k} z^{2k}$

(c) $\sum_{k=0}^{\infty} \frac{k+1}{k!} z^k$

(d) $\sum_{k=0}^{\infty} \frac{k!}{k^k} z^k$

When applicable, you may freely use the fact that $(n!)^{1/n} / n \to e^{-1}$ as $n \to \infty$.

4. Use the fact that for $|z| < 1$ we have $\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}$ to find an explicit closed form expression for each of the following power series.

(a) $\sum_{k=0}^{\infty} (-1)^k z^k$

(b) $\sum_{k=0}^{\infty} z^{2k}$

(c) $\sum_{k=1}^{\infty} k z^k$

(d) $\sum_{k=1}^{\infty} \frac{1}{k} z^k$

5. Use the fact that $\sum_{k=0}^{\infty} z^k / k! = e^z$ to determine the Taylor series around $z_0 = 0$ for the following analytic $f(z)$.

(a) $f(z) = e^{-z}$

(b) $f(z) = e^{z^2}$

(c) $f(z) = z e^z$

(d) $f(z) = \cosh z$