Section 6.6 – Bayes’ Theorem

In section 6.5, we were interested in finding the probability of the second event on a tree diagram if we know that the first event had occurred. In this section, we reverse the roles. If we know the second event has occurred, what is the probability of a given outcome of the first event? 

\[ P(U_1 \mid E) = \frac{P(U_1 \cap E)}{P(E)} \]

\( U_1 \) and \( U_2 \) are mutually exclusive events. We would like to know:\n
\[ P(U_1 \mid E) = \frac{\text{Product of the branch probabilities leading to } E \text{ through } U_1}{\text{Sum of all branch products leading to } E} \]

**Example 1:** A company produces 1,000 refrigerators a week at three plants. Plant A produces 350 refrigerators a week, plant B produces 250 refrigerators a week, and plant C produces 400 refrigerators a week. Production records indicate that 5% of the refrigerators produced at plant A will be defective, 3% of those produced at plant B will be defective, and 7% of those produced at plant C will be defective. All the refrigerators are shipped to a central warehouse.

- **a.** What is the probability that a refrigerator chosen at random from the warehouse will be defective?

\[ P(D) = \frac{350 \times 0.05}{1000} + \frac{250 \times 0.03}{1000} + \frac{400 \times 0.07}{1000} = 0.053 \]

- **b.** If a refrigerator at the warehouse is found to be defective, what is the probability that it was produced at plant B?

\[ P(B \mid D) = \frac{P(B \cap D)}{P(D)} = \frac{\frac{250}{1000} \times 0.03}{0.053} = 0.1415 \]

**Example 2:** In a random sample of 1,000 people it was found that 7% have a liver ailment. Of those who have a liver ailment, 40% are heavy drinkers, 50% are moderate drinkers and 10% are nondrinkers.
Of those who do not have a liver ailment, 10% are heavy drinkers, 70% are moderate drinkers and 20% are nondrinkers. If a person is chosen at random and it is found that he or she is a heavy drinker, what is the probability of that person having a liver ailment?

Example 3: Suppose that from a well-shuffled deck of 52 playing cards two cards are drawn in succession, without replacement. What is the probability that the first card was a king, given that the second card was not a king?
Example 4: A placement test is given by a certain high school to predict student success in a particular math course. On average, 70% of students who take the test pass it, and 87% of those who pass the test also pass the course, whereas 8% of those who fail the test pass the course. If a student passed the course, what is the probability that he or she passed the test?

\[
p(\text{pass test} \mid \text{pass course}) = \frac{p(\text{pass test} \cap \text{pass course})}{p(\text{pass course})} = \frac{0.70 \times 0.87}{0.70(0.87) + 0.30(0.08)} = 0.9621
\]
A rule that assigns a number to each outcome of an experiment is called a random variable.

We can construct the **probability distribution** associated with a random variable

If $x_1, x_2, x_3, ..., x_n$ are values assumed by the random variable $X$ with associated probabilities $P(X = x_1), P(X = x_2), ..., P(X = x_n)$, respectively, then the probability distribution of $X$ may be expressed in the following way.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X=x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1$</td>
<td>$P(X = x_1) = p_1$</td>
</tr>
<tr>
<td>$x_2$</td>
<td>$P(X = x_2) = p_2$</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>$x_n$</td>
<td>$P(X = x_n) = p_n$</td>
</tr>
</tbody>
</table>

We can also graphically represent the probability distribution of a R.V.

A bar graph which represents the probability distribution of a random variable is called a histogram.

**EX:**

Example 1: The probability distribution of the random variable $X$ is shown in the accompanying table:

<table>
<thead>
<tr>
<th>$x$</th>
<th>$P(X=x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>0.01</td>
</tr>
<tr>
<td>-1</td>
<td>0.11</td>
</tr>
<tr>
<td>0</td>
<td>0.20</td>
</tr>
<tr>
<td>1</td>
<td>0.32</td>
</tr>
<tr>
<td>2</td>
<td>0.21</td>
</tr>
<tr>
<td>3</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Find:

a. $P(X = -2) = 0.01$

b. $P(-1 \leq X < 1) = 0.11 + 0.20 = 0.31$

c. $P(X > 0) = 0.32 + 0.21 + 0.15 = 0.68$
Example 2: A survey was conducted by the Public Housing Authority in a certain community among 920 families to determine the distribution of families by size.

The results follow:

<table>
<thead>
<tr>
<th>Family Size</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency of Occurrence</td>
<td>350</td>
<td>200</td>
<td>245</td>
<td>125</td>
</tr>
</tbody>
</table>

a. Let $X$ denote the number of persons in a randomly chosen family. Find the probability distribution for this experiment.

\[
\begin{array}{c|cccc}
X & 2 & 3 & 4 & 5 \\
\hline
P(X) & \frac{350}{920} & \frac{200}{920} & \frac{245}{920} & \frac{125}{920} \\
& 0.38 & 0.217 & 0.266 & 0.136 \\
\end{array}
\]

b. Draw the histogram corresponding to the probability distribution in part a.

c. What is the $P(3 < X \leq 5)$?

\[
P(X = 4) + P(X = 5) = 0.402
\]

d. What is the $P(X > 2)$?

\[
P(X = 3) + P(X = 4) + P(X = 5) = 0.620
\]

e. What is the $P(2 \leq X \leq 5)$?

\[
P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5)
\]

\[
(\text{approx.} P(X)) \approx 1
\]
Example 3: A coin is tossed twice. Let the random variable $X$ denote the number of tails that occur in the two tosses. Find the probability distribution for $X$ and then draw the histogram corresponding to the probability distribution of $X$. 

$$\mathbb{E} X = 2^2 = 4$$

$\begin{array}{c|c|c|c}
X & 0 & 1 & 2 \\
\hline
p(X) & \frac{1}{4} & \frac{2}{4} & \frac{1}{4} \\
\hline
0.25 & 0.50 & 0.25 \\
\end{array}$

$\begin{array}{c}
H \, H \rightarrow 0 \, \text{tail} \, \frac{1}{4} \\
H \, T \rightarrow 1 \, \text{tail} \, \frac{2}{4} \\
T \, H \rightarrow 1 \, \text{tail} \, \frac{1}{4} \\
T \, T \rightarrow 2 \, \text{tail} \, \frac{1}{4} \\
\end{array}$
Use for questions 1, 2 and 3.
An automobile manufacturer obtains the microprocessors used to regulate fuel consumption in its automobiles from three microelectronic firms: A, B, and C. The quality-control department of the company has determined that 1% of the microprocessors produced by firm A are defective, 2% of those produced by firm B are defective, and 1.5% of those produced by firm C are defective. Firms A, B, and C supply 45%, 25%, and 30%, respectively, of those microprocessors used by the company. An automobile is selected at random. Draw a tree diagram.

1. What is the probability it was manufactured at firm B and it was found to be defective?
   A. .0050   B. .0150   C. .0140   D. .0200
   \[ P(B \cap D) = (0.25)(0.02) \]

2. What is the probability it is defective?
   A. .0038   B. .0150   C. .0140   D. .0200
   \[ P(D) = (0.45)(0.01) + (0.25)(0.02) - (0.30)(0.015) \]

3. What is the probability it was defective given it was manufactured at firm C?
   A. .0038   B. .0150   C. .0140   D. .0200
   \[ P(D \mid C) = \frac{P(D \cap C)}{P(C)} = \frac{(0.30)(0.015)}{0.30} \]

Use for questions 4 and 5.
An urn contains 10 red and 13 blue marbles. Two marbles are chosen at random, in succession and without replacement. Draw tree diagram.

4. What is the probability a marble is blue?
   A. .4348   B. .5652   C. .5504   D. .5455
   \[
   \frac{10}{23} \cdot \frac{13}{22} + \frac{13}{23} \cdot \frac{12}{22}
   \]

5. What is the probability that the first marble was red, given that the second one was blue?
   A. .4545   B. .5652   C. .5504   D. .5455
   \[
   P(R_1 \mid B_2) = \frac{P(R_1 \cap B_2)}{P(B_2)} = \frac{10 \cdot 13}{23 \cdot 22} + \frac{13 \cdot 12}{23 \cdot 22}
   \]