Math 1313  Section 3.5
Section 3.5: The Inverse of a Matrix

Over the set of real number we have what we call the multiplicative inverse or reciprocal. The multiplicative inverse of a number is a second number that when multiplied by the first number yields the multiplicative identity 1.

This is where the Identity Matrix comes in.

Let $A$ be a square matrix of size $n$ and another square matrix $A^{-1}$ of size $n$ such that $AA^{-1} = A^{-1}A = I_n$ is called the inverse of $A$.

Note: Not every square matrix has an inverse. A matrix with no inverse is called singular.

Finding the Inverse of a Matrix

Given the $n \times n$ matrix $A$:

1. Adjoin the $n \times n$ identity matrix $I$ to obtain the augmented matrix $(A \mid I)$

2. Use the Gauss-Jordan elimination method to reduce $(A \mid I)$ to the form $(I \mid B)$, if possible.

The matrix $B$ is the inverse of $A$.

**Example 1:** Find the inverse, if possible and check:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$
Example 2: Find the inverse of a 3 x 3 matrix. (Use Gauss-Jordan)

\[ C = \begin{pmatrix} 1 & 4 & -1 \\ 2 & 3 & -2 \\ -1 & 2 & 3 \end{pmatrix} \]
Example 3: Find the inverse.

\[
B = \begin{pmatrix}
4 & 2 & 2 \\
-1 & -3 & 4 \\
3 & -1 & 6
\end{pmatrix}
\]

Matrices That Have No Inverses

If there is a row to the left of the vertical line in the augmented matrix containing all zeros, then the matrix does not have an inverse. Example 3 has this problem and does not have an inverse.
Math 1313   Section 3.5

**Formula for the Inverse of a 2X2 Matrix**

Let \( A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). Suppose \( D = ad - bc \) is not equal to zero. Then \( A^{-1} \) exists and is given by

\[
A^{-1} = \frac{1}{D} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\]

**Example 4:** Find the inverse of the following matrices.

a. \( A = \begin{pmatrix} -5 & 10 \\ 2 & 7 \end{pmatrix} \)

b. \( B = \begin{pmatrix} 8 & -4 \\ -4 & 2 \end{pmatrix} \)

**Matrix Representation**

A system of linear equations may be written in a compact form with the help of matrices.

**Example 5:** Given the following system of equations, write it in matrix form.

\[
\begin{align*}
2x - 4y + z &= 6 \\
-3x + 6y - 5z &= -1 \\
x - 3y + 7z &= 0
\end{align*}
\]
Example 6: Write each system of equations as a matrix equation and then solve the system using the inverse of the coefficient matrix.

\[ \begin{align*}
2x + 3y &= 5 \\
3x + 5y &= 8 \\
\end{align*} \]

The use of inverses to solve systems of equations is advantageous when we are required to solve more than one system of equations \( AX = B \), involving the same coefficient matrix, \( A \), and different matrices of constants, \( B \).

Example 7: A performance theatre has 10,000 seats. The ticket prices are either $25 or $35, depending on the location of the seat. Assume every seat can be sold.

a. How many tickets of each type should be sold to bring in a return of $275,000?
b. How many tickets of each type should be sold to bring in a return of $300,000?

Let \( x \) = number of $25 tickets and \( y \) = number of $35 tickets