Topics on Complex Geometry and Analysis

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1 Complex Manifolds

What is complex analysis and complex geometry? One of the leaders in differential geometry of the twentieth century Shing-Shen Chern (1911-2004) wrote:

“Euclidean’s Elements of Geometry (300 B.C.) is one of the great achievements of the human mind. It makes geometry into a deductive science and the geometrical phenomena as the logical conclusions of a system of axioms and postulates. The concept is not restricted to geometry as we now understand the term. Its main geometrical results are:

(a) Pythagoras’ Theorem: \( c^2 = a^2 + b^2 \).
(b) Angle-sum of a triangle: \( \alpha + \beta + \gamma = \pi \).

Euclid realized that the parallel postulate was not as transparent as the other axioms and postulates. Efforts were made to prove it as a consequence. Their failure led to the discovery of non-Euclidean geometry by C.F. Gauss, John Bolyai and N.I. Lobachevski in early 19th century.

Contemporary geometry is thus a far cry from Euclid. To summarize, I would like to consider the following as the major developments in the history of geometry:

1. Axioms (Euclid);
2. Coordinates (Descartes, Fermat);
3. Calculus (Newton, Leibniz);
4. Manifolds (Riemann);
5. Fiber bundles (Elie Cartan, Whitney).”

Outline of the topics Therefore, in this lecture we are going to talk about manifolds (complex, Kähler, Hermitian, pseudoconvex manifolds, varieties, almost complex manifolds, orbifolds) and fiber bundles (vector and line bundles, sheaves) with analytic techniques (\(L^2\) estimate, currents, group cohomology, Hodge theorem, Bochner type techniques, etc.).

Origin of the notion of complex manifolds

• It was a difficulty problem how to define \( \log a \) when \( a \) is a negative number. Between 1712 and 1713, Bernoulli held the view that \( \log a = \log(-a) \) because \( \frac{d(-x)}{-x^2} = \frac{dx}{x^2} \), while Leibniz believed that \( \log(-a) \) must be imaginary. Between 1727 and 1731, the question was taken up by Bernoulli and Euler without a solution. It was only in the period between 1749

\[ \text{S.S. Chern, } \textit{What is geometry?} \text{ A. M. Monthly 97(1990), 548-555.} \]
and 1751, Euler developed ideas far enough to study logarithm so that it leads to a satisfying solution. Euler’s approach is to deal with logarithm of general complex numbers. Euler’s argument is as follows: by the definition, \( \log z = \lim_{n \to \infty} n(\sqrt[n]{z} - 1) \). For each positive integer \( n \), \( n(\sqrt[n]{z} - 1) \) has \( n \) different roots (Euler realized it!) so that the limit indeed has infinitely many different values. When \( z = 1 \), he got \( \log 1 = \pm 2\pi ki, \ k = 1, 2, 3, ... \).^2

In 1857, Riemann study analytic continuation of holomorphic functions (e.g. \( f(z) = \log z \)). It leads to consider multi-valued holomorphic functions, and Riemann introduced the concept of Riemann surfaces. The concept of “manifolds”, including the real case, originated from this work of Riemann.

Let \( U \) be a ball in \( \mathbb{C} - \{0\} \). On \( U \), let \( z = re^{i\theta} \), where \( \theta \) is a continuous function on \( U \). Then \( \sqrt{z} \) takes two values \( f_1 = \sqrt{r}e^{\frac{1}{2}i\theta} \) and \( f_2 = \sqrt{r}e^{-\frac{1}{2}i\theta} \). According to Weierstrass, \( \sqrt{z} \) is defined as the set of pairs \( (U, f_1) \) and \( (U, f_2) \), where \( f_1 \) and \( f_2 \) are power series of \( f_1 \) and \( f_2 \), respectively. By using equivalent classes, we can suitably “glue” \((U, f_j)’\)’s together to form a topological space \( X \) on which \( \sqrt{z} : X \to \mathbb{C} \) is a well-defined holomorphic function. Such \( X \) is a two branched covering space of \( \mathbb{C} - \{0\} \).^3 \( X \) is called the Riemann surface of the function \( \sqrt{z} \).

• A half-century after Riemann, Hermann Weyl (1885-1955) provided a rigorous construction of Riemann surfaces using techniques of topology. The basic strategy of this construction is to construct the surface as a subspace of a larger topological space. The elements of the underlying set of are pairs consisting of a point of the complex plane and a convergent power series about that point. It leads to the concept of “complex manifolds.”

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^3The above result shows that even in the simple cases, the notion of manifold is naturally introduced.
Paul Koebe (1882 - 1945) proved the uniformization theorem of Riemann surfaces in a series of four papers in 1907-1909: every simply connected Riemann surface is biholomorphic to either \( \mathbb{C} \), or \( \mathbb{B} \) or \( \mathbb{CP}^1 \). He did his thesis at Berlin, where he worked under Herman Schwarz.

We call a manifold to be a complex manifold if there are coordinates charts \((z_1, ..., z_n)\) so that their coordinate transformations are holomorphic. More precisely, a complex manifold \( M \) of dimension \( n \) is a Hausdorff topological space equipped with an atlas with values in \( \mathbb{C}^n \). An atlas (or a coordinate map) is a collection of homeomorphisms \( \tau_\alpha : U_\alpha \rightarrow V_\alpha, \alpha \in I \), called holomorphic charts (or holomorphic coordinate map), such that \( \{U_\alpha\}_{\alpha \in I} \) is an open covering of \( M \) and \( V_\alpha \) is an open subset of \( \mathbb{C}^n \), and such that for all \( \alpha, \beta \in I \) the transition map

\[
\tau_{\alpha \beta} := \tau_\alpha \circ \tau_\beta^{-1} : \tau_\beta(U_\alpha \cap U_\beta) \rightarrow \tau_\alpha(U_\alpha \cap U_\beta)
\]

\[
z \mapsto \tau_{\alpha \beta}(z) \tag{1}
\]

is a holomorphic map from an open subset of \( V_\beta \) onto an open subset of \( V_\alpha \). \(^4\)

When \( n = 1 \), a complex manifold is called a Riemann surface.

If \( M \) is a complex manifold, a function \( f : M \rightarrow \mathbb{C} \) is called a holomorphic function if for each \( \alpha, f \circ \tau_\alpha^{-1} : \tau_\alpha(U_\alpha) \rightarrow \mathbb{C} \) is a holomorphic function.

A continuous map \( f : M \rightarrow N \), where \( M \) and \( N \) are complex manifolds with coordinate coverings \( \{U_i, \Phi_i\}, \{V_j, \Psi_j\} \), \( \dim M = m \), \( \dim N = n \), respectively, is called a holomorphic map if

\[
\tau_j \circ f \circ \tau_i^{-1} : \tau_i(U_i \cap f^{-1}(V_j)) \subset \mathbb{C}^n \rightarrow \tau_j(V_j)
\]

is a holomorphic map.

**Examples of complex manifolds**

- For a compact Riemann surface, if the genus is 0, it is biholomorphic to Riemann sphere \( S = \mathbb{CP}^1 = \mathbb{C} \cup \{\infty\} \), which is generalized to \( n \)-dimensional complex projective space \( \mathbb{CP}^n \) in higher dimensional case.

If the genus is 1, it is biholomorphic to a complex torus \( \mathbb{T}/\Lambda \), which is generated to abelian varieties \( \mathbb{T}^n/\Lambda \) in higher dimensional case.

\(^4\)Note that the transition maps \( \tau_{\alpha \beta} \) is holomorphic \( \forall \alpha, \beta \) if and only if \( \tau_{\alpha \beta} \) is biholomorphic \( \forall \alpha, \beta \).
If the genus $\geq 2$, it is a hyperbolic surface. For every hyperbolic Riemann surface, the universal covering space is $\mathbb{B}^1$, the unit disk, or $\mathbb{H}^1$, the upper-half plane, and the fundamental group is isomorphic to a Fuchsian group, and thus the surface can be modeled by a Fuchsian model $\mathbb{H}/\Gamma$ where $\mathbb{H}$ is the upper half-plane and $\Gamma$ is the Fuchsian group. The set of representatives of the cosets of $\mathbb{H}/\Gamma$ are free regular sets and can be fashioned into metric fundamental polygons. Quotient structures as $\mathbb{H}/\Gamma$ are generalized to Shimura varieties.

- For dimension 2 compact complex manifolds, we have the Enriques-Kodaira classification which states that every non-singular minimal compact complex surface is of exactly one of the 10 types of rational, ruled (genus $> 0$), type VII, K3, Enriques, Kodaira, toric, hyperelliptic, properly quasi-elliptic, or general type surfaces.

For the 9 classes of surfaces other than general type, there is a fairly complete description of what all the surfaces look like (which for class VII depends on the global spherical shell conjecture, still unproved in 2009). For surfaces of general type not much is known about their explicit classification, though many examples have been found.

- The most popular version of String theory requires the universe to be 10-dimensional for this quantization process to work. The space we live in looks locally like

$$M = \mathbb{R}^4 \times X$$

where $\mathbb{R}^4$ is the Minkowski space and $X$ is a compact Calabi-Yau 3 dimensional complex manifold with radius of order $10^{-33}$ cm, the Planck length. Since the Planck length is so small, space then appears to macroscopic observers to be 4-dimensional.

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5In modern physics, general relativity says that space/time is curved; quantum mechanics holds that space/time itself is inherently uncertain. String theory is an approach to a possible solution to the main inconsistency between the two great pillars of 20th century physics: Einstein’s theory of general relativity, and quantum field theory.

Originally formulated in the 1970s, string theory hypothesizes that particles such as electrons and quarks are not actually point-like objects. Rather, **if viewed through a very powerful microscope, those particles would look like little, tiny, closed loops of string**. According to the theory, these strings can vibrate and rotate, and as they do so, they take on different configurations. Professor Strominger compares the process to the way that sounds are made by a guitar string. If you pluck a guitar string, it will vibrate and emit a certain sound. If you put your finger over a fret and pluck the same string again, the frequency will be different. The string is the same, but the sound is not. Similarly, in physics, if you change the vibration of one of these theoretical strings, it will still be a string, but it will look different.

So, the basic idea is that the fundamental constituents of reality are strings of the Planck length (about $10^{-33}$ cm) which vibrate at resonant frequencies.

String theory

Why $X$ is a Calabi - Yau 3- fold?

To explain it, concerning the notion of holonomy group of a connection in a vector bundle. We recall Berger’s classification:

**Theorem 1.1** (Berger, 1955, [Beg55]) Let $M$ be a simply connected manifold of dimension $n$, and $g$ be a Riemannian metric on $M$ such that $g$ is irreducible and non-symmetric. Then exactly one of the following seven cases holds

1. $\text{Hol}(g) = \text{SO}(n)$.  
2. $n = 2m, m \geq 2$, $\text{Hol}(g) = \text{U}(m)$ in $\text{SO}(2m)$.  
3. $n = 2m, m \geq 2$, $\text{Hol}(g) = \text{SU}(m)$ in $\text{SO}(2m)$.  
4. $n = 4m, m \geq 2$, $\text{Hol}(g) = \text{Sp}(m)$ in $\text{SO}(4m)$.  
5. $n = 4m, m \geq 2$, $\text{Hol}(g) = \text{Sp}(m)\text{Sp}(1)$ in $\text{SO}(4m)$.  
6. $n = 7$ and $\text{Hol}(g) = \text{G}_2$ in $\text{SO}(7)$.  
7. $n = 8$ and $\text{Hol}(g) = \text{Spin}(7)$ in $\text{SO}(8)$,

where $\text{Hol}(g)$ is the holonomy group.

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7Let $E$ be a rank $k$ vector bundle over a smooth manifold $M$ and let $\nabla$ be a connection on $E$. Given a piecewise smooth loop $\gamma : [0,1] \to M$ based at $x$ in $M$, the connection defines a parallel transport map $P_\gamma : E_x \to E_x$. This map is both linear and invertible and so defines an element of $\text{GL}(E_x)$. The holonomy group of $\nabla$ based at $x$ is defined as

$$\text{Hol}_x(\nabla) = \{ P_\gamma \in \text{GL}(E_x) \mid \gamma \text{ is a loop based at } x \}.$$
<table>
<thead>
<tr>
<th>Hol(g)</th>
<th>dim(M)</th>
<th>Type of manifold</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>SO(n)</td>
<td>n</td>
<td>Orientable manifold</td>
<td>Kähler</td>
</tr>
<tr>
<td>U(n)</td>
<td>2n</td>
<td>Kähler manifold</td>
<td>Kähler</td>
</tr>
<tr>
<td>SU(n)</td>
<td>2n</td>
<td>Calabi-Yau manifold</td>
<td>Ricci flat, Kähler</td>
</tr>
<tr>
<td>Sp(n) · sp(1)</td>
<td>4n</td>
<td>Quaternion-Kähler manifold</td>
<td>Einstein</td>
</tr>
<tr>
<td>Sp(n)</td>
<td>4n</td>
<td>Hyperkähler manifold</td>
<td>Ricci-flat, Kähler</td>
</tr>
<tr>
<td>G₂</td>
<td>7</td>
<td>G₂ manifold</td>
<td>Ricci-flat</td>
</tr>
<tr>
<td>Spin(7)</td>
<td>8</td>
<td>Spin(7) manifold</td>
<td>Ricci-flat</td>
</tr>
</tbody>
</table>

$X$ is a Calabi-Yau 3-fold because of supersymmetry, so that the holonomy group is $SU(3)$, in the String theory.

In three complex dimensions, classification of the possible Calabi-Yau manifolds is an open problem, although Yau suspects that there is a finite number of families (although it would be a much bigger number than his estimate from 20 years ago). One example of a three-dimensional Calabi-Yau manifold is a non-singular quintic threefold in $\mathbb{CP}^4$, which is the algebraic variety consisting of all of the zeros of a homogeneous quintic polynomial in the homogeneous coordinates of the $\mathbb{CP}^4$. Another example is a smooth model of the Barth-Nieto quintic. Some discrete quotients of the quintic by various $\mathbb{Z}_5$ actions are also Calabi-Yau and have received a lot of attention in the literature. One of these is related to the original quintic by mirror symmetry.