17 Chern connection on Hermitian vector bundles

Hermitian connection A Hermitian structure h_E in a smooth complex vector bundle E is a smooth field of Hermitian inner products $\langle\ ,\ \rangle_{h_E}$ in the fibres of E. With respect to a local frame, a Hermitian structure is given by a Hermitian matrix-valued function $H = (H_{ij})$, with $H_{ij} = \langle s_i, s_j\rangle_{h_E}$ which transforms according to $H = A \cdot H'\overline{A}^t$. [39] h_E is also called a hermitian metric on E. We call (E, h_E) a *Hermitian vector bundle*.

If h_E is a Hemitian metric of a complex vector bundle E on a smooth manifold X. A connection $\bigtriangledown_E$ is called a *Hermitian connection* if for any $s_1, s_2 \in \mathcal{C}^\infty(X, E)$,

$$d\langle s_1, s_2\rangle_{h_E} = \langle \bigtriangledown_E s_1, s_2\rangle_{h_E} + \langle s_1, \bigtriangledown_E s_2\rangle_{h_E}.$$

There always exist Hermitian connections.

Chern connection on E Let E be a holomorphic vector bundle over a complex manifold (X, J) where J is the complex structure. Let (E, h_E) be a *Hermitian vector bundle* over X.

There exists a natural $\mathbb{C}$-linear operator $\overline{\partial} : A^{p,q}(E) \to A^{p,q+1}(E)$ with $\overline{\partial}^2 = 0$ and which is defined locally by $\overline{\partial}(f \cdot \alpha) = \overline{\partial} f \otimes \alpha$.

Using the decomposition $\mathcal{A}^1(E) = \mathcal{A}^{1,0}(E) \oplus \mathcal{A}^{0,1}(E)$ we can decompose any connection D on E in its two components $D^{1,0}0$ and $D^{0,1}$, i.e. $D = D^{1,0} \oplus D^{0,1}$ with

$$D^{1,0} : \mathcal{A}^0(E) \to \mathcal{A}^{1,0}(E)$$

and

$$D^{0,1} : \mathcal{A}^0(E) \to \mathcal{A}^{0,1}(E).$$

Note that $D^{0,1}$ satisfies $D^{0,1}(fs) = \overline{\partial} f \otimes s + f \cdot D^{0,1}(s)$, i.e. it behaves similarly to the operator $\overline{\partial}$. Indeed, the decomposition $D = D^{1,0} \oplus D^{0,1}$ makes sense even when E is not holomorphic. A connection D on a holomorphic vector bundle E is *compatible with the holomorphic structure* if $D^{0,1} = \overline{\partial}$.

If a connection h_E on a holomorphic vector bundle E is both Hermitian and compatible with the holomorphic structure is called *Chern connection*.

Proposition 17.1 *Chern connection exists uniquely.*

[39] A Hermitian structure in T_X is exactly a Hermitian metric on X.

Proof: With respect to a local holomorphic frame $S = (s_\alpha) = (s_1, ..., s_q)$ on U, We have $DS = \omega S$, i.e., $Ds_\alpha = \omega_\alpha^\gamma s_\gamma$ on U. By the facts that s_α are holomorphic, $D = D^{1,0} + D^{0,1} = D^{1,0} + \overline{\partial}$ and $\overline{\partial} s_\alpha = 0$, we have $Ds_\alpha = D^{1,0} s_\alpha = \omega_\alpha^\gamma s_\gamma$ so that ω_α^γ are all $(1,0)$-forms.

By the condition, we have

$$d\langle s_\alpha, s_\beta\rangle = \langle Ds_\alpha, s_\beta\rangle + \langle s_\alpha, Ds_\beta\rangle = \langle \omega_\alpha^\gamma s_\gamma, s_\beta\rangle + \langle s_\alpha, \omega_\beta^\gamma s_\gamma\rangle = h_{\beta\overline{\gamma}}\omega_\alpha^\gamma + h_{\alpha\overline{\gamma}}\overline{\omega_\beta^\gamma}$$

where we denote the smooth function $h_{\alpha\overline{\beta}} := \langle s_\alpha, s_\beta\rangle$.

On the other hand, $d\langle s_\alpha, s_\beta\rangle = dh_{\alpha\overline{\beta}} = (\partial + \overline{\partial})h_{\alpha\overline{\beta}} = \partial h_{\alpha\overline{\beta}} + \overline{\partial} h_{\alpha\overline{\beta}}$. Since all ω_β^α are $(1,0)$-form, it implies

$$\partial h_{\alpha\overline{\beta}} = \omega_\alpha^\gamma h_{\gamma\overline{\beta}},$$

i.e.,

$$\omega_\alpha^\gamma = \partial h_{\alpha\overline{\beta}} h^{\overline{\beta}\gamma}$$

where $H^{-1} = (h^{\overline{\beta}\gamma}) = (h_{\alpha\overline{\beta}})^{-1}$ is the inverse matrix of $H = (h_{\alpha\overline{\beta}})$, i.e.,

$$\omega = \partial H \cdot H^{-1}$$

is uniquely determined by H, i.e., the connection matrix is completely determined by the metric. $\square$

[Example] Let (L, h) be a Hermitian holomorphic line bundle over a complex manifold X. For any point $a \in X$, let s be a local frame on a neighborhood U of a in X, then the corresponding curvature form is

$$\Omega = -\partial\overline{\partial} log\ H.$$

where $H = \langle s, s\rangle$. We call $c_1(L, h) = -\frac{i}{2\pi}\partial\overline{\partial} log\ H$ the Chern curvature form of (L, h).

In fact, $\omega = \frac{\partial H}{H} = \frac{\overline{\partial}\partial H - \overline{H}\wedge\partial H}{H^2}$,

$$-\partial\overline{\partial} log\ H = -\partial\frac{\overline{\partial} H}{H} = -\frac{\partial\overline{\partial} H}{H} + \frac{\partial H \wedge \overline{\partial} H}{H^2} = d\frac{\partial H}{H} = d\omega = d\omega - \omega\wedge\omega = \Omega.$$

If we denote $\|s\|_h = \sqrt{h}|s|$, we have $H = h|s|^2$ so that

$$\frac{i}{2\pi}\Omega = -\frac{i}{2\pi}\partial\overline{\partial} log\ h|s|^2 = -\frac{i}{2\pi}\partial\overline{\partial} log\ h = c_1(L, h).$$

Chern connection and Levi-Civita connection Recall that a Hermitian structure on a complex manifold X is just a Riemannian metric g on the underlying real manifold compatible with the complex structure J defining X. Recall that the complexified tangent bundle $T_{\mathbb{C}}X$ decomposes as $T_{\mathbb{C}}X = T_X^{1,0} \oplus T_X^{0,1}$. We still denote by g the $\mathbb{C}$-bilinear form on $T_{\mathbb{C}}X$ induced by g. Notice that $g(\ ,\ )$ vanishes on $T^{1,0}X \times T^{1,0}X$ and on $T^{0,1}X \times T^{0,1}X$. Recall that the bundle $T_X^{1,0}$ is the complex bundle underlying the holomorphic tangent bundle and hence is the same as the holomorphic tangent bundle. Moreover, the Hermitian extension $g_{\mathbb{C}}$ of g to $T_{\mathbb{C}}X$ restricted to $T_X^{l,0}$ is

$$\frac{1}{2}(g - i\omega),$$

where ω is the fundamental form $g(J(\ ), (\ ))$. The complex vector bundles $T_X^{1,0}$ and (T_X, J) are identified via the isomorphism

$$\xi : T_X \to T_X^{1,0}, \quad u \mapsto \frac{1}{2}\big(u - iJ(u)\big).$$

Let $\{w_j\}_{j=1}^n$ be a local orthnomal frame of $T_X^{1,0}$. Then $\{\frac{1}{\sqrt{2}}(w_j + \overline{w_j}), \quad \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w_j})\}$ form an orthonormal frame of T_X.

Under the natural isomorphism ξ, any Hermitian connection V on $T_X^{1,0}$ induces a metric connection $\widetilde{V}$ on the Riemannian manifold (X, g). In general, a Hermitian connection V on $(T_X^{1,0}, g)$ will not necessarily induce the Levi-Civita connection on the Riemannian manifold (X, g). In fact, this could hardly be true, as the Levi-Civita connection is unique, but there are many hermitian connections $(T^{1,0}, g)$. But even for the Chern connection on the holomorphic tangent bundle (T_X, g), which is unique, the induced connection may not be the Levi-Civita connection in general.

Theorem 17.2 [40] *Let (X, g) be a Kähler manifold. Then under the isomorphism $\xi : T_X \to T_X^{l,0}$ the Chern connection D on the holomorphic tangent bundle $T_X^{l,0}$ corresponds to the Levi-Civita connection $\bigtriangledown$.*

[40] Daniel Huybrechts, *Complex Geometry - an introduction*, Springer, 2005. , p. 219.

18 Chern Classes

Bianchi Identity Let M be an m-dimensional smooth manifold and (E, M, π) a q-dimensional complex vector bundle on M. We'll define the important invariants: Chern classes.

Suppose $\{s_\alpha, 1 \leq \alpha \leq q\}$ is a local frame field of the complex vector bundle E in a neighborhood $U \subset M$. Then the action of a connection D on E can be expressed in U by

$$Ds_\alpha = \sum_\beta \omega_\alpha^\beta s_\beta,$$

where the connection matrix ω_α^β is a complex-valued differential 1-form. If we use the matrix notation, then the above can be written as

$$DS = \omega \cdot S,$$

where

$$S =^t (s_1, ..., s_q),$$

where

$$\omega = \begin{pmatrix} \omega_1^1 & ... & \omega_1^q \\ \vdots & \ddots & \vdots \\ \omega_q^1 & ... & \omega_q^q \end{pmatrix}$$

The curvature matrix for the connection is

$$\Omega = (\Omega_\alpha^\beta) = d\omega - \omega \wedge \omega.$$

Taking exteriorly differential, we then get the *Bianchi identity*:

$$d\Omega = \omega \wedge \Omega - \Omega \wedge \omega. \tag{55}$$

If we choose another local frame field S' and assume that

$$S' = A \cdot S,$$

where $detA \neq 0$, then we have the transformation formula for curvature matrices

$$\Omega' = A \cdot \Omega \cdot A^{-1}. \tag{56}$$