The singular homology groups $H_{n,\text{sing}}(X)$ are defined by using the singular simplicial chain complex $C(X)_{\text{sing}}$, with $C(X)_{n,\text{sing}}$ the free abelian group generated by the singular n-simplices of $X$. More precisely, a singular n-simplex is a continuous mapping $\sigma$ from the standard n-simplex $\Delta_n$ to a topological space $X$.

By a fundamental result from differential topology, simplicial homology and singular homology are isomorphic: $H_p(X) \cong H_{p,\text{sing}}(X)$.

**Cohomology**  
Cohomology is fundamental to modern algebraic topology, i.e., $C^p(X, G) := \text{Hom}(C_p(X), G)$ where $G$ is an abelian group. Then we have the cochain complex which has opposite direction by comparing with homology chain complex:

$$
\cdots \leftarrow C^{n+1}(X, G) \xleftarrow{\delta} C^n(X, G) \xleftarrow{\delta} \cdots \leftarrow C^0(X, G) \leftarrow 0
$$

and we can similarly define cohomology group:

$$
H^p(X) = \ker(\partial^p) / \text{im}(\partial^{p-1}).
$$

Its importance was not seen for some 40 years after the development of homology. The concept of dual cell structure, which Henri Poincaré used in his proof of his Poincaré Duality theorem, contained the germ of the idea of cohomology, but this was not seen until later.

**Historic remark**

- In 1930, Alexander defined a first cochain notion.
- In 1931, Georges de Rham related homology and exterior differential forms, proving De Rham’s theorem. In fact, De Rham’s original work was to define relative de Rham groups and to prove that the resulting homology theory satisfied the axioms of Eilenberg and Steenrod. This result is now understood to be more naturally interpreted in terms of cohomology.
- In 1934, L. Pontryagin proved the Pontryagin duality theorem; a result on topological groups. This (in rather special cases) provided an interpretation of Poincaré duality and Alexander duality in terms of group characters.
- In 1935, A. Kolmogorov and Alexander both introduced cohomology and tried to construct a cohomology product structure.
• In 1936 Norman Steenrod published a paper constructing Čech cohomology by dualizing Čech homology.

• From 1936 to 1938, Hassler Whitney and Eduard Čech developed the cup product (making cohomology into a graded ring) and cap product, and realized that Poincaré duality can be stated in terms of the cap product. Their theory was still limited to finite cell complexes.

• In 1944, Samuel Eilenberg overcame the technical limitations, and gave the modern definition of singular homology and cohomology.

**Stokes theorem** E. Cartan developed the theory of exterior differentials in 1920s. In 1936-1937 in Paris, E. Cartan discovered the general Stokes theorem [Katz79] for any dimension, which gives a link between analysis and geometry

\[
\int_D d\omega = \int_{\partial D} \omega
\]

where \( \omega \) is a smooth \( p \)-form, and \( D \) is a \((p+1)\)-dimensional orientable submanifold.

In classical complex, Cauchy integral theorem, Cauchy integral formulas, Residue theorem are consequence of Stokes' theorem.

**De Rham cohomology groups** ([GH78], p.44) The **de Rham complex** is the cochain complex of exterior differential forms on some smooth manifold \( M \), with the exterior derivative as the differential.

\[
0 \to A_0(M) \xrightarrow{d} A_1(M) \xrightarrow{d} A_2(M) \xrightarrow{d} A_3(M) \to \ldots
\]

where \( A^0(M) \) is the space of smooth functions on \( M \), \( A^1(M) \) is the space of 1-forms, and so forth. The **de Rham cohomology group** of \( M \) in degree \( p \) is

\[
H^{p}_{DR}(M, \mathbb{R}) := \frac{\text{Ker}(d : A^p(M) \to A^{p+1}(M))}{\text{Im}(d : A^{p-1}(M) \to A^{p}(M))}.
\]

The **Betti number** of \( M \) is \( b_p := \dim_{\mathbb{R}} H^{p}_{DR}(M, \mathbb{R}) \).

We claim that the map

\[
H^k_{DR}(M, \mathbb{R}) \times H_k(M, \mathbb{R}) \to \mathbb{R}
\]

\[
(\omega, \sigma) \mapsto \int_{\sigma} \omega
\]
is well-defined. In fact, if $\omega, \omega'$ are in the same cohomology class, i.e., $\omega - \omega' = d\delta$ for some $\delta \in \Omega^{k-1}(M)$, then $\int_\sigma (\omega - \omega') = \int_\sigma d\delta = \int_{d\sigma} \delta = \int_{\{0\}} \delta = 0$, so that $\int_\sigma \omega = \int_\sigma \omega'$. Here we have used Stokes' theorem.

If $\sigma, \sigma'$ are in the same homology class, i.e., $\sigma - \sigma' = d\psi$ for some $(k+1)$-chain $\psi$. Then $\int_\sigma \omega - \int_{\sigma'} \omega = \int_{d\psi} \omega = \int_{\psi} d\omega = 0$, so that $\int_\sigma \omega = \int_{\sigma'} \omega$. Here we used Stokes' theorem. Our claim is proved.

In other words, Stokes' theorem is an expression of duality between de Rham cohomology and the homology of chains. The above pairing of differential forms and chains, via integration, gives a homomorphism from de Rham cohomology $H^k_{\text{DR}}(M, \mathbb{R})$ to singular cohomology groups $H^k_{\text{sing}}(M; \mathbb{R}) := \text{Hom}_\mathbb{R}(H_k(M, \mathbb{R}), \mathbb{R})$:

$$H^k_{\text{DR}}(M, \mathbb{R}) \rightarrow H^k_{\text{sing}}(M, \mathbb{R}) \quad \omega \quad \mapsto \quad (\sigma \mapsto \int_\sigma \omega)$$

De Rham’s theorem, proved by Georges de Rham in 1931, states that for a smooth manifold $M$, this map is in fact an isomorphism.

The wedge product endows the direct sum of these groups with a ring structure:

$$H^k_{\text{DR}}(M, \mathbb{R}) \times H^r_{\text{DR}}(M, \mathbb{R}) \rightarrow H^{k+r}_{\text{DR}}(M, \mathbb{R}) \quad \gamma \wedge \delta \quad \mapsto \quad \int_\sigma (\omega + \delta)$$

A further result of the theorem is that the two cohomology rings are isomorphic (as graded rings), where the analogous product on singular cohomology is the cup product.

De Rham’s theorem says

$$H^p_{\text{DR}}(M, \mathbb{R}) \simeq H^p_{\text{sing}}(M, \mathbb{R})$$
where $H^p_{\text{sing}}(M, \mathbb{R})$ is the simplicial cohomology group of $M$ with respect to $\mathbb{R}$. Also,

$$H^p_{DR}(M, \mathbb{R}) \simeq \check{H}^p(M, \mathbb{R})$$

where $\check{H}^p(M, \mathbb{R})$ is the Čech cohomology group of $M$ for the constant sheaf $\mathbb{R}$. (cf., [GH78], p.44).

If we consider complex-valued forms, we take the complex $K^p := C^\infty(M, \mathbb{C} \otimes \wedge^p T^*M)$ and can similarly define $H^p_{DR}(M, \mathbb{C}) := Z^p(M, \mathbb{C})/B^p(M, \mathbb{C})$. We have

$$H^p_{DR}(M, \mathbb{C}) = \mathbb{C} \otimes H^p_{DR}(M, \mathbb{R}).$$

Dolbeault cohomology groups ([GH78], p.44) The complex analogy of the de Rham theorem was discovered by Dolbeault in 1953.

Let $X$ be a complex manifold. The Dolbeault cohomology is the vector space

$$H^{p,q}(X) := H^q(A^{p,\bullet}(X), \overline{\partial}) := \frac{\text{Ker}(\overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X))}{\text{Im}(\overline{\partial} : A^{p,q-1}(X) \to A^{p,q}(X))}.$$ 

Dolbeault isomorphism theorem says that ([GH78], p.45)

$$H^{p,q}(X, \mathbb{C}) \simeq H^q(X, \Omega^p_X).$$

(5)

where $\Omega^p_X$ is the sheaf $\Omega^p$ of holomorphic $p$-forms over $X$. We’ll discuss in details on $A^{p,q}$. 

16