Chapter 1
Complex Numbers

1.1 Complex Numbers

**Origin of Complex Numbers** Where did the notion of complex numbers came from? Did it come from the equation

\[ x^2 + 1 = 0 \]  \hspace{1cm} (1.1)

as \( i \) is defined today?

No. Very long times ago people had no problem to accept the fact that an equation may have no solution. When Brahmagupta (598-668) introduced a general solution formula

\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

for the quadratic equation \( ax^2 + bx + c = 0 \), he only recognized positive real root.

The starting point of emerging the notion of complex number indeed came from the theory of cubic equation. In the 16th century, cubic equations was solved by the del Ferro-Tartaglia-Cardano formula: a general cubic equation can be reduced into a special one: \( y^3 = py + q \) which can be solved by

\[ y = \sqrt[3]{\frac{q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}} + \sqrt[3]{\frac{q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3}}. \]  \hspace{1cm} (1.2)

Cardano (1501-1576) was the first to introduce complex numbers \( a + \sqrt{-b} \) into algebra, but had misgivings about it. He made a comment that dealing with \( \sqrt{-1} \) "involves mental
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tortures and is truly sophisticated” and these numbers were “as subtle as they are useless.”

Here comes a problem. A cubic equation always has a solution $y$ because we can consider the graph: $y^3 - py - q > 0$ as $y$ is a large positive number and $y^3 - py - q < 0$ as $y$ is a large negatively number so that the graph curve must intersect the $x$-axis. On the other hand, the the number inside the square root in (1.2), $(\frac{q}{2})^2 - (\frac{p}{3})^3$, could be negative. How could the formula (1.2) produce a real solution in this case?

Concerning on this problem, in 1569, Rafael Bombelli (1526-1572) observed that the cubic equation $x^3 = 15x + 4$ does have a root $x = 4$, but by the formula (1.2) gives

$$x = \sqrt[3]{2 + \sqrt{-121}} + \sqrt[3]{2 - \sqrt{-121}}.$$  (1.3)

Bombelli tried to give an explanation. He sets $\sqrt[3]{2 + \sqrt{-121}} = a + ib$ from which he deduces $\sqrt[3]{2 - \sqrt{-121}} = a - ib$ and obtains $a = 2$ and $b = 1$ so that $x = a + bi + a - bi = 2a = 4$. He may consider the equations $2 + \sqrt{-121} = (a + ib)^3$ and $2 - \sqrt{-121} = (a - ib)^3$ and found that $(a, b) = (2, 1)$ is a solution.

This is the first time that the notion of “complex number” appeared. However, Bombelli did not really understood it. After doing this, Bombelli commented: “At first, the thing seemed to me to be based more on sophism than on truth, but I searched until I found the proof.”

John Napier (1550-1617), who invented logarithm, called complex numbers “nonsense.”

Rene Descartes (1596-1650), who was a pioneer to work on analytic geometry and used equation to study geometry, called complex numbers “impossible.” In fact, the terminology “imaginary number” came from Descartes.

Issac Newton (1643-1727) agreed with Descartes. He wrote: “But it id just that the Roots of Equations should be often impossible (complex), lest they should exhibit the cases of Problems that are impossible as if they were possible.”

Gottfried Wilhelm Leibniz (1646-1716), who and Newton established calculus, remarked that imaginary numbers are lide the Holy Ghost of Christian scriptures-a sort of amphibian, midway between existence and nonexistence.

As time passes, mathematicians gradually redefine their thinking and began to believe that complex numbers existed, and set out to make them understood and accepted.

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3 http://library.thinkquest.org/22584/temh3016.htm
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Wallis tried in 1673 to give a geometric representation which failed but was quite close. Johann Bernoulli noted in 1702 that

\[
\frac{dz}{1 + z^2} = \frac{dz}{2(1 + z\sqrt{-1})} + \frac{sz}{2(1 - z\sqrt{-1})}
\]

ans he could have got

\[
tan^{-1} z = \frac{1}{2t} \log \frac{i - z}{i + z}.
\]

In 1732, Leonhard Euler (1707-1783) introduced the notation \( i = \sqrt{-1} \), and visualized complex numbers as points with rectangular coordinates. Euler used the formula

\[
x + iy = r(\cos \theta + i \sin \theta), \quad r = \sqrt{x^2 + y^2}
\]

and visualized the roots of \( z^n = 1 \) as vertices of a regular polygon, which was used before by Cotes (1714) \(^4\). Euler defined the complex exponential, and proved the famous identity

\[
e^{i\theta} = \cos \theta + i \sin \theta.
\]

In 1799 Carl Friedrich Gauss gave the first of his four proofs for the well-known Fundamental Theorem of Algebra: *Any polynomial equation*

\[
a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0 = 0 \quad (a_n \neq 0)
\]

*has exactly \( n \) complex roots.*

In 1811 Gauss wrote to Bessel to indicate that many properties of the classical functions are only fully understood when complex arguments are allowed. In this letter, Gauss described the Cauchy integral theorem but this result was unpublished.

Caspar Wessel (1745-1818) first gave the geometrical interpretation of complex numbers

\[
z = x + iy = r(\cos \theta + i \sin \theta)
\]

where \( r = |z| \) and \( \theta \in \mathbb{R} \) is the polar angle. Wessel’s approach used what we today call vectors. He uses the geometric addition of vectors (parallelogram law) and defined multiplication of vectors in terms of what we call today adding the polar angles and multiplying the magnitudes. Wessel’s paper, written in Danish in 1797.

The same fate awaited the similar geometric interpretation of complex numbers put forth by the Swiss bookkeeper J. Argand (1768-1822) in a small book published in 1806.

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It was only because Gauss used the same geometric interpretation of complex numbers in his proofs of the fundamental theorem of algebra and in his study of quartic residues that this interpretation gained acceptance in the mathematical community.  

William Rowan Hamilton (1805-65) in an 1831 memoir defined ordered pairs of real numbers \((a, b)\) to be a couple. He defined addition and multiplication of couples: 
\[
(a, b) + (c, d) = (a + c, b + d) \\
(a, b)(c, d) = (ac - bd, bc + ad)
\]
This is the first algebraic definition of complex numbers.

**Algebraic Definition of Complex Numbers**

Since the equation 
\[
x^2 + 1 = 0
\]
admits no real solution, we define \(i\) to be a formal solution: \(i^2 + 1 = 0\). We define a complex number to be \(z = x + iy\) where \(x, y \in \mathbb{R}\) are real numbers. \(Re(z) := x\) is called the real part of \(z\), and \(Im(z) := y\) is called the imaginary part of \(z\). Then we define the space of complex numbers 
\[
\mathbb{C} := \{z = x + iy \mid x, y \in \mathbb{R}\}.
\]

For two complex numbers \(z = x + iy\) and \(w = u + iv\), we define

1. (equality) \(z = w \iff x = u \text{ and } y = v\).
2. (addition) \(z + w = (x + u) + i(y + v)\).
3. (multiplication) \(zw = (x + iy)(u + iv) = (xu - yv) + i(xv + yu)\).  
4. (division) \(\frac{w}{z} = w\overline{z}^{-1}\), where \(z \neq 0\) and 
\[
\frac{1}{z} := \frac{1}{x + iy} = \frac{x - iy}{(x + iy)(x - iy)} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}.
\]

As a result, the space of complex numbers \(\mathbb{C}\) is a field.

**Example** Find the value of \(\frac{5}{-3 + 4i}\).

**Solution:** 
\[
\frac{5}{-3 + 4i} = \frac{5(-3 - 4i)}{(-3 + 4i)(-3 - 4i)} = \frac{-15 - 20i}{9 + 16} = \frac{-15 + 20i}{25}.
\]

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6. Here we used the fact that \(i^2 = -1\).
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Geometric Interpretation of Complex Numbers  We can regard a complex number $z = x + iy$ as a vector $(x, y)$ in $\mathbb{R}^2$ so that it has polar representation:

$$z = x + iy = r(\cos \theta + i \sin \theta) = re^{i\theta}$$  \hspace{1cm} (1.4)

where $r = |z| = \sqrt{x^2 + y^2}$ and $\theta$ is the polar angle.

Since (1.4) still holds if we replace $\theta$ by $\theta + 2\pi k$ where $k$ is any integer, $\theta$ is uniquely determined up to $2k\pi$. We also denote $\theta$ by $\arg(z)$. We call $\arg(z)$ the *argument* of $z$. Geometrically $\arg(z)$ simply means the geometric angle of the vector $z$ and the positive $x$-axis.

We use the notation $z = re^{i\theta}$ because of Euler’s formula $e^{i\theta} = \cos \theta + i \sin \theta$ and $z = r(\cos \theta + i \sin \theta)$.

Addition and multiplication  Recall we have defined the addition and multiplication of addition and multiplication of two complex numbers.

Geometrically the addition of complex numbers can be visualized as addition of vectors. Geometrically the multiplication of two complex numbers satisfies:

$$|z_1z_2| = |z_1||z_2|, \quad \arg(z_1z_2) = \arg(z_1) + \arg(z_2), \quad \text{mod}(2\pi).$$  \hspace{1cm} (1.5)

In fact, write $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$. Then $z_1z_2 = r_1r_2[(\cos \theta_1\cos \theta_2 - \sin \theta_1\sin \theta_2) + i(\sin \theta_1\cos \theta_2 + \cos \theta_1\sin \theta_2)] = r_1r_2[\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)].$

\footnote{By $A = B \mod(2\pi)$, we mean that $A - B = 2\pi k$ for some integer $k.$}