CR Submanifolds of a Sphere

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This lecture notes is an extended version of my lecture series given at the workshop that took place at the Department of Mathematics, Seoul National University, in Seoul in 2009.

It will survey the theory of proper holomorphic mappings between balls and its past and recent development. This theory was originated from Poincaré’s work in 1807: any non-constant holomorphic map \( f : U \to V \) satisfying \( f(U \cap \partial B^2) \subset V \partial B^2 \) is a map in \( \text{Aut}(\partial B^2) \), where \( U, V \) are open subsets of \( \mathbb{C}^2 \). Over time many mathematicians made contribution to this theory.

In Chapter 1, we introduce some background information and introduce one problem: the first gap theorem. This theorem started from 1979 by Webster, and is an accumulative result by many mathematicians over 20 years. There are two approaches for this theorem: analytic one and geometric one. We shall discuss the analytic approach in Chapter 1.

In Chapter 2, from the first gap theorem, we introduce the second and the third gap theorem, and also a lots of specific examples of proper holomorphic mappings between balls, from which a general conjecture about gap phenomenon is formulated. All constructed examples seem to be all polynomial maps, nevertheless, not every proper rational map between balls are equivalent to polynomial maps. A criterion, which tells when a proper rational map can be equivalent to a polynomial one, is introduced. As a result, explicit such examples will be provided. Although general classification seems to be far from reaching, there are complete classification for such rational maps from \( B^2 \) to \( B^N \) with degree 2. To illustrate the method that used to study such classification, we shall show a new proof for Faran’s theorem on classification of maps from \( B^2 \) to \( B^3 \).

In Chapter 3, we shall start with a result on maps from \( B^n \) to \( B^{2n-1} \). We list main ingredient of the proof, and discuss its generalization for higher codimensional case. As a result, we shall demonstrate applications of these generalizations, including the rationality problems, and the proof of the second gap theorem. Besides the analytic approach, we also introduce a geometric approach, namely, the Cartan’s moving frame theory in differential geometry, as well as its applications.

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Chapter 1

Earlier Result: The First Gap Theorem

1.1 Domains and Their Boundaries

Geometry on a domain in $\mathbb{C}^n$ and on its boundary are closely related. We start with several theorems concerning domains in $\mathbb{C}^n$ and their boundaries.

Theorem 1.1.1 [Fe74][B43] Let $D_1, D_2 \subset \mathbb{C}^n$ be smooth strongly pseudoconvex domains with $C^\infty$ boundaries. Then the following statements are equivalent:

(i) There exists a biholomorphic map $f : D_1 \to D_2$.
(ii) There is a $C^\infty$ CR isomorphism $F : \partial D_1 \to \partial D_2$.

Theorem 1.1.2 (i) [CJ96] If $\Omega$ is a bounded simply connected domain in $\mathbb{C}^{n+1}$ with connected smooth spherical real analytic boundary, then $\Omega$ is globally biholomorphic to the unit ball $\mathbb{B}^{n+1}$.

(ii) [HJ98] The “simply connected” condition can be dropped if the boundary is defined by a real polynomial.

Let us denote by

$$\mathbb{B}^n = \{ z = (z_1, ..., z_n) \in \mathbb{C}^n \mid |z|^2 = |z_1|^2 + ... + |z + n|^2 < 1 \}$$

the unit ball, denote by

$$\mathbb{H}^n := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im}(w) > |z|^2 \}$$
the Siegel upper-half space and denote by
\[ \partial \mathbb{H}^n := \{(z,w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im}(w) = |z|^2\} \]
the Heisenberg hypersurface. By the Cayley transformation, a biholomorphic map,
\[ \rho_n : \mathbb{H}^n \to \mathbb{B}^n, \quad \rho_n(z,w) = \left( \frac{2z}{1-iw}, \frac{1+iw}{1-iw} \right), \]
we can identify \( \mathbb{B}^n \) with \( \mathbb{H}^n \) and identify \( \partial \mathbb{H}^n \) with \( \partial \mathbb{B}^n \).

Also we have \( \text{Aut}(\partial \mathbb{B}^n) = \text{Aut}(\partial \mathbb{H}^n) \). Denote
\[ \text{Aut}_0(\partial \mathbb{H}^n) = \{ F \in \text{Aut}(\partial \mathbb{H}^n) \mid F(0) = 0 \}. \]

It is known that any \( F = (f,g) \in \text{Aut}_0(\partial \mathbb{H}^n) \) is of the form
\[ f(z) = \frac{\lambda(z + \bar{a}w)U}{1 - 2i\langle z, \bar{a} \rangle - (r + \langle \bar{a}, \bar{a} \rangle)w}, \]
\[ g(z) = \frac{\sigma \lambda^2 w}{1 - 2i\langle z, \bar{a} \rangle - (r + \langle \bar{a}, \bar{a} \rangle)w} \]
where \( \sigma = \pm, \lambda > 0, r \in \mathbb{R}, \bar{a} \in \mathbb{C}^{n-1}, U \) is an \((n-1) \times (n-1)\) unitary matrix.

Let us verify \((f,g) \in \text{Aut}_0(\partial \mathbb{H}^n)\), i.e., to verify
\[ \text{Im}(g) = |f|^2, \quad \forall \text{Im}(w) = |z|^2, \]
i.e. to verify that for any \( \text{Im}(w) = |z|^2, \)
\[ \frac{\sigma \lambda^2 w}{1 - 2i\langle z, \bar{a} \rangle - (r + \langle \bar{a}, \bar{a} \rangle)w} - \frac{\sigma \lambda^2 \bar{w}}{1 + 2i\langle \bar{z}, \bar{a} \rangle - (r - \langle \bar{a}, \bar{a} \rangle)\bar{w}} = 2i \left| \frac{\lambda(z + \bar{a}w)U}{1 - 2i\langle z, \bar{a} \rangle - (r + \langle \bar{a}, \bar{a} \rangle)w} \right|^2, \]
i.e., to verify
\[ \sigma \lambda^2 w \left[ 1 + 2i\langle z, \bar{a} \rangle - (r - \langle \bar{a}, \bar{a} \rangle)\bar{w} \right] - \sigma \lambda^2 \bar{w} \left[ 1 - 2i\langle \bar{z}, \bar{a} \rangle - (r + \langle \bar{a}, \bar{a} \rangle)w \right] = 2i \left| \lambda(z + \bar{a}w)U \right|^2, \quad \forall \text{Im}(w) = |z|^2. \quad (1.1) \]

Notice
\[ \left| \lambda(z + \bar{a}w)U \right|^2 = \langle \lambda(z + \bar{a}w)U, \lambda(z + \bar{a}w)\bar{U} \rangle. \]
Motivated from the equation \( Im(w) = |z|^2 \), we define the weighted degree:

\[
deg(z^j) = deg(\overline{z}^j) = j \quad \text{and} \quad deg(w^k) = deg(\overline{w}^k) = 2k.
\]

To prove the equality in (1.1), we first prove the equality involving all terms of weighted degree 2 (i.e., the \( z^2, \overline{z}^2, w \) and \( \overline{w} \) terms) in (1.1):

\[
\sigma \lambda^2 w - \sigma \lambda^2 \overline{w} = 2i\lambda \langle zU, \overline{zU} \rangle, \quad \forall \: Im(w) = |z|^2.
\]

Since \( U \) is unitary, we need to show

\[
\sigma \lambda^2 w - \sigma \lambda^2 \overline{w} = 2i\lambda^2 \sigma \langle z, \overline{z} \rangle, \quad \forall \: Im(w) = |z|^2,
\]

which is true.

Secondly, we prove the equality involving all terms of weighted degree 3 (i.e., the \( zw, \overline{zw}, \overline{z}w \) and \( z\overline{w} \) terms) in (1.1):

\[
\sigma \lambda^2 w 2i\langle z, \overline{a} \rangle - \sigma \lambda^2 \overline{w} - 2i\langle z, \overline{a} \rangle = 2i\langle \lambda zU, \lambda \overline{aw}U \rangle + 2i\langle \lambda \overline{aw}U, \lambda \overline{zU} \rangle, \quad \forall \: Im(w) = |z|^2,
\]

Since \( U \) is unitary, the above is equivalent to

\[
\sigma \lambda^2 w 2i\langle \overline{z}, \overline{a} \rangle - \sigma \lambda^2 \overline{w} - 2i\langle \overline{z}, \overline{a} \rangle = 2i\lambda^2 \sigma \langle \overline{z}, \overline{aw} \rangle + 2i\lambda^2 \sigma \langle \overline{aw}, \overline{z} \rangle, \quad \forall \: Im(w) = |z|^2,
\]

which is true.

Finally we prove the equality involving all terms of weighted degree 4 (i.e., the \( w\overline{w} \) terms) in (1.1), which is the highest weighted degree case:

\[
-\sigma \lambda^2 (r - i|\overline{a}|^2)|w|^2 + \sigma \lambda^2 (r + i|\overline{a}|^2)|w|^2 = 2i\langle \lambda \overline{aw}U, \lambda \overline{aw}U \rangle, \quad \forall \: Im(w) = |z|^2.
\]

Since \( U \) is unitary, divided by \( |w|^2 \), the above is equivalent to

\[
-\sigma \lambda^2 (r - i|\overline{a}|^2) + \sigma \lambda^2 (r + i|\overline{a}|^2) = 2i\sigma \lambda^2 |\overline{a}|^2 \quad \forall \: Im(w) = |z|^2,
\]

which is true.
1.2 CR Geometry

CR geometry originated from a work by Poincaré in 1907 [P07]: any non-constant holomorphic map \( f : U \to V \) satisfying \( f(U \cap \partial \mathbb{B}^2) \subset V \partial \mathbb{B}^2 \) is a map in \( \text{Aut}(\partial \mathbb{B}^2) \), where \( U, V \) are open subsets of \( \mathbb{C}^2 \). N. Tanaka [T62] extended this result to high dimensional case. Poincaré-Tanaka theorem could be regarded as a CR analogue of the following classical Liouville’s Conformality Theorem. In the Euclidean space \( \mathbb{E}^n \) with \( n \geq 3 \), the only conformal mappings are inversions, similarity transformations, and congruence transformations. More precisely, let \( U, V \) be open subsets in \( \mathbb{R}^n \) with \( n \geq 3 \), equipped with the flat metric \( \omega \), and \( f : U \to V \) a smooth map. Then \( f \) is conformal (i.e., if \( f^* (\omega) = e^u \omega \) for some continuous function \( u \)) if and only if \( f \) is a Mobius transformation: A composition of the following type of transformations: (i) translations, (ii) rotations, (iii) scalings and inversions.

By E. Cartan [Ca32]-Chern-Moser [CM74]’s work, complete invariants for local Levi non-degenerate real hypersurfaces are constructed.

These two pieces of work laid down the foundation of CR geometry.

A **CR manifold** is a differentiable manifold together with a geometric structure modeled on that of a real hypersurface in \( \mathbb{C}^n \). More precisely, a CR manifold is a differentiable \( (2n + 1) \)-dimensional manifold \( M \) together with a subbundle of the complexified tangent bundle \( CTM = TM \otimes \mathbb{C} \) such that

\[
[L, L] \subseteq L, \quad \text{and} \quad L \cap \overline{L} = \{0\}.
\]

1.3 CR Submanifold in a Sphere

\[
\{\text{CR submanifolds in hyperquadratic}\} \subset \{\text{Embeddable CR manifolds}\} \subset \{\text{CR manifolds}\}
\]

It has long been known that generic 3-dimensional CR manifolds are locally not embeddable, and that all strictly pseudoconvex CR manifolds of dimension 7 and higher are locally embeddable, but the 5- dimensional strictly pseudoconvex case remains open.

Forstnerič [Fo86b] and Faran [Fa88] proved the existence of real analytic strictly pseudoconvex hypersurfaces \( M^{2n+1} \subset \mathbb{C}^{n+1} \) which do not admit any germ of holomorphic mapping
1.3. CR SUBMANIFOLD IN A SPHERE

taking \( M \) into sphere \( \partial \mathbb{B}^{N+1} \) for any \( N \). We may compare this with the Cartan-Janet theorem which asserted that for any analytic Riemannian manifold \((M^n, g)\), there exist local isometric embeddings of \( M^n \) into Euclidean space \( \mathbb{E}^N \) as \( N \) is sufficiently large.

On the other hand, by Webster [W78b], any Levy-nondegenerate real-algebraic hypersurface is holomorphically embeddable into a nondegenerate hyperquadric. Here by nondegenerate hyperquadric, we mean

\[
\partial \mathbb{H}^n_{\ell} := \left\{ (z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} \mid \text{Im}(w) = -\sum_{j=1}^{\ell} |z_j|^2 + \sum_{j=\ell+1}^{n-1} |z_j|^2 \right\}.
\]

We can assume \( \ell \leq \frac{n-1}{2} \), and \((\ell, n-1-\ell)\) is called the signature. When \( \ell = 0 \), it is the standard Heisenberg hypersurface \( \partial \mathbb{H}^n \).

From above, it leads us to concentrate on a subclass of the set of all CR manifolds:

\[
\{ \text{CR submanifolds in a sphere} \partial \mathbb{B}^{N+1} \}
\]

S.-Y. Kim and J.-W. Oh [KO06] gave a necessary and sufficient condition for local embeddability into a sphere \( \partial \mathbb{B}^{N+1} \) of a generic strictly pseudoconvex pseudohermitian CR manifold \((M^{2n+1}, \theta)\) in terms of its Chern-Moser curvature tensors and their derivatives.

Zaitsev [Za08] constructed explicit examples for the Forstnerič and Faran phenomenon above.

Ebenfelt, Huang and Zaitsev [EHZ04] proved rigidity of CR embeddings of general \( M^{2n+1} \) into spheres with CR co-dimension \( < \frac{n}{2} \), which generalizes a result of Webster that was for the case of co-dimension 1 [W79]. Here by rigidity, we mean that for any two smooth CR immersions \( f \) and \( \tilde{f} : M^{2n+1} \to \partial \mathbb{B}^{n+d+1} \) with \( d < \frac{n}{2} \), there exists \( \phi \in \text{Aut}(\partial \mathbb{B}^{n+1+d}) \) such that \( \tilde{f} = \phi \circ f \).

Very recently, Ji and Yuan [JY09] proved that if a CR submanifold \( M \) with hypersurface type of \( \partial \mathbb{B}^N \) and with zero CR second fundamental form, then \( M \) is the image of a sphere by a linear map.

The most basic and non-trivial example of CR submanifolds in a sphere \( \partial \mathbb{B}^{N+1} \) is the image \( M = F(\partial \mathbb{B}^{n+1}) \) where

\[
F : \mathbb{B}^{n+1} \to \partial \mathbb{B}^{N+1}
\]
is a proper holomorphic map that is $C^2$-smooth up to the closed ball $\mathbb{B}^{n+1}$. Here the $C^2$-smooth condition allows the map $F$ restricted on the sphere to become a CR mapping

$$F : \partial \mathbb{B}^{n+1} \to \partial \mathbb{B}^{N+1}.$$ 

1.4 Proper Holomorphic Maps Between Balls

It leads us to concentrate on a subclass of the set of CR submanifolds in a sphere:

$$\text{Prop}(\mathbb{B}^n, \mathbb{B}^N) := \{\text{proper holomorphic map } F : \mathbb{B}^n \to \mathbb{B}^N\},$$

$$\text{Prop}_k(\mathbb{B}^n, \mathbb{B}^N) := \text{Prop}(\mathbb{B}^n, \mathbb{B}^N) \cap C^k(\mathbb{B}^{n+1}),$$

$$\text{Rat}(\mathbb{B}^n, \mathbb{B}^N) := \text{Prop}(\mathbb{B}^n, \mathbb{B}^N) \cap \{\text{rational maps}\}.$$

$$\text{Poly}(\mathbb{B}^n, \mathbb{B}^N) := \text{Prop}(\mathbb{B}^n, \mathbb{B}^N) \cap \{\text{polynomial maps}\}.$$

We say that $F, G \in \text{Prop}(\mathbb{B}^n, \mathbb{B}^N)$ are equivalent, denoted as $F \equiv G$, if there are automorphisms $\sigma \in \text{Aut}(\mathbb{B}^n)$ and $\tau \in \text{Auto}(\mathbb{B}^N)$ such that $F = \tau \circ G \circ \sigma$, i.e., the following diagram commutes

$$\begin{array}{c}
\mathbb{B}^n \xrightarrow{G} \mathbb{B}^N \\
\uparrow \sigma \quad \bigcirc \quad \downarrow \tau \\
\mathbb{B}^n \xrightarrow{F} \mathbb{B}^N.
\end{array}$$

H. Alexander [A77] further proved that any proper holomorphic map from $\mathbb{B}^n$ onto $\mathbb{B}^n$ must be an automorphism when $n \geq 2$.

The condition that $n \geq 2$ is crucial. In fact, when $n = 1$,

$$\text{Prop}(\mathbb{B}^1, \mathbb{B}^1) = \left\{F(z) = \prod_{j=1}^m \frac{z - a_j}{1 - \overline{a_j}z}, \text{ with } |a_j| < 1 \right\}.$$ 

Bochner and Martin [BM48] found a necessary and sufficient condition for mappings in $\text{Prop}(\mathbb{B}^n, \mathbb{B}^N)$ in terms of its power series centered at the origin. More precisely, if $F = (f_1, \ldots, f_h)$ is written as power series

$$f_j(z) = \sum a_{n_1\cdots n_k}^{(j)} z_1^{n_1} \cdots z_k^{n_k}, \quad j = 1, \ldots, h,$$
1.4. PROPER HOLOMORPHIC MAPS BETWEEN BALLS

then $F$ maps $\partial B^k$ into $\partial B^h$ if and only if

$$
\sum_{j=1}^{h} a_{m_1 \ldots m_k}^{(j)} \overline{a_{n_1 \ldots n_k}^{(j)}} = 0, \text{ for } (m_1 - n_1)^2 + \ldots + (m_k - n_k)^2 > 0,
$$

and

$$
\sum_{j=1}^{h} \left| a_{m_1 \ldots m_k}^{(j)} \right|^2 = \frac{(n_1 + \ldots + n_k)!}{n_1! \cdots n_k!} A_{n_1 + \ldots + n_k},
$$

where $A_N$ are suitable nonnegative numbers.

It was discovered in the early 80’s (cf. [Fo93][H99]) that $\text{Prop}(\mathbb{B}^n, \mathbb{B}^N)$ is much larger than $\text{Prop}_k(\mathbb{B}^n, \mathbb{B}^N)$ in general. In fact, there are some mappings $F \in \text{Prop}(\mathbb{B}^n, \mathbb{B}^{n+1}) \cap C^0(\mathbb{B}^n)$ but they are neither in $\text{Prop}_2(\mathbb{B}^n, \mathbb{B}^{n+1})$ nor in $\text{Rat}(\mathbb{B}^n, \mathbb{B}^{n+1})$.

For any $F \in \text{Prop}_2(\mathbb{B}^{n+1}, \mathbb{B}^{N+1})$, it induces a $C^2$ smooth CR map from $\partial \mathbb{B}^{n+1}$ into $\partial \mathbb{B}^{N+1}$.

Webster was the first to investigate the geometric structure of proper holomorphic maps between balls in complex spaces of different dimensions. In 1979, he showed [W79] that a proper holomorphic map $F \in \text{Prop}_3(\mathbb{B}^n, \mathbb{B}^{n+1})$ with $n > 2$ is indeed a linear fractional embedding.

Forstnerič shown [Fo86] that

$$
\text{Prop}_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N) = \text{Rat}(\mathbb{B}^n, \mathbb{B}^N).
$$

Moreover, such $F$ has no poles on $\partial \mathbb{B}^n$ by Cima-Suffridge [CS90].

J.P. D’Angelo did lots of work on polynomial and monomial mappings in $\text{Prop}_k(\mathbb{B}^n, \mathbb{B}^N)$ [DA88][DA92][DA93], in particular he found the structure of proper holomorphic polynomial mappings between balls.
1.5 The First Gap Theorem

Theorem 1.5.1 (The First Gap Theorem) For $N < 2n - 1$, any map $F \in \text{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ is equivalent to the linear map $(z, 0, w)$.

This theorem is a result by many mathematicians over 20 years.

In 1979, S. Webster proved [W79] that any mapping in $\text{Prop}_3(\mathbb{B}^n, \mathbb{B}^{n+1})$ with $n \geq 3$ must be equivalent to a linear map $(z, 0, w)$.

In 1982, J. Faran [Fa82] proved that there are exactly four maps in $\text{Prop}_3(\mathbb{B}^2, \mathbb{B}^3)$, up to equivalence class.

Next year, A. Cima and T.J. Suffridge [CS83] improved the above results of Webster and Faran by replacing “$\text{Prop}_3$” with “$\text{Prop}_2$”. In the same paper [CS83], A. Cima and T. J. Suffridge conjectured that any mapping in $\text{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ with $n \geq 3$ and $N \leq 2n - 2$ should be equivalent to the linear map $(z, 0, w)$.

In 1986, Faran [Fa86] proved the Cima-Suffridge’s conjecture under the assumption that $F$ is holomorphic in a neighborhood of $\mathbb{B}^n$.

In the same year, F. Forstnerič [Fo86] proved $\text{Prop}_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N) = \text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ and later Cima and Suffridge [CS90] shown that any mapping in $\text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ must be holomorphic on the boundary. As a consequence, the First Gap Theorem is proved for any $F \in \text{Prop}_{N-n+1}(\mathbb{B}^n, \mathbb{B}^N)$ with $N < 2n - 1$. 

\[ \begin{array}{cccc}
0 & 1 & 2 & n \\
& 2n - 1 \\
& 2n
\end{array} \]
In 1999 X. Huang [Hu99] proved that any mapping in \( \text{Prop}_2(\mathbb{B}^n, \mathbb{B}^N) \) with \( N \leq 2n - 2 \) is equivalent to the linear map \((z, 0, w)\).

**Outline of the Proof for the First Gap Theorem:**

**Step 1.** if \( N < 2n - 1 \), it implies that its geometric rank \( \kappa_0 = 0 \).

- *(analytic proof)* Use Uniqueness theorem (see Corollary 1.15.1 and Theorem 1.15.2 below).
- *(geometric proof)* Use the formula

\[
N \geq n + \frac{(2n - \kappa_0 - 1)\kappa_0}{2}
\]

for any \( F \in \text{Prop}_2(\mathbb{B}^n, \mathbb{B}^N) \) with geometric rank \( \kappa_0 \). In fact, if \( N < 2n - 1 \), the above inequality forces \( \kappa_0 = 0 \).

**Step 2.** Show: \( \kappa_0 = 0 \iff F \) is a linear fractional map.

- *(analytic proof)* The first order PDE argument (see Theorem 1.14.1 below).
- *(geometric proof)* \( \kappa_0 = 0 \iff \) the CR second fundamental form \( II_M = 0 \iff F \) is a linear fractional map. \( \square \)

We need to explain the following:

1. What is the geometric rank \( \kappa_0 \) of a map \( F \)? (see (1.71) below, or [HJ01])

2. Why \( N \geq n + \frac{(2n - \kappa_0 - 1)\kappa_0}{2} \)? (see Corollary 3.2.2, or [H03])

3. Why \( \kappa_0 \) if and only if \( II_M = 0 \)? (see Corollary 3.14.3, [JY09][HJ09])

4. Why \( II_M = 0 \) if and only if \( F \) is a linear fractional map (see Theorem 3.8.1, [JY09]).
1.6 Passing from $\partial B^n$ to $\partial H^n$

Recall the Heisenberg hypersurface

$$\partial H^n := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im}(w) = |z|^2\}$$

and the Cayley transformation

$$\rho_n : H^n \to B^n, \quad \rho_n(z, w) = \left(\frac{2z}{1 - iw}, \frac{1 + iw}{1 - iw}\right).$$

We can define the space $\text{Prop}(H^n, H^N)$, $\text{Prop}^k(H^n, H^N)$ and $\text{Rat}(H^n, H^N)$. We can identify a map $F \in \text{Prop}^k(B^n, B^N)$ or $\text{Rat}(B^n, B^N)$ with $\rho_N^{-1} \circ F \circ \rho_n$ in the space $\text{Prop}^k(H^n, H^N)$ or $\text{Rat}(H^n, H^N)$, respectively.

We say that $F$ and $G \in \text{Prop}(H^n, H^N)$ are equivalent if there are automorphisms $\sigma \in \text{Aut}(H^n)$ and $\tau \in \text{Aut}(H^N)$ such that $F = \tau \circ G \circ \sigma$.

![Diagram](image)

1.7 Differential Operators on $\partial H^n$

The vector fields $\{L_1, \ldots, L_{n-1}\}$, where $L_j := 2i x_j \frac{\partial}{\partial w} + \frac{\partial}{\partial z_j}$, form a global basis for the complex tangent bundle $\mathbb{C}T^{1,0}H^n$ over $\partial H^n$, and their conjugates $\{\overline{L}_1, \ldots, \overline{L}_{n-1}\}$, called CR vector fields, form a global basis for the complex tangent bundle $\mathbb{C}T^{0,1}H^n$ over $\partial H^n$. Recall that for $z_j = x_j + iy_j$ and for $w = u + iv$, we have

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

and

$$\frac{\partial}{\partial w} = \frac{1}{2} \left( \frac{\partial}{\partial u} - i \frac{\partial}{\partial v} \right), \quad \frac{\partial}{\partial \overline{w}} = \frac{1}{2} \left( \frac{\partial}{\partial u} + i \frac{\partial}{\partial v} \right).$$

There is a real vector field which is transversal to $\mathbb{C}T^{(1,0)}H^n + \mathbb{C}T^{(0,1)}H^n$

$$T = \frac{\partial}{\partial \text{Re}(w)} = \frac{\partial}{\partial u} = \frac{\partial}{\partial w} + \frac{\partial}{\partial \overline{w}}$$

(1.2)
1.7. DIFFERENTIAL OPERATORS ON $\partial \mathbb{H}^N$

which is called the Reeb vector field.

The vector fields $\{L_1, \ldots, L_{n-1}, L_1, \ldots, L_{n-1}, T\}$ forms a basis of $CT\partial \mathbb{H}^n$.

Lemma 1.7.1 (i) $TL_j = L_jT$, $T\bar{L}_j = \bar{L}_jT$, and $L_jL_k = L_kL_j$ for all $1 \leq j, k \leq n - 1$.

(ii) For any continuous CR function $h$ over an open subset $M_1 \subset \partial \mathbb{H}^n$, $T h$ is a CR distribution over $M_1$. For any $1 \leq j, k \leq n - 1$, $\bar{L}_k(L_jh) = -[L_j, \bar{L}_k]h = 2i\delta_{kj}T h$.

(iii) Let $h$ be a $C^2$ CR function over $\partial \mathbb{H}^n$ and $\chi$ a $C^1$ function over $\partial \mathbb{H}^n$. Then for any integer $k > 0$, we have

\[
\bar{L}_k(L_jh) = 4iL_k(T(h))\chi + L_k^2(h)\bar{L}_k(\chi),
\]

\[
L_k(L_k(T(h)))\chi = 2iT^2(h)\chi + L_k(T(h))L_k(\chi)
\]
in the sense of currents.

(iv) For any $k, l, j$ and any $C^2$ CR function $h$, we have

\[
\bar{L}_kL_lL_jh = -2z_l\delta_{kj}\frac{\partial^2 h}{\partial u^2} - 2z_j\delta_{kl}\frac{\partial^2 h}{\partial u^2} + 2i\delta_{kj}\frac{\partial^2 h}{\partial u\partial z_l} + 2i\delta_{kl}\frac{\partial^2 h}{\partial u\partial z_j}
\]
in the sense of currents. In particular, we have

\[
\bar{L}_kL_lL_jh = \begin{cases} 
0, & \text{when } k \neq l \text{ and } k \neq j; \\
2iT(L_ih), & \text{when } k = j \neq l; \\
2iT(L_jh), & \text{when } k = l \neq j; \\
4iT(L_kh), & \text{when } k = j = l.
\end{cases}
\]

Proof of Lemma 1.7.1 : (i) For any differentiable function $f(z, \bar{z}, w, \bar{w})$,

\[
T(L_jf) = \left(\frac{\partial}{\partial w} + 2i\bar{z}_j\frac{\partial}{\partial \bar{w}}\right)\left(\frac{\partial f}{\partial z_j} + 2i\bar{z}_j\frac{\partial f}{\partial \bar{w}}\right) = \frac{\partial^2 f}{\partial w\partial z_j} + 2i\bar{z}_j\frac{\partial^2 f}{\partial w\partial \bar{w}} + \frac{\partial^2 f}{\partial \bar{w}\partial z_j} + 2i\bar{z}_j\frac{\partial^2 f}{\partial \bar{w}\partial \bar{w}},
\]

\[
L_j(Tf) = \left(\frac{\partial}{\partial \bar{z}_j} + 2i\bar{z}_j\frac{\partial}{\partial \bar{w}}\right)\left(\frac{\partial f}{\partial \bar{w}} + \frac{\partial f}{\partial \bar{w}}\right) = \frac{\partial^2 f}{\partial \bar{w}\partial z_j} + 2i\bar{z}_j\frac{\partial^2 f}{\partial \bar{w}\partial \bar{w}} + \frac{\partial^2 f}{\partial \bar{w}\partial z_j} + 2i\bar{z}_j\frac{\partial^2 f}{\partial \bar{w}\partial \bar{w}}.
\]

Then $T L_j = L_j T$. Similarly, $L_jL_k = L_kL_j, \forall 1 \leq j, k \leq n - 1$. 
(ii) The first statement follows from (i): \( Th \) is CR because \( \overline{L}_j Th = T \overline{L}_j h = 0 \). The second statement follows from the following calculation:

\[
[L_j, \overline{L}_k] = \left( \frac{\partial}{\partial \overline{z}_j} + 2i \overline{z}_j \frac{\partial}{\partial w} \right) \left( \frac{\partial}{\partial \overline{z}_k} - 2i z_k \frac{\partial}{\partial w} \right) - \left( \frac{\partial}{\partial \overline{z}_k} - 2i z_k \frac{\partial}{\partial w} \right) \left( \frac{\partial}{\partial \overline{z}_j} + 2i \overline{z}_j \frac{\partial}{\partial w} \right) = -2i \delta_{jk} \frac{\partial}{\partial w} - 2i \delta_{jk} \frac{\partial}{\partial w} = -2i \delta_{kj} T.
\]

(iii) It is sufficient to prove (iii) for any holomorphic polynomial \( h \) by a lemma below. By (ii), we know that \( Th \) is CR and that \( \overline{L}_k L_k h = 2i Th \). This follows the second identity. To prove the first identity, it is sufficient to prove \( \overline{L}_k L_k^2 h = 4i L_k Th, \ \forall \ C^2 \ CR \ function \ h. \) (1.3)

In fact, \( \overline{L}_k L_k^2 h \) equals to

\[
([\overline{L}_k, L_k] + L_k \overline{L}_k) L_k h = 2i T L_k h + L_k ([\overline{L}_k, L_k] + L_k \overline{L}_k) h = 2i T L_k h + 2i L_k Th + 0 = 4i T L_k h.
\]

(iv) It is sufficient to prove (iv) for any holomorphic polynomial \( h \) as above. Consider

\[
[\overline{L}_k, L_\ell] L_j h = ([\overline{L}_k, L_\ell] + L_\ell \overline{L}_k) L_j h
= 2i \delta_{k\ell} T L_j h + L_\ell ([\overline{L}_k, L_j] + L_j \overline{L}_k) h
= 2i \delta_{k\ell} T L_j h + L_\ell 2i \delta_{kj} T h + 0 = 2i \delta_{k\ell} T L_j h + 2i \delta_{kj} T L_\ell h
= \begin{cases} 0, & \text{if } k \neq j, k \neq \ell, \\ 2i TL_\ell h, & \text{if } k = j, j \neq \ell, \\ 2i TL_\ell ll h, & \text{if } k = \ell \neq j, \\ 4i TL_k h, & k = j = \ell. \end{cases}
\]

by using the similar computation. \( \square \)

### 1.8 Equations Associated with \( F \)

Let \( F = (f, \phi, g) = (\tilde{f}, g) : M_1 \cap \partial \mathbb{H}^n \to \partial \mathbb{H}^N \) be a non-constant \( C^2 \)-smooth CR map with \( F(0) = 0 \), where \( M_1 \) is an open subset of \( \partial \mathbb{H}^n \). We denote \( f = (f_1, \ldots, f_{n-1}) \), \( \phi = (\phi_1, \ldots, \phi_{N-n}) \) and \( \tilde{f} = (f, \phi) \). The basic equation is

\[
\frac{Im g}{2i} = \tilde{f} \cdot \overline{f} = \langle \tilde{f}, \overline{f} \rangle, \ \forall (z, w) \in M_1
\]
1.8. EQUATIONS ASSOCIATED WITH $F$

i.e.,

$$\frac{g - \bar{g}}{2i} = \sum_{j=1}^{n-1} |f_j|^2 + \sum_{j=1}^{N-n} |\phi_j|^2, \quad \forall (z, w) \in M_1 \text{ with } \text{Im}(w) = |z|^2. \quad (1.4)$$

By the Lewy Extension Theorem, $F$ extends holomorphically to a certain pseudoconvex side of $M_1$ denoted by $\Omega$.

Let us differentiate (1.4) by $L_j$ and $T$. First we consider the first order differential operators: $L_l$ and $T$, $1 \leq l \leq n - 1$: $L_l g = \sum_j L_l f_j \cdot \bar{f_j} + \sum_j L_l \phi_j \cdot \bar{\phi_j}$, $\forall (z, w) \in M_1$,

$$\frac{L_l g}{2i} = \sum_j L_l f_j \cdot \bar{f_j} + \sum_j L_l \phi_j \cdot \bar{\phi_j}, \quad \forall (z, w) \in M_1, \quad (1.5)$$

We also apply $\overline{L_l}$ to (1.5) and use $\overline{L_l L_l} = 2iT$ to obtain

$$Tg = 2i\langle T\bar{f}, f \rangle + |L_l \bar{f}|^2, \quad \forall (z, w) \in M_1. \quad (1.7)$$

We consider the second order differential operators $L_k L_l, TL_l$ and $T^2$, $1 \leq k, l \leq n - 1$:

$$\frac{L_k L_l g}{2i} = \sum_j L_k(L_l \bar{f}) \cdot \bar{f_j}, \quad \forall (z, w) \in M_1. \quad (1.8)$$

$$\frac{1}{2i}T(L_l g) = T(L_l \bar{f}) \cdot \bar{f} + L_l(\bar{f}) \cdot T\bar{f}, \quad \forall (z, w) \in M_1 \quad (1.9)$$

$$\text{Im}(T^2 g) = 2 \text{Im}(iT^2 \bar{f} \cdot \bar{f}^t) + 2|T\bar{f}|^2, \quad \forall (z, w) \in M_1. \quad (1.10)$$

Next we consider the third order differential operators $\overline{L_k L_j L_l}$, $1 \leq k, j, l \leq n - 1$:

$$\frac{1}{2i}\overline{L_k L_j (L_l g)} = \overline{L_k L_j (L_l \bar{f})} \cdot \bar{f} + L_j(L_l \bar{f}) \cdot \overline{L_k \bar{f}}, \quad \forall (z, w) \in M_1. \quad (1.11)$$

When $k \neq j$ and $k \neq l$, by Lemma 1.7.1(iv), (1.12) becomes

$$L_j(L_l \bar{f}) \cdot \overline{L_k \bar{f}} = 0. \quad (1.13)$$
When \( k = j \neq l \), by Lemma 1.7.1 (iv), (1.12) becomes
\[
T(L_l g) = 2iT(L_l \tilde{f}) \cdot \tilde{f} + L_j(L_l \tilde{f}) \cdot L_l \tilde{f}.
\] (1.14)

When \( k = l \neq j \), by Lemma 1.7.1 (iv), (1.12) becomes
\[
T(L_j g) = 2iT(L_j \tilde{f}) \cdot \tilde{f} + L_l(L_j \tilde{f}) \cdot L_j \tilde{f}.
\] (1.15)

When \( k = j = l \), by Lemma 1.7.1(iv) again, we have
\[
2T(L_k g) = 4iT(L_k \tilde{f}) \cdot \tilde{f} + L_k(L_k \tilde{f}) \cdot L_k \tilde{f}.
\] (1.16)

Since \( F(0) = 0 \), by (1.5) and (1.8), we obtain
\[
\frac{\partial g}{\partial z_j} \bigg|_0 = \frac{\partial^2 g}{\partial z_k \partial z_l} \bigg|_0 = 0.
\] (1.17)

### 1.9 The Associated Map \( F^* \) of \( F \)

From (1.11), since \( F(0) = 0 \), we have
\[
\frac{1}{2i}L_k L_j g \bigg|_0 = L_j \tilde{f} \bigg|_0 \cdot L_k \tilde{f} \bigg|_0.
\]

By Lemma 1.7.1, we have
\[
\frac{1}{2i}L_k L_j g \bigg|_0 = \frac{1}{2i}2i \delta_{kj} Tg \bigg|_0 = \lambda \delta_{kj}
\]
where
\[
\lambda = Tg \bigg|_0 > 0.
\] (1.18)

In fact, by (1.7), \( T g \bigg|_0 = 2i \langle T \tilde{f}, \tilde{f} \rangle \bigg|_0 + \| L_l \tilde{f} \|^2 \bigg|_0 = \| L_l \tilde{f} \|^2 \bigg|_0 > 0. \)

Then we have the orthogonal property:
\[
L_j \tilde{f} \bigg|_0 \cdot L_k \tilde{f} \bigg|_0 = \lambda \delta_{kj}.
\]

Denoting
\[
E_l = \left( \frac{\partial \tilde{f}}{\partial z_l} \right) \bigg|_0 = \left( \frac{\partial f_1}{\partial z_l}, \ldots, \frac{\partial f_{n-1}}{\partial z_l}, \frac{\partial \phi_1}{\partial z_l}, \ldots, \frac{\partial \phi_{N-n}}{\partial z_l} \right) \bigg|_0.
\]
Then it has orthogonal property:

\[ E_j E_k^* = \lambda \delta_{kj}. \]  

(1.19)

We extend \( \{E_1/\sqrt{\lambda}, ..., E_{n-1}/\sqrt{\lambda}\} \) to a certain orthonormal basis of \( \mathbb{C}^{N-1} \):

\[
\left\{ \frac{E_1}{\sqrt{\lambda}}, ..., \frac{E_{n-1}}{\sqrt{\lambda}}, C_1, ..., C_{N-n} \right\}.
\]  

(1.20)

Now we define a new map \( F^* = (f^*_l, \phi^*_k, g^*) = H \circ F \) where \( H \in Aut(\mathbb{H}^N) \), which is equivalent to \( F \), defined by

\[ f^*_l = \frac{1}{\lambda} \tilde{f} \cdot \overline{E_l}, \quad \phi^*_k = \frac{1}{\sqrt{\lambda}} \tilde{f} \cdot \overline{C_k}, \quad g^* = \frac{1}{\lambda} g. \]  

(1.21)

\( F^* \) satisfies some initial conditions at 0:

\[
F^*(0) = 0, \quad \frac{\partial f^*_l}{\partial z_l} \bigg|_0 = \delta^l_j, \quad \frac{\partial \phi^*_k}{\partial z_l} \bigg|_0 = 0, \quad \frac{\partial g^*}{\partial z_l} \bigg|_0 = 0, \quad \frac{\partial g^*}{\partial w} \bigg|_0 = 1.
\]  

(1.22)

It is not good enough because we need to take care of the terms \( \frac{\partial f^*_l}{\partial w} \bigg|_0 \) and \( \frac{\partial \phi^*_k}{\partial w} \bigg|_0 \). We need further normalization.

Taking differential and by the chain rule, we have

\[
\left( f^*_l \right)' \bigg|_{z_l} = \frac{1}{\lambda} L_k \tilde{f} \cdot E^*_l \bigg|_{0} = \frac{1}{\lambda} L_k(\tilde{f}) \cdot \overline{L_l(\tilde{f})} \bigg|_{0} = \delta^l_k,
\]

\[
\left( f^*_l \right)' \bigg|_{w} = \frac{1}{\lambda} E_w \cdot E^*_l \bigg|_{0} = \frac{1}{\lambda} T(\tilde{f}) \cdot \overline{L_l(\tilde{f})} \bigg|_{0},
\]

\[
\left( \phi^*_k \right)' \bigg|_{z_l} = \frac{1}{\sqrt{\lambda}} L_k \tilde{f} \cdot C^*_l \bigg|_{0} = 0,
\]

\[
\left( \phi^*_k \right)' \bigg|_{w} = \frac{1}{\sqrt{\lambda}} E_w \cdot C^*_l \bigg|_{0} = \frac{1}{\sqrt{\lambda}} T(\tilde{f}) \cdot \overline{C^*_k} \bigg|_{0},
\]

\[
\left( g^* \right)' \bigg|_{z_l} = \frac{1}{\lambda} \left( L_l g - 2iL_l \tilde{f} \cdot \overline{f} \right) \bigg|_{0} = 0, \quad (By \ (1.5))
\]

\[
\left( g^* \right)' \bigg|_{w} = \frac{1}{\lambda} \left( T g - 2iT \tilde{f} \cdot \overline{f} \right) \bigg|_{0} = 1, \quad (By \ (1.7))
\]
Then (1.22) has been proved. Besides, other formulas up to degree 2 are given as follows.

\[
\left( f_j \right)^{''} \bigg|_{z_{k}z_{l}} = \frac{1}{\lambda} L_k L_l \tilde{f} \cdot L_j \tilde{f} \bigg|_{0}, \quad \left( f_l \right)^{''} \bigg|_{z_{j}w} = \frac{1}{\lambda} L_l T(\tilde{f}) \cdot L_i (\tilde{f}) \bigg|_{0},
\]

\[
\left( f_j \right)^{''} \bigg|_{w^2} = \frac{1}{\lambda} L_k L_l \tilde{f} \cdot C_j \bigg|_{0}, \quad \left( \phi_j \right)^{''} \bigg|_{z_{k}z_{l}} = \frac{1}{\sqrt{\lambda}} L_k L_l \tilde{f} \cdot \tilde{f} \bigg|_{0},
\]

\[
\left( \phi_j \right)^{''} \bigg|_{z_{j}w} = \frac{1}{\sqrt{\lambda}} T L_k \tilde{f} \cdot C_j \bigg|_{0}, \quad \left( \phi_j \right)^{''} \bigg|_{w^2} = \frac{1}{\sqrt{\lambda}} T^2 \tilde{f} \cdot C_j \bigg|_{0},
\]

\[
\left( g^{*} \right)^{''} \bigg|_{z_{l}z_{k}} = \frac{1}{\lambda} \left( L_l L_k g - 2i L_k L_k \tilde{f} \cdot \tilde{f} \right) \bigg|_{0} = 0, \quad (B y (1.8))
\]

\[
\left( g^{*} \right)^{''} \bigg|_{z_{j}w} = \frac{1}{\lambda} L_l \left( T g - 2iT \tilde{f} \cdot \tilde{f} \right) = \frac{2i}{\lambda} L_l \tilde{f} \cdot T \tilde{f} \bigg|_{0},
\]

\[
\left( g^{*} \right)^{''} \bigg|_{w^2} = \frac{1}{\lambda} \left( T^2 g - 2iT^2 \tilde{f} \cdot \tilde{f} - 2iT \tilde{f} \cdot T \tilde{f} \right) \bigg|_{0}.
\]

### 1.10 The Associated Map $F^{**}$ of $F$

We want to define $F^{**} = (\tilde{f}^{**}, g^{**}) = (f^{**}, \phi^{**}, g^{**}) = (f_l^{**}, \phi_k^{**}, g^{**}) = G \circ F^*$, for some $G \in Aut(\partial \mathbb{H}^N)$, such that this normalization $F^{**}$ satisfies the following properties:

\[
F^{**}(0) = 0, \quad \frac{\partial f^{**} - z}{\partial z_j} \bigg|_{0} = \frac{\partial f^{**}}{\partial w} \bigg|_{0} = \delta^l_j, \quad \frac{\partial \phi^{**}}{\partial z_l} \bigg|_{0} = \frac{\partial \phi^{**}}{\partial w} \bigg|_{0} = 0, \quad g^{**} \bigg|_{0} = \frac{g^{**} - w}{\partial w} \bigg|_{0} = 1, \quad (1.23)
\]

and

\[
\frac{\partial^2 g^{**}}{\partial z_j \partial z_k} \bigg|_{0} = Re \frac{\partial^2 g^{**}}{\partial w^2} \bigg|_{0} = 0. \quad (1.24)
\]

This can be done by defining (cf. [H99])

\[
G = \frac{(z^* - aw^*, w^*)}{1 + 2i(z^*, a) + (r - i\|a\|^2)w^*} \in Aut_0(\partial \mathbb{H}^N) \quad (1.25)
\]
where
\[ a := \left( \tilde{f}^* \right)' \bigg|_{w} = \left( \cdots, \frac{T \tilde{f} \cdot L_j \tilde{f}}{\lambda}, \cdots, \frac{T \tilde{f} \cdot C_j^t}{\sqrt{\lambda}}, \cdots \right) \bigg|_{0}, \]
\[ r := \frac{1}{2} Re \left( g^* \right)'' \bigg|_{w} = \frac{1}{2\lambda} Re \left( T^2 g - 2iT^2 \tilde{f} \cdot \tilde{f} \right) \bigg|_{0}. \]  

(1.26)

Taking differential and by the chain rule, we have
\[ \left( f_j^{**} \right)'' \bigg|_{z_j w} = \left( f_j^{**} \right)'' \bigg|_{z_j w} - 2i\delta_j^k \overline{a_i} - 2i\delta_j^k \overline{a_k}, \]
\[ = \frac{1}{\lambda} L_k L_f \tilde{f} \cdot L_j \tilde{f} \bigg|_{l} - \frac{2i\delta_j^k}{\lambda} T \tilde{f} \cdot L_i \tilde{f} \bigg|_{l} - \frac{2i\delta_j^k}{\lambda} T \tilde{f} \cdot L_k \tilde{f} \bigg|_{l}. \]  

(1.27)

We can say more about this important formula which will be used to define geometric rank \( \kappa_0 \). Applying \( T^2 \) to the basic equation \( Im(g) = |\tilde{f}|^2 \), we get \( 0 = 2iIm(T^2 \tilde{f} \cdot \tilde{f}^*) + 2i|T \tilde{f}|^2 - i Im(T^2 g) \) on \( \partial \mathbb{H}^n \) by (1.10). Combining this to the above, we get
\[ \left( f_j^{**} \right)'' \bigg|_{z_j w} = \frac{1}{\lambda} L_j T \tilde{f} \cdot \overline{L_i \tilde{f}} \bigg|_{l} - \frac{2i}{\lambda} \left( T \tilde{f} \cdot \overline{L_i \tilde{f}} \right) \left( L_j \tilde{f} \cdot \overline{T \tilde{f}} \right) \bigg|_{l} \]
\[ - \frac{\delta_j^k}{2\lambda} \left( T^2 g \left( -2iT \tilde{f} \cdot \tilde{f}^* \right) \right) \bigg|_{l}. \]  

(1.28)

We also have
\[ \left( f_i^{**} \right)'' \bigg|_{w^2} = \left( f_i^{**} \right)'' \bigg|_{w^2} - a_i \left( g^* \right)'' \bigg|_{w^2} \]
\[ = \frac{1}{\lambda} T^2 \tilde{f} \cdot L_i \tilde{f} \bigg|_{l} - \frac{1}{\lambda^2} \left( T \tilde{f} \cdot L_i \tilde{f} \right) \left( T^2 g - 2iT \tilde{f} \cdot \tilde{f} - 2i|T \tilde{f}|^2 \right) \bigg|_{l}. \]  

(1.29)
\[
\left( \phi^*_t \right)''_{zjz_k} |_0 = \left( \phi^*_t \right)''_{zjz_k} |_0 - b_t (g^*)''_{zjz_k} = \left( \phi^*_t \right)''_{zjz_k} |_0 = \frac{1}{\sqrt{\lambda}} L_j L_k \tilde{f} \cdot \overline{C_t} |_0. \tag{1.30}
\]

Here we used the fact that \((g^*)''_{zjz_k} |_0 = 0\).

\[
\left( \phi^*_l \right)''_{w^2} |_0 = \left( \phi^*_l \right)''_{w^2} |_0 - b_j (g^*)''_{w^2} |_0 = \frac{1}{\sqrt{\lambda}} T L_j \tilde{f} \cdot \overline{C_t} |_0 - \frac{1}{\lambda^{3/2}} \left( T \tilde{f} \cdot \overline{C_t} \right) \left( T^2 g - 2i T^2 \tilde{f} \cdot \overline{\tilde{f}} \right) |_0. \tag{1.31}
\]

\[
\left( \phi^*_l \right)''_{w^2} |_0 = \left( \phi^*_l \right)''_{w^2} |_0 - 2i \left( g^* \right)''_{w^2} |_0 = \frac{1}{\sqrt{\lambda}} T^2 \tilde{f} \cdot \overline{C_t} |_0 - \frac{1}{\lambda} \left( T^2 \tilde{f} \cdot \overline{\tilde{f}} \right) |_0 - 2i \left| T \tilde{f} \cdot \overline{\tilde{f}} \right|^2 |_0 = 0. \tag{1.32}
\]

\[
\left( g^* \right)''_{zjz_k} |_0 = 0,
\]

\[
\left( g^* \right)''_{zjw} |_0 = 0 - 2i a_j = \frac{2i}{\lambda} L_j \tilde{f} \cdot \overline{\tilde{f}} |_0 - \frac{2i}{\lambda} T \tilde{f} \cdot \overline{L_j \tilde{f}} |_0.
\]

\[
\left( g^* \right)''_{w^2} |_0 = \left( g^* \right)''_{w^2} |_0 - 2i \left( g^* \right)''_{w^2} |_0 - 2 \left[ i \left| a_j \right|^2 + r \right] |_0 = \frac{1}{\lambda} \left( T^2 g - 2i T^2 \tilde{f} \cdot \overline{\tilde{f}} \right) |_0 - \frac{2}{\lambda} \left[ i \left| T \tilde{f} \cdot \overline{\tilde{f}} \right|^2 + \frac{1}{2} \text{Re} \left( T^2 g - 2i T^2 \tilde{f} \cdot \overline{\tilde{f}} \right) \right] |_0 = 0.
\]

This implies \(\text{Re} \left( g^* \right)''_{w^2} |_0 = 0\). Then (1.23) and (1.24) are proved.
1.11 The Cheren-Moser Operator

If $F = F^* \in \text{Prop}_2(\partial \mathbb{H}^n, \partial \mathbb{H}^N)$, then we have

$$f = z + \hat{f}, \ g = w + \hat{g} \ \text{with} \ \hat{f}, \hat{g}, \phi = O(||(z, w)||^2), \ \frac{\partial^2 \hat{g}}{\partial z_i \partial z_k} \big|_0 = \text{Re} \frac{\partial^2 \hat{g}}{\partial w^2} \big|_0 = 0. \quad (1.33)$$

Then we obtain

$$\text{Im}(w + \hat{g}) = \sum_{j=1}^{n-1} |z_j + \hat{f}_j|^2 + \sum_{j=1}^{N-n} |\phi_j|^2, \ (z, w) \in \partial \mathbb{H}^n. \quad (1.34)$$

Let $M_1 \subset \partial \mathbb{H}^n$ be an open subset. For a function $f$ on $M_1$, we denote $h \in o_{wt}(s)$ if

$$\lim_{t \to 0^+} \frac{h(tz, t^2w, t\overline{z}, t^2\overline{w})}{t^s} \to 0$$

uniformly with respect to $(z, w) \approx (0', 0) \in \mathbb{C}^{n_1} \times \mathbb{C}$. In other words, we define \emph{weighted degree} by

$$\deg_{wt}(z^kw^l) = k + 2l.$$

We write $F$ as

$$\hat{f}_j = \sum_{s=2}^{m-1} f_j^{(s)} + o_{wt}(m-1), \ \hat{g} = \sum_{s=3}^{m} \hat{g}^{(s)} + o_{wt}(m), \ \phi_j = \sum_{s=l}^{m-l} \phi_j^{(s)} + o_{wt}(m-l), \ l \geq 2, \quad (1.35)$$

where we denote by $h^{(s)}$ the homogeneous polynomial of $(z, w)$ of weighted degree $s$.

Substituting these into (1.34), we obtain

$$\text{Im}(w) + \text{Im}(\hat{g}) = \sum_j (z_j + \hat{f}_j)(\overline{z}_j + \overline{\hat{f}}_j) + \sum_k (\sum_s \phi_k^{(s)})(\sum_t \overline{\phi}_k^{(t)})$$

$$= |z|^2 + \sum_j (z_j \overline{\hat{f}}_j + \hat{f}_j \overline{z}_j + |\hat{f}_j|^2) + \sum_k (\sum_s \phi_k^{(s)})(\sum_t \overline{\phi}_k^{(t)})$$

$$= |z|^2 + \sum_j \text{Im}(2i(z_j, \hat{f}_j)) + \sum_j |\hat{f}_j|^2 + \sum_k (\sum_s \phi_k^{(s)})(\sum_t \overline{\phi}_k^{(t)}), \ \forall \text{Im}(w) = |z|^2.$$ 

Here we used the fact $a + \overline{a} = \text{Im}(2ia)$ for any $a \in \mathbb{C}$. Then

$$\text{Im}(\hat{g}) = \text{Im}(2i(\overline{z}, \hat{f})) + |\hat{f}|^2 + \sum_k (\sum_s \phi_k^{(s)})(\sum_t \overline{\phi}_k^{(t)}), \ \forall \text{Im}(w) = |z|^2.$$


Then for any \( l \leq s \leq m \), we collect terms in the above equation of weighted degree \( s \) to obtain the following equation:

\[
\text{Im}(\hat{g}^{(s)} - 2i\langle z, \hat{f}^{(s-1)} \rangle) = \sum_{j=1}^{N-n} \sum_{p=l}^{s-l} \phi_j^{(s-p)} \overline{\phi_j^{(p)}} + G^{(s)}, \quad \forall (z, w) \in \partial \mathbb{H}^n
\]  

(1.36)

where \( G^{(s)} \) is weighted homogeneous polynomial of weighted degree \( s \) contributed by \( \hat{f}^{(\sigma-1)} \) and \( g^{(\sigma)}, \sigma \leq s - 1 \). Here we denote \( \phi^{(s)} \equiv 0 \) if \( s < 0 \).

The operator

\[
\mathcal{L}(f, g) := \text{Im}(\hat{g} - 2i\langle z, \hat{f} \rangle)
\]

is called the Chern-Moser operator.

We notice \( G^{(s)} \equiv 0 \) if \( \hat{f}^{(\sigma-1)} \equiv g^{(\sigma)} \equiv 0 \) for \( \sigma \leq s - 1 \). Let us consider the following two cases.

**Case 1: \( s = 2k \)**

We suppose \( s = 2k \leq m \). If the following additional conditions are satisfied

\[
\hat{f}^{(\sigma-1)} \equiv \phi^{(\sigma)} \equiv 0, \quad \text{for } \sigma \leq 2k - 1,
\]

then

\[
\text{Im}(\hat{g}^{(2k)}(z, w) - 2i\langle z, \hat{f}^{(2k-1)}(z, w) \rangle) = \sum_{j=1}^{N-n} \phi_j^{(k)} \overline{\phi_j^{(k)}}, \quad \forall (z, w) \in M_1.
\]

(1.38)

**Case 2: \( s = 2k + 1 \)**

We suppose \( s = 2k + 1 \leq m \). If the following conditions are satisfied

\[
\hat{f}^{(\sigma-1)} \equiv \phi^{(\sigma)} \equiv 0, \quad \text{for } \sigma \leq 2k,
\]

then

\[
\text{Im}(\hat{g}^{(2k+1)}(z, w) - 2i\langle z, \hat{f}^{(2k)}(z, w) \rangle) = 0, \quad \forall (z, w) \in M_1.
\]

(1.40)

**Lemma 1.11.1** Let \( F \in \text{Prop}_2(\partial \mathbb{H}^n, \partial \mathbb{H}^N) \) be as above. Then

(i) \( f^{(2)} \equiv 0, \quad f^{(3)} \equiv a^{(1)}(z)w, \quad \hat{f}^{(2)}(z, w) = \phi^{(2)}(z), g^{(3)} = g^{(4)} \equiv 0. \)

(ii) \(-2i\langle a^{(1)}(z), \overline{z} \rangle |z|^2 = \sum_{j=1}^{N-n} |\phi_j^{(2)}(z)|^2. \)

**Proof:** Consider \( s = 2 \) and (1.38). since \( \hat{f}^{(\sigma_1)} \equiv \phi^{(\sigma_2)} \equiv 0 \), \( \sigma_1 \leq 0, \sigma_2 \leq 0 \) always hold, the both sides of the identity (1.38) are zero as \( k = 1 \). So the equation (1.36) is trivial.

Consider \( s = 3 \) and \( m = 3 \) in the identity (1.40):

\[
\text{Im}(\hat{g}^{(3)} - 2i\langle z, \hat{f}^{(2)} \rangle) \equiv 0 \quad \text{on } \partial \mathbb{H}^n.
\]

(1.41)
1.11. THE CHEREN-MOSER OPERATOR

We claim
\[
\hat{g}^{(3)}(z) \equiv 0 \text{ and } \hat{f}^{(2)}(z) \equiv 0.
\] (1.42)

In fact, write \( \hat{f}^{(2)}(z, w) = a^{(2)}(z) \) and \( \hat{g}^{(3)}(z, w) = c^{(3)}(z) + c^{(1)}(z)w \). Substituting into (1.40), we have
\[
\text{Im}(c^{(3)}(z) + c^{(1)}(z)w - 2i\langle \overline{z}, a^{(2)}(z) \rangle) \equiv 0, \quad \forall \text{Im}(w) = |z|^2.
\]

Since \( w = u + i|z|^2 \), it follows that \( c^{(1)}(z) \equiv 0, c^{(3)}(z) \equiv 0 \) and \( a^{(2)}(z) \equiv 0 \). Hence Claim is proved.

Consider \( s = 4 \) and \( m = 4 \) in (1.38):
\[
\text{Im}(g^{(4)} - 2i\langle \overline{z}, \hat{f}^{(3)} \rangle) = \sum_{j=1}^{N-n} |\phi_j^{(2)}|^2, \quad \forall \text{Im}(w) = |z|^2.
\] (1.43)

We claim
\[
g^{(4)} \equiv 0, \quad \phi_j^{(2)}(z) \equiv \phi_j^{(2)}(z), \quad f^{(3)} \equiv a^{(1)}(z)w,
\]
\[
-2i\langle a^{(1)}(z), \overline{z} \rangle |z|^2 = \sum_{j=1}^{N-n} |\phi_j^{(2)}|^2,
\] (1.44)

where \( a^{(1)}(z) \) is a certain holomorphic polynomial of degree one. In fact, write
\[
f^{(3)}(z, w) = a^{(1)}(z)w + a^{(3)}(z), \quad \phi_j^{(2)}(z, w) = b^{(2)}(z)
\]
and \( g(z, w) = c^{(4)}(z) + c^{(2)}(z)w + c_0 w^2 \). Substituting into (1.38),
\[
\text{Im}(c^{(4)}(z) + c^{(2)}(z)w + c_0 w^2 - 2i\langle \overline{z}, a^{(1)}(z) \rangle w - 2i\langle \overline{z}, a^{(3)}(z) \rangle) = \sum_{j=1}^{N-n} |b_j^{(2)}|^2, \quad \forall (z, w) \in M_1.
\]

Since \( w = u + i|z|^2 \) and \( z, u \) are independent variables, we consider \( u^0, u \) and \( u^2 \) terms to get three identities:
\[
\text{Im}(c^{(4)}(z) + ic^{(2)}(z)|z|^2 - c_0|z|^4 + 2\langle \overline{z}, a^{(1)}(z) \rangle w - 2i\langle \overline{z}, a^{(3)}(z) \rangle) = \sum_{j=1}^{N-n} |b_j^{(2)}|^2,
\]
\[
\text{Im}(c^{(2)}(z) + 2ic_0|z|^2 - 2i\langle \overline{z}, A^{(1)}(z) \rangle)u = 0,
\]
\[
\text{Im}(c_0)u^2 = 0.
\]

Then \( c_0 \equiv 0 \) and \( c^{(2)}(z) \equiv 0 \) and \( \text{Im}(2i\langle \overline{z}, a^{(1)}(z) \rangle) \equiv 0 \). Thus from the first one, \( c^{(4)}(z) \equiv 0 \) and \( a^{(3)}(z) \equiv 0 \) so that the claim is proved. \( \Box \)

By considering the weighted degree 4 in the equation (1.36) and by the property of \( F^{**} \), we obtain:
Theorem 1.11.2 ([H99], Lemma 5.3) Let $F$ be a non-constant map in $Prop_2(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$, $1 \leq n \leq N$ with $F(0) = 0$. Then there is an automorphism $\tau^{**} \in Aut_0(\mathbb{H}^N)$ such that $F^{**} := \tau^{**} \circ F = (f^{**}, \phi^{**}, g^{**})$ satisfies the following normalization:

$$f^{**} = z + \frac{i}{2} a^{**(1)}(z) w + o_{wt}(3), \quad \phi^{**} = \phi^{**}_{p}(z) + o_{wt}(2), \quad g^{**} = w + o_{wt}(4), \quad \langle z, a^{**(1)}(z) \rangle |z|^2 = |\phi^{**}_{p}(z)|^2.$$

1.12 The Associated Map $F_{p}$ of $F$

Let $F = (f, \phi, g) = (\tilde{f}, g) = (f_1, ..., f_{n-1}, \phi_1, ..., \phi_{N-n}, g)$ be a non-constant $C^2$ smooth CR map from $M_1 \subset \partial \mathbb{H}^n$ into $M_2 \subset \partial \mathbb{H}^N$ as above.

For any point $p \in M_1$, we have an associated CR map $F_{p}$ from a small neighborhood of $0 \in \partial \mathbb{H}^n$ to $\partial \mathbb{H}^N$ with $F_{p}(0) = 0$, defined by

$$F_p = \tau_{F}^{p} \circ F \circ \sigma_{p}^{0},$$

$$\begin{array}{cccc}
p \in \partial \mathbb{H}^n & \overset{F}{\rightarrow} & \partial \mathbb{H}^N \ni F(p) \\
\uparrow \sigma_{p}^{0} & & & \downarrow \tau_{F}^{p} \\
0 \in \partial \mathbb{H}^n & \overset{F_{p} := \tau_{F}^{p} \circ F \circ \sigma_{p}^{0}}{\rightarrow} & \partial \mathbb{H}^N \ni 0
\end{array}$$

where $\sigma_{p}^{0} \in Aut(\mathbb{H}^n)$, $p = (z_0, w_0)$, given by

$$\sigma_{p}^{0}(z, w) = (z + z_0, w + w_0 + 2i \langle z, z_0 \rangle),$$

and $\tau_{F}^{p} \in Aut(\mathbb{H}^N)$ is given by

$$\tau_{F}^{p}(z^*, w^*) = (z^* - \tilde{f}(z_0, w_0), w^* - g(z_0, w_0) - 2i \langle z^*, \bar{f}(z_0, w_0) \rangle).$$

Notice that $F(0)$ may not be 0, but we always have $F_{p}(0) = 0$. By the similar calculation of $F^{*}$ and $F^{**}$, we have the following formulas.
1.12. THE ASSOCIATED MAP $F_P$ OF $F$

\[
\left( \tilde{f}_p \right)_{z_l}^{'} |_{0} = L_l(\tilde{f})(p) := E_l(p),
\]
\[
\left( \tilde{f}_p \right)_{w}^{'} |_{0} = T(\tilde{f})(p) := E_w(p),
\]
\[
\lambda(p) := |L_j \tilde{f}|^2(p), \text{ for any } j \in \{1, ..., n-1\},
\]
\[
(g_p)_{z_l}^{'} |_{0} = L_l g(p) - 2iL_l \tilde{f}(p) \cdot \bar{\tilde{f}}(p) = 0 \quad (because \ (1.5)),
\]
\[
(g_p)_{w}^{'} |_{0} = T g(p) - 2iT \tilde{f}(p) \cdot \bar{\tilde{f}}(p) = |L_j \tilde{f}_p(0)|^2, \quad 1 \leq j \leq n-1,
\]

\[
\left( \tilde{f}_p \right)^{''} |_{z_l z_k} = L_l L_k(\tilde{f})(p),
\]
\[
\left( \tilde{f}_p \right)^{''} |_{z_l w} = TL_l(\tilde{f})(p),
\]
\[
\left( \tilde{f}_p \right)^{''} |_{w^2} = T^2(\tilde{f})(p),
\]
\[
(g_p)^{''} |_{z_l z_k} = L_l L_k g(p) - 2iL_l L_k \tilde{f}(p) \cdot \bar{\tilde{f}}(p) = 0, \quad (By \ (1.8))
\]
\[
(g_p)^{''} |_{z_l w} = L_l \left( T g(p) - 2iT \tilde{f}(p) \cdot \bar{\tilde{f}}(p) \right) = 2iL_l \tilde{f}(p) \cdot T \bar{\tilde{f}}(p),
\]
\[
(g_p)^{''} |_{w^2} = T^2 g(p) - 2iT^2 \tilde{f}(p) \cdot \bar{\tilde{f}}(p) - 2iT \tilde{f}(p) \cdot T \bar{\tilde{f}}(p).
\]

Here for the second equality about $(g_p)^{''}_{w^2}$, we used the fact that $g - \bar{g} = 2i \tilde{f} \cdot \bar{\tilde{f}}$ and then $TL_l g = 2iT L_p \tilde{f} \cdot \bar{\tilde{f}} + 2iL_l \tilde{f} \cdot T \bar{\tilde{f}}$. Notice that there are two formulas for $(g_p)^{''}_{w^2}|_{0}$.

We define $F^*_p = (\tilde{f}^*_p, g^*_p)$ given by

\[
F^*_p = (f^*_p, \phi^*_p, g^*_p) = \left( f^*_p, \phi^*_p, g^*_p \right)
\]

(1.49)

where

\[
f^*_p = \frac{1}{\lambda_p} \tilde{f}_p \cdot \bar{E}_l(p), \quad \phi^*_p = \frac{1}{\sqrt{\lambda_p}} \tilde{f}_p \cdot \bar{C}_k(p), \quad g^*_p = \frac{1}{\lambda_p} \tilde{f}_p,
\]

(1.50)
where $1 \leq l \leq n - 1$ and $1 \leq k \leq N - n$. $F_p^*$ satisfies the following properties:

$$F_p^*(0) = 0, \left(f_{p,l}^*\right)'_{z_l} |_{z_l} = \delta^l, \left(\phi_{p,j}^*\right)'_{z_l} |_{z_l} = 0, \left(g_p^*\right)'_{z_l} |_{z_l} = 0, \left(g_p^*\right)''_{w} |_{w} = 1. \quad (1.51)$$

As before, we can choose vectors $C_1(p), ..., C_{N-n}(p) \in \mathbb{C}^{N-1}$ so that

$$\left\{ \frac{E_1(p)^t}{\sqrt{\lambda}}, ..., \frac{E_{n-1}(p)^t}{\sqrt{\lambda}}; C_1(p)^t, ..., C_{N-n}(p)^t \right\} \quad (1.52)$$

form an $(N - 1) \times (N - 1)$ unitary matrix.

\[
\begin{align*}
\left( f_{p,l}^* \right)'_{z_l} |_{z_l} &= \frac{1}{\lambda(p)} L_k \tilde{f}(p) \cdot E_l(p)^t = \frac{1}{\lambda(p)} L_k (\tilde{f}(p) \cdot \overline{L_l(f)(p)}) = \delta^l, \\
\left( f_{p,l}^* \right)''_{w} |_{w} &= \frac{1}{\lambda(p)} E_w(p) \cdot E_l(p)^t = \frac{1}{\lambda(p)} T(\tilde{f}(p) \cdot \overline{L_l(f)(p)}), \\
\left( \phi_{p,l}^* \right)'_{z_k} |_{z_k} &= \frac{1}{\sqrt{\lambda(p)}} L_k \tilde{f}(p) \cdot \overline{C_l(p)} = 0, \\
\left( \phi_{p,k}^* \right)'_{w} |_{w} &= \frac{1}{\sqrt{\lambda(p)}} E_w(p) \cdot \overline{C_k(p)} = \frac{1}{\sqrt{\lambda(p)}} T(\tilde{f}(p) \cdot \overline{C_k(p)}), \\
\left( g_p^* \right)'_{z_l} |_{z_l} &= \frac{1}{\lambda(p)} \left( L_l g(p) - 2i L_l \tilde{f}(p) \cdot \overline{f(p)} \right) = 0, \quad (By \ (1.5)) \\
\left( g_p^* \right)''_{w} |_{w} &= \frac{1}{\lambda(p)} \left( T g(p) - 2i T \tilde{f}(p) \cdot \overline{f(p)} \right) = 1, \quad (By \ (1.7)) \\
\left( f_{p,j}^* \right)''_{z_k z_l} |_{z_k z_l} &= \frac{1}{\lambda(p)} L_k L_l \tilde{f}(p) \cdot \overline{L_j \tilde{f}(p)} , \quad \left( f_{p,l}^* \right)''_{z_l w} |_{z_l w} = \frac{1}{\lambda(p)} L_l T(\tilde{f}(p) \cdot \overline{L_l(f)(p)}), \\
\left( f_{p,j}^* \right)''_{w^2} |_{w^2} &= \frac{1}{\lambda(p)} T^2 \tilde{f}(p) \cdot \overline{L_j \tilde{f}(p)} , \quad \left( \phi_{p,j}^* \right)''_{z_k z_l} |_{z_k z_l} = \frac{1}{\lambda(p)} L_k \tilde{f}(p) \cdot \overline{C_j(p)} , \\
\left( \phi_{p,j}^* \right)''_{z_k z_l} |_{z_k z_l} &= \frac{1}{\lambda(p)} TL \tilde{f}(p) \cdot \overline{C_j(p)} , \quad \left( \phi_{p,j}^* \right)''_{w^2} |_{w^2} = \frac{1}{\lambda(p)} T^2 \tilde{f}(p) \cdot \overline{C_j(p)} ,
\end{align*}
\]
We define
\[
G_p = \frac{z^* - a(p)w^*}{1 + 2i\langle z^*, a(p) \rangle + (r(p) - i |a(p)|^2)w^*}
\] (1.53)
where
\[
a(p) := \left. \left( \begin{array}{c} \tilde{f}_p^* \\ \phi^* \end{array} \right) \right|_w = (a(p), b(p)) = (a_1(p), \ldots, a_{n-1}(p), b_1(p), \ldots, b_{N-n}(p)) = \\
\left( \frac{\ldots, T\tilde{f}(p) \cdot L_j \tilde{f}(p)}{\lambda(p)} \right), \ldots, \frac{\ldots, T\tilde{f}(p) \cdot C_j(p)^t}{\sqrt{\lambda(p)}} \right),
\] (1.54)
\[
r(p) := \frac{1}{2} \text{Re} \left. \left( \left( g_p^* \right)^{''} \right) \right|_w = \frac{1}{2\lambda(p)} \text{Re} \left( T^2 g(p) - 2i T^2 \tilde{f}(p) \cdot \overline{f(p)} \right).
\] (1.55)
In particular, because \( A = \left( \frac{E_i}{\sqrt{\lambda}}, C_k \right) \) is a unitary matrix,
\[
|a(p)|^2 = \frac{1}{\lambda(p)} |E_w(p)|^2 = \frac{1}{\lambda(p)} |T\tilde{f}(p)|^2.
\] (1.56)
We then define the normalization
\[
F_{p}^{**} = (\tilde{f}_p^{**}, g_p^{**}) = (f_p^{**}, \phi_p^{**}, g_p^{**}) := G_p \circ F_{p}^{*}.
\] (1.57)
\[
f_{p,j}^{**} = \frac{f_{p,j}^* - a_j(p)g_p^*}{1 + 2i\langle f_{p}^*, a(p) \rangle - (r(p) - i|a(p)|^2)g_p^*},
\] (1.58)
\[
\phi_{p,j}^{**} = \frac{\phi_{p,j}^* - b_j(p)g_p^*}{1 + 2i\langle f_{p}^*, a(p) \rangle - (r(p) + i|a(p)|^2)g_p^*}.
\] (1.59)
Here we used the fact that $F_p^{**}$ must satisfy the following properties:

$$F_p^{**}, \left( f_p^{**} - z \right)_{z_i}, \left( f_p^{**} \right)'_{z_i}, \left( f_p^{**} \right)''_{z_i}, \left( g_p^{**} \right)'_{z_i}, \left( g_p^{**} \right)''_{z_i}, \left( g_p^{**} - w \right)'_{w}, \left( g_p^{**} \right)''_{w},$$

and $\left( g_p^{**} \right)''_{w}$ all vanish at $(z, w) = 0$. (1.61)

From (1.58) (1.59) and (1.60), we have

$$(f^{*}_{p,j})_{|z_i} = 0 = (f^{*}_{p,j})'_{|0} = (f^{*}_{p,j})''_{|0} - a_j(p) = 0, (\phi^{*}_{p,j})_{|z_i} = 0 = (\phi^{*}_{p,j})'_{|0} = (\phi^{*}_{p,j})''_{|0} - b_j(p) = 0,$$

and $(g^{*}_{p})''_{|z_i} = 0 = (g^{*}_{p})'_{|0} = 0$. (1.62)

Here we used the fact that $(g^{*}_{p})''_{z_jz_k} = 0$. The last equality holds because of Lemma 1.11.1 (i).
We can say more about this important formula which will be used to define geometric rank $\kappa_0$. Applying $T^2$ to the basic equation $Im(g) = |\tilde{f}|^2$, we get $0 = 2iIm(iT^2\tilde{f} \cdot \overline{\tilde{f}}) + 2i|T\tilde{f}|^2 - i Im(T^2g)$ on $\partial \mathbb{H}^m$ by (1.10). Combining this to the above, we get

$$
\left( f_{p,l}^{**} \right)'' \bigg|_{z,j} = \left( f_{p,l}^{*} \right)'' \bigg|_{z,j} - \frac{2i}{\lambda(p)^2} \left( T\tilde{f}(p) \cdot L_1\tilde{f}(p) \right) \left( L_j\tilde{f}(p) \cdot \overline{T\tilde{f}(p)} \right) 
$$

(1.63)

$$
\left( f_{p,l}^{**} \right)'' \bigg|_{w^2} = \left( f_{p,l}^{*} \right)'' \bigg|_{w^2} - \frac{1}{\lambda(p)^2} \left( T\tilde{f}(p) \cdot L_1\tilde{f}(p) \right) \left( T^2g(p) - 2iT^2\tilde{f} \cdot \overline{\tilde{f}} - 2i|T\tilde{f}|^2 \right) \bigg|_{p}.
$$

(1.64)

$$
\left( \phi_{p,l}^{**} \right)'' \bigg|_{z,j} = \left( \phi_{p,l}^{*} \right)'' \bigg|_{z,j} - \frac{1}{\sqrt{\lambda(p)}} \frac{1}{\lambda(p)} L_j L_k \tilde{f}(p) \cdot C_i(p) \bigg|_{z,j}.
$$

(1.65)

Here we used the fact that $(g_p^*)''_{z,j} = 0$.

$$
\left( \phi_{p,l}^{**} \right)'' \bigg|_{z,j} = \left( \phi_{p,l}^{*} \right)'' \bigg|_{z,j} - b_l(g_p^*)''_{z,j} = \left( \phi_{p,l}^{*} \right)'' \bigg|_{z,j} - \frac{1}{\sqrt{\lambda(p)}} L_j L_k \tilde{f}(p) \cdot C_i(p) \bigg|_{z,j}.
$$

(1.66)

$$
\left( \phi_{p,l}^{**} \right)'' \bigg|_{w^2} = \left( \phi_{p,l}^{*} \right)'' \bigg|_{w^2} - \frac{1}{\lambda(p)^2} \left( T\tilde{f}(p) \cdot C_i(p) \right) \left( T^2g(p) - 2iT^2\tilde{f} \cdot \overline{\tilde{f}} - 2i|T\tilde{f}|^2 \right) \bigg|_{w^2}.
$$

(1.67)
CHAPTER 1. EARLIER RESULT: THE FIRST GAP THEOREM

\[
\left( g_p^{**} \right)^{\prime\prime}_{zjz_k} |_0 = 0,
\]

\[
\left( g_p^{**} \right)^{\prime\prime}_{zjw} |_0 = \left( g_p^{*} \right)^{\prime\prime}_{zjw} |_0 - 2ia_j(p) = \frac{2i}{\lambda(p)}L_j \tilde{f}(p) \cdot \overline{\tilde{f}(p)} - \frac{2i}{\lambda(p)}T \tilde{f}(p) \cdot L_j \tilde{f}(p) = 0,
\]

\[
\left( g_p^{**} \right)^{\prime\prime}_{w^2} |_0 = \left( g_p^{*} \right)^{\prime\prime}_{w^2} |_0 - 2 \left[ i|a_j(p)|^2 + r(p) \right]
\]

\[
= \frac{1}{\lambda(p)} \left[ T^2g(p) - 2iT^2\tilde{f}(p) \cdot \overline{\tilde{f}(p)} \right]
\]

\[
- \frac{2}{\lambda(p)} \left[ i|T \tilde{f}(p)|^2 + \frac{1}{2} Re \left( T^2g(p) - 2iT^2\tilde{f}(p) \cdot \overline{\tilde{f}(p)} \right) \right] = 0.
\]

The above two equalities equal to zero because of Lemma 1.11.1 (i).

By the similar calculation of \( F^* \) and \( F^{**} \), we can define \( F_p^* \) and \( F_p^{**} \) with the following theorem.

**Theorem 1.12.1** ([H99], Lemma 5.3) Let \( F \) be a non-constant map in \( Prop_2(\mathbb{H}^{n+1}, \mathbb{H}^{N+1}) \), \( 1 \leq n \leq N \) with \( F(0) = 0 \). For each \( p \in \partial \mathbb{H}^{n+1} \), there is an automorphism \( \tau_p^{**} \in Aut_0(\mathbb{H}^N) \) such that \( F_p^{**} := \tau_p^{**} \circ F_p \) satisfies the following normalization:

\[
f_p^{**} = z + \frac{i}{2}a_p^{**(1)}(z)w + o_{wt}(3), \quad \phi_p^{**} = \phi_p^{**(2)}(z) + o_{wt}(2), \quad g_p^{**} = w + o_{wt}(4), \quad (1.68)
\]

\[
\langle \overline{z}, a_p^{**(1)}(z) \rangle |z|^2 = |\phi_p^{**(2)}(z)|^2. \quad (1.69)
\]

### 1.13 Geometric Rank of \( F \)

We denote \( a_p^{**(1)}(z) = zA(p) \) where

\[
A(p) = -2i \left( \frac{\partial^2 f_p^{**}}{\partial z_j \partial w^l} \right) |_0 1 \leq j, l \leq n-1
\]
1.14. Maps With Geometric Rank $\kappa_0 = 0$

is an $(n - 1) \times (n - 1)$ hermitian matrix. This matrix is semi-positive because of (1.69).

We define [H03]

$$Rk_F(p) := \operatorname{Rank}(A(p)),$$

which is called the geometric rank of $F$ at $p$ and is a lower semi-continuous function on $p$. We also define

$$\kappa_0 = \kappa_0(F) := \max_{p \in \partial\mathbb{H}^n} \operatorname{Rk}_F(p)$$

which is called the geometric rank of $F$.

Remarks (i) $\kappa_0(F)$ is an invariant.

(ii) $0 \leq \kappa_0(F) \leq n - 1$.

(iii) $\kappa_0(F) = \kappa_0$ if and only if at a generic point $p \in \partial\mathbb{H}^n$, $F \cong F^{**}$ that satisfies

\[
\begin{align*}
  f^{**}_{j,p} &= z_j + \frac{\mu_j(p)}{2} z_j w + o_{wt}(3), & 1 \leq j \leq \kappa_0, \mu_j(p) > 0 \\
  f^{**}_{j,p} &= z_j + o_{wt}(3), & \kappa_0 + 1 \leq j \leq n - 1, \\
  \phi^{**}_p &= \phi^{(2)**}_p(z) + o_{wt}(2), \\
  g^{**}_p &= w + o_{wt}(4).
\end{align*}
\]

(iv) When $\kappa_0(F) = n - 1$, the image submanifold $F(\partial\mathbb{H}^n)$ "occupies more room" in the target space $\partial\mathbb{H}^N$ so that it is the most complicated case. In fact, when $\kappa_0(F) \leq n - 2$, $F$ has "semi-linearity" properties.

### 1.14 Maps With Geometric Rank $\kappa_0 = 0$

**Theorem 1.14.1** (Linearity Criterion, [H99])

$$\kappa_0 = 0 \iff F \text{ is equivalent to the linear map}.$$

To prove this theorem, let us first prove two lemmas.

**Lemma 1.14.2** Let $m$ and $n$ be any positive integers. Let $X = (f_1, ..., f_m)$ be a vector-valued differentiable function defined in a neighborhood of 0 in $\mathbb{R}^n$ satisfying

$$DX = A(x)X', \quad X(0) = 0,$$

where $D = \left(\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\right)$ and $A(x)$ is a matrix of continuous functions. Then $X \equiv 0$ holds in some neighborhood of 0 in $\mathbb{R}^n$. 

CHAPTER 1. EARLIER RESULT: THE FIRST GAP THEOREM

Proof of Lemma 1.14.2: For any \( p \) near 0 in \( \mathbb{R}^n \), we denote \( X_p(t) := X(tp) \) for \( 0 \leq t \leq 1 \). Then \( \frac{dx_p}{dt} = pA(tp)X_p(t)t \). Since \( X_p(0) = 0 \), we get \( X_p(t) = \int_0^t pA(\tau p)X_p(\tau)t d\tau \). Hence \( \|X_p\| \leq C\|p\|\|X_p\| \) for some constant \( C > 0 \) which is independent of \( p \). It follows that \( X_p \equiv 0 \) once \( \|p\| < \frac{1}{C} \). □

Lemma 1.14.3 We have

(i) For any \( p \in \partial \mathbb{H}^n \),

\[
L_kL_l\tilde{f}(p) \cdot \frac{\partial \tilde{f}}{\partial t}(p) = 2\sqrt{-1}\delta_k^l \left( \overline{f'_w(p) \cdot L_l\tilde{f}(p)} \right) + 2\sqrt{-1}\delta_l^j \left( \overline{f'_w(p) \cdot L_k\tilde{f}(p)} \right).
\]

(ii) For any fixed \( j \) and \( k \), if \( (\phi_p^*)''_{z_k} |_0 = 0 \) for any \( p \in \partial \mathbb{H}^n \), then

\[
L_jL_k\tilde{f}(p) = \frac{2\sqrt{-1}}{\lambda} \left( \overline{f'_w(p) \cdot (L_j\tilde{f})^t(p)} \right) L_k\tilde{f}(p) + \frac{2\sqrt{-1}}{\lambda} \left( \overline{f'_w(p) \cdot (L_k\tilde{f})^t(p)} \right) L_j\tilde{f}(p).
\]

Proof (i) By the construction of \( F^{**} \), we know that \( \left( f^{**}_{p,ij} \right)''_{z_jz_k} |_0 = 0 \). By (1.62), we have

\[
\left( f^{**}_{p,ij} \right)''_{z_jz_k} |_0 = \frac{1}{\lambda(p)} L_kL_l\tilde{f}(p) \cdot \frac{\partial \tilde{f}}{\partial t}(p) - \frac{2i\delta_j^k}{\lambda(p)} T\tilde{f}(p) \cdot \frac{\partial \tilde{f}}{\partial t}(p) - \frac{2i\delta_k^j}{\lambda(p)} T\tilde{f}(p) \cdot \frac{\partial \tilde{f}}{\partial t}(p) = 0.
\]

Then (i) follows.

(ii) By the construction of \( F^{**} \), we see that \( (\phi_p^*)''_{z_k} |_0 = 0 \) if and only if \( L_kL_l\tilde{f}(p) \cdot C(p) = 0 \). Then \( L_kL_l\tilde{f}(p) \) is perpendicular to the subspace \( \text{span}\{C(p)\} \) so that they are linear combination of the vectors \( E_s(p) \): \( L_kL_l\tilde{f}(p) = \sum_{s=1}^{n-1} \lambda_{kl}^s E_s(p) \), and hence \( L_kL_l\tilde{f}(p) \cdot E_j(p) = \sum_{s=1}^{n-1} \lambda_{kl}^s E_s(p) \cdot E_j(p) = \lambda \delta_{sj} \). Here we have used the orthogonal property: \( E_s \cdot E_j = \lambda \delta_{sj} \) in (1.19). Finally we use (i) to obtain the desired identity. □

Proof of Theorem 1.14.1: If we can show \( \phi \equiv 0 \), then \( (f, g) : \partial \mathbb{H}^n \rightarrow \partial \mathbb{H}^n \) is a \( C^2 \)-smooth CR map. By Poincaré-Tanaka theorem, \( (f, g) \in \text{Aut}(\mathbb{H}^n) \) so that \( (f, g) \) must be linear fractional. By the normalization condition, we assume \( F = F^{**} \). This implies that \( F(z, w) \equiv (z, 0, w) \).

Since \( \phi(0) = 0 \), it suffices to show \( X\phi \equiv 0 \) for any vector field \( X \in T(\partial \mathbb{H}^n) \). Since \( L_j \) and \( T \) form a basis for \( T(\partial \mathbb{H}^n) \) and \( \phi \) is CR, it suffices to show that \( L_j\phi \equiv 0 \) and \( T\phi \equiv 0 \) for all \( 1 \leq j \leq n-1 \).
1.14. MAPS WITH GEOMETRIC RANK $\kappa_0 = 0$

By applying Lemma 1.14.2, it is enough for us to prove

$$\begin{align*}
L_j(L_k(\phi)) &= A_j(z,w)L_k(\phi) + A_k(z,w)L_j(\phi); \\
TL_k\phi &= B_{k,1}(z,w)L_k(\phi) + B_{k,2}(z,w)T(\phi); \\
T^2\phi &= C_{k,1}(z,w)L_k(\phi) + C_{k,2}(z,w)T(\phi),
\end{align*}$$

(1.72)

where $A_k, B_{k,1}, B_{k,2}, C_{k,1}$ and $C_{k,2}$ are continuous function defined in a neighborhood of 0 in $\partial\mathbb{H}^n$.

Notice $\kappa_0 = 0 \iff (\phi^{**}_p)_{z_1z_k} = 0$, $\forall p \in \partial\mathbb{H}^n$. From Lemma 1.14.3(ii), we obtain

$$L_j(L_k(\phi)) = A_j(z,w)L_k(\phi) + A_k(z,w)L_j(\phi),$$

(1.73)

where $A_k := \frac{2\phi''(\phi^*)}{\phi''}$ which are $C^1(\partial\mathbb{H}^n)$. Then the first equality of (1.72) is proved.

Putting $j = k$ in (1.73), we get $L_k^2(\phi) = 2A_kL_k(\phi)$. Applying $\overline{T_k}$ and by Lemma 1.7.1(ii), we have

$$\overline{T_k}\phi = \frac{\overline{T_k}A_k}{2i}L_k(\phi) + A_kT(\phi) = B_{k,1}L_k(\phi) + B_{k,2}T(\phi),$$

(1.74)

where $B_{k,1} := \frac{\overline{T_k}A_k}{2i} \in C^0(\partial\mathbb{H}^n)$ and $B_{k,2} := A_k \in C^1(\partial\mathbb{H}^n)$. We have proved the second equality of (1.72).

Applying $\overline{T_k}$ again to (1.74), we obtain

$$2iT^2(\phi) = (\overline{T_k}B_{k,1})L_k(\phi) + (B_{k,1}2i + \overline{T_k}B_{k,2})T(\phi) = C_{k,1}L_k(\phi) + C_{k,2}T(\phi),$$

(1.75)

where $C_{k,2} := B_{k,1}2i + \overline{T_k}B_{k,2} \in C^0(\partial\mathbb{H}^n)$ because of $B_{k,2} \in C^1(\partial\mathbb{H}^n)$, and $C_{k,1} := \overline{T_k}B_{k,1}$.

It remains to prove the following claim: $C_{k,1}$ is continuous. In fact, when $j = k$, apply Lemma 1.14.2(ii) and take the component $f_k$, as we did for (1.73), we get $A_k = \frac{L_k^2(f_k)}{2\overline{T_k}(f_k)}$.

Then

$$\begin{align*}
B_{k,1} &= \frac{1}{4}\overline{T_k}(A_k) = \frac{1}{2i}\overline{T_k}\left(\frac{1}{(L_k(f_k))^2}\right)L_k^2(f_k) + \frac{1}{L_k(f_k)}TL_k(f_k) \\
&= \frac{T(f_k)}{(L_k(f_k))^2}L_k^2(f_k) + \frac{1}{L_k(f_k)}TL_k(f_k) \\
&= b_{k,1}L_k^2(f_k) + b_{k,2}TL_k(f_k),
\end{align*}$$

(1.76)

where $b_{k,1}, b_{k,2} \in C^1(\partial\mathbb{H}^n)$. Thus

$$\begin{align*}
C_{k,1} &= \overline{T_k}B_{k,1} = \overline{T_k}(b_{k,1}L_k^2(f_k) + b_{k,2}TL_k(f_k)) \\
&= \overline{T_k}b_{k,1}L_kT(f_k) + 2ib_{k,1}T^2f_k + \overline{T_k}b_{k,2}L_k^2(f_k) + 2ib_{k,2}TL_k(f_k),
\end{align*}$$

(1.77)

Hence the claim is proved so that the third equality in (3.11) is proved. □
1.15 Analytic Proof Of The First Gap Theorem

By Theorem 1.14.1, in order to complete the proof of the First Gap Theorem, we need to show

**Corollary 1.15.1** Let $F \in \text{Prop}_2(\mathbb{B}_n, \mathbb{B}_N)$ with $2 \leq n \leq N \leq n - 2$. Then $F$ has geometric rank $\kappa_0 = 0$.

**Proof:** Let $F \in \text{Prop}_2(\mathbb{B}_n, \mathbb{B}_N)$ with $2 \leq n \leq N \leq n - 2$. Then for any $p \in \partial \mathbb{H}_n$, $F^{**}$ satisfies the normalization condition in (1.69) and

$$\langle z, a_p^{**(1)}(z) \rangle |z|^2 = |\phi_p^{**(2)}(z)|^2.$$

Since $n \leq 2n - 2$, by a uniqueness theorem 1.15.2 below, it implies

$$\phi_p^{**(2)} \equiv 0 \text{ and } a_p^{**(1)} \equiv 0.$$  \hspace{1cm} (1.78)

Thus $\kappa_0(F) = 0$.  \hfill $\Box$

**Theorem 1.15.2 ([H99], [EHZ05])** Let $\phi_j, \psi_j$ be holomorphic function near the origin of $\mathbb{C}^n$, $1 \leq j \leq k$, $n > 1$. Suppose that $H(z, \bar{z})$ is a real analytic function defined in a neighborhood of $0 \in \mathbb{C}^n$ such that

$$H(z, \bar{z}) \langle z, \bar{z} \rangle_l = \sum_{j=1}^{k} \phi_j(z) \bar{\psi}_j(z) \text{ for } z \in \mathbb{C}^n \text{ near } 0.$$  \hspace{1cm} (1.79)

Suppose $k \leq n - 1$. Then $H(z, \bar{z}) \equiv 0$ and $\sum_{j=1}^{k} \phi_j(z) \bar{\psi}_j(z) \equiv 0$.

**Proof:** Complexifying the identity, we have

$$H(z, \bar{\zeta}) \langle z, \bar{\zeta} \rangle_l = \sum_{j=1}^{k} \phi_j(z) \bar{\psi}_j(\zeta)$$  \hspace{1cm} (1.80)

where $z, \zeta$ are independent variables. Assume that $\phi_j \not\equiv 0$ for each $1 \leq j \leq k$. We can find a point $z_0$ near the origin such that $\phi_j(z_0) = \epsilon_j \not\equiv 0$ for each $j$.

Consider the complex variety $V_{z_0} = \{ z | \phi_j(z) = \phi_j(z_0), 1 \leq j \leq k \}$. Since $k \leq n - 1$, this variety $V_{z_0}$ has complex dimension at least 1. For each $z^* \in V_{z_0}$, there exists a complex hyperplane $K_{z^*} = \{ \zeta | \langle z^*, \zeta \rangle = 0 \}$. Then for any $\zeta \in K_{z^*}$, we have $\sum_{j=1}^{k} \epsilon_j \bar{\psi}_j(\zeta) = 0$. 
Since $\dim_{\mathbb{C}} V_{z_0} \geq 1$ and $\dim_{\mathbb{C}} K_{z_0} = n - 1$, such $\zeta$ fills in an open subset of $\mathbb{C}^n$. Hence $\sum_{j}^{k} \epsilon_j \overline{\psi_j}(\zeta) = 0$, or $\overline{\psi_k}(z) + \sum_{j=1}^{k-1} \epsilon_j \overline{\psi_j}(z) = 0$. Multiplying with $\psi_k(z)$ and subtracting this to (1.79), we obtain

$$H(z, \overline{z}) \langle z, \overline{z} \rangle = \sum_{j=1}^{k-1} \left( \psi_j(z) - \frac{\epsilon_j}{\epsilon_k} \psi_k(z) \right) \overline{\psi_j(z)}.$$  

Then applying an induction argument, it follows easily that $\sum \phi_j \overline{\psi_j} \equiv 0$ and $H \equiv 0$. □

**Theorem 1.15.2** can be extended into a more general version by induction as follows.

**Corollary 1.15.3** Let $\phi_{jp}, \psi_{jp}, 1 \leq j \leq n - 1, 0 \leq p \leq q$, be holomorphic functions near the origin of $\mathbb{C}^n$ with $n > 1$. Suppose that $H(z, \overline{z})$ is a real analytic function defined in a neighborhood of $0 \in \mathbb{C}^n$ such that

$$H(z, \overline{\zeta}) \langle z, \overline{\zeta} \rangle_{l}^{q+1} = \sum_{p=0}^{q} \left( \sum_{j=1}^{n-1} \phi_{jp}(z) \overline{\psi_{jp}(\zeta)} \right) \langle z, \overline{\zeta} \rangle_{k}^{p}, \text{ for } z \sim 0 \text{ and } \zeta \sim 0.$$  

Then $H(z, \overline{z}) \equiv 0$ and $\sum_{j=1}^{n-1} \phi_{jp}(z) \overline{\psi_{jp}(\zeta)} \equiv 0, 1 \leq p \leq q$.  


Chapter 2

Construction and Classification of Rational Maps

2.1 Gap Phenomenon

A map $F \in \text{Prop}(\mathbb{B}^n, \mathbb{B}^N)$ is called \textit{minimum} if $F$ is not equivalent to a map of the form $(G, 0)$ where $G \in \text{Prop}(\mathbb{B}^n, \mathbb{B}^{N'})$ with $N' < N$.

Recall the First Gap Theorem in Lecture 1:

Any $F \in \text{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ where $N < 2n - 1$ is equivalent to a linear map $(z, w) \mapsto (z, 0, w)$.

This theorem can be restated as

\textbf{Theorem 2.1.1} \textit{(The First Gap Theorem)} There is no minimum map in $\text{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ if

$$N \in \mathcal{I}_1 = \{m \mathbb{Z}^+ \mid n < m < 2n - 1\}.$$ 

Furthermore, we have

\[
\begin{array}{cccccccc}
0 & 1 & 2 & n & 2n - 1 & 3n & 4n - 6 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} & \text{\textbullet} \\
2n & 3n - 3 & \end{array}
\]
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Theorem 2.1.2 (The Second Gap Theorem) (Huang-Ji-Xu, [HJX06]) There is no minimum map in $\text{Prop}_3(\mathbb{B}^n, \mathbb{B}^N)$ if $n \geq 4$ and

$$N \in \mathcal{I}_2 = \{ m \in \mathbb{Z}^+ \mid 2n < m < 3n - 3 \}.$$ 

Theorem 2.1.3 (The Third Gap Theorem, Huang-Ji-Yin, preprint) There is no minimum map in $\text{Prop}_3(\mathbb{B}^n, \mathbb{B}^N)$ if $n \geq 7$ and

$$N \in \mathcal{I}_3 = \{ m \in \mathbb{Z}^+ \mid 3n < m < 4n - 6 \}.$$ 

In general, we formulate the following: Consider $\text{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$. For the integer $n > 0$, let

$$K(n) := \max \{ t \in \mathbb{Z}^+ \mid \frac{t(t+1)}{2} < n \}.$$ 

For integer $k$ with $1 \leq k \leq K(n)$, let

$$\mathcal{I}_k := \left\{ m \in \mathbb{Z}^+ \mid kn < m < (k+1)n - \frac{k(k+1)}{2} \right\}.$$ 

[Example]

If $n \geq 2$, then $K(n) \geq 1$. Take $k = 1$ and $\mathcal{I}_1 = \{ m \in \mathbb{Z}^+ \mid n < m < 2n - 1 \}$.  
If $n \geq 4$, then $K(n) \geq 2$. Take $k = 2$ and $\mathcal{I}_2 = \{ m \in \mathbb{Z}^+ \mid 2n < m < 3n - 3 \}$.  
If $n \geq 7$, then $K(n) \geq 3$. Take $k = 3$ and $\mathcal{I}_3 = \{ m \in \mathbb{Z}^+ \mid 3n < m < 4n - 6 \}$. 

Theorem 2.1.4 (Huang-Ji-Yin, [HJY09]) For $n > 2$, let $K(n)$ be as above. For each $k$ with $1 \leq k \leq K(n)$, let $\mathcal{I}_k$ be as above. Then for each $N > n$ with

$$N \notin \bigcup_{k=1}^{K(n)} \mathcal{I}_k,$$

there exists a minimum monomial map in $\text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$. 

Conjecture: For $n > 2$, let $K(n)$ be as above. For each $k$ with $1 \leq k \leq K(n)$, let $\mathcal{I}_k$ be as above. Then for each $N > n$, the following two statements are equivalent:

(i) There exists no minimum maps in $\text{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$.  
(ii) $N \in \mathcal{I}_k$ for some $k$ with $1 \leq k \leq K(n)$. 

Recently, D’Angelo and Lebl (2007) found out that there is no gap phenomenon for mappings in $\text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ when $N \geq T(n) = n^2 - 2n + 2$.  
Based on the above conjecture, it would imply that there is no gap phenomenon for mappings in $\text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ when $N > n^{3/2}$. 
2.2 Construction of Minimum Maps

The proof of Theorem 2.1.4 is based on the construction of following minimum maps.

[Example A][HJY09] Let

\[
\begin{align*}
\psi_1 &= (z_1, \sqrt{2}z_2, \ldots, \sqrt{2}z_k, z_{k+1}, \ldots, z_n), \\
\psi_2 &= (z_2, \sqrt{2}z_3, \ldots, \sqrt{2}z_k, z_{k+1}, \ldots, z_n), \\
&\quad \ldots, \\
\psi_{k-1} &= (z_{k-1}, \sqrt{2}z_k, z_{k+1}, \ldots, z_n), \\
\psi_k &= (z_k, z_{k+1}, \ldots, z_n), \\
\psi_{k+1} &= (z_{k+1}, \ldots, z_n).
\end{align*}
\]

Let

\[ W_{n,k}(z_1, \ldots, z_n) := (z_1\psi_1, \ldots, z_k\psi_k, \psi_{k+1}). \]

This map, called \textit{generalized Whitney map}, is a quadratic polynomial minimum map in Prop(\(\mathbb{B}^n, \mathbb{B}^N\)) where \(N = (k + 1)n - \frac{k(k + 1)}{2}\).

[Example B] [HJY09] Let \(\psi_j\) be defined as above. Let \(\tau\) be an integer with \(1 \leq \tau \leq k\) and \(\lambda_j \in (0, 1)\) with \(1 \leq j \leq \tau\). We define

\[ W_{n,k}(\lambda_1, \ldots, \lambda_\tau) := (z_1\tilde{\psi}_1, \ldots, z_k\tilde{\psi}_k, \psi_{k+1}, \lambda_1z_1, \ldots, \lambda_\tau z_\tau) \]

where

\[
\begin{align*}
\tilde{\psi}_1 &= (\sqrt{1 - \lambda_1^2}z_1, \sqrt{1 - \lambda_1^2 + \mu_{12}^2}z_2, \ldots, \sqrt{1 - \lambda_1^2 + \lambda_2^2 + \lambda_1^2 z_k, \sqrt{1 - \lambda_1^2 + \lambda_1^2 z_{k+1}, \ldots, \sqrt{1 - \lambda_1^2 + \lambda_1^2 z_n}),} \\
\tilde{\psi}_2 &= (\sqrt{1 - \lambda_2^2}z_2, \sqrt{1 - \lambda_2^2 + \mu_{23}^2}z_3, \ldots, \sqrt{1 - \lambda_2^2 z_k, \sqrt{1 - \lambda_2^2 + \lambda_3^2}z_{k+1}, \ldots, \sqrt{1 - \lambda_2^2 + \lambda_2^2 z_n}),} \\
&\quad \ldots, \\
\tilde{\psi}_\tau &= (\sqrt{1 - \lambda_\tau^2}z_\tau, \sqrt{1 - \lambda_\tau^2 + \mu_{(\tau+1)}^2}z_{\tau+1}, \ldots, \sqrt{1 - \lambda_\tau^2 + \lambda_{\tau+1}^2}z_k, \sqrt{1 - \lambda_\tau^2 + \lambda_\tau^2 z_{k+1}, \ldots, \sqrt{1 - \lambda_\tau^2 + \lambda_\tau^2 z_n}),} \\
\tilde{\psi}_\tau &= (\sqrt{1 - \lambda_\tau^2}z_k, \sqrt{1 - \lambda_\tau^2 z_{k+1}, \ldots, \sqrt{1 - \lambda_\tau^2 z_n}),} \quad \text{for } \tau < k, \\
\tilde{\psi}_\tau &= (\sqrt{1 - \lambda_\tau^2}z_{k+1}, \ldots, \sqrt{1 - \lambda_\tau^2}z_n), \quad \text{for } \tau = k, \\
\tilde{\psi}_j &= \psi_j \quad \text{if } \tau < j \leq k.
\end{align*}
\]

where \(\mu_{jl} = \sqrt{1 - \lambda_l^2}\) for \(j \leq l \leq \tau\) and \(\mu_{jl} = 1\) for \(l > \tau\).

This map is a quadratic polynomial minimum map in Prop(\(\mathbb{B}^n, \mathbb{B}^N\)) where

\[ N = (k + 1)n - \frac{k(k + 1)}{2} + \tau. \]
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[Example C] [HJY09] Let $F : \mathbb{B}^n \rightarrow \mathbb{B}^{N^*}$ be a proper polynomial minimum map with $F(0) = 0$. Then we define a new map $W_{n,k}(\lambda_1, ..., \lambda_r, F)$ by modifying the map $W_{n,k}(\lambda_1, ..., \lambda_r)$ in the following way: while keeping all other components the same, replacing $\psi_1$ with

$$
\tilde{\psi}_1 = (\sqrt{1 - \lambda_1^2 z_1}, \sqrt{1 - \lambda_1^2 + \mu_1^2 z_2}, ..., \sqrt{1 - \lambda_1^2 + \lambda_k^2 z_k}, \sqrt{1 - \lambda_1^2 z_{k+1}}, ..., \sqrt{1 - \lambda_1^2 z_n}).
$$

This map is a polynomial minimum map in $\text{Prop}(\mathbb{B}^n, \mathbb{B}^N)$ where

$$
N = N^* - 1 + (k+1)n - \frac{k(k+1)}{2} + \tau.
$$

Lemma 2.2.1 [HJY09] Let $F : \mathbb{B}^n \rightarrow \mathbb{B}^{n(k-k_0)}$ be a minimum proper polynomial map with $k > k_0 > 0$ and $F(0) = 0$. Then a new map

$$
W_{n,k_0}(\lambda_1, ..., \lambda_r, F) : \mathbb{B}^n \rightarrow \mathbb{B}^N,
$$

with $N = (k+1)n - \frac{k_0(k_0+1)}{2}$, and $0 \leq \tau \leq k_0 \leq n$

is a proper polynomial minimum map.

Proof of Theorem 2.1.4: We need to construct minimum proper monomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ under the assumption that either $(k+1)n - k(k+1)/2 \leq N \leq (k+1)n$ with $k \leq K(n)$ or $N \geq (K(n)+1)n - K(n)(K(n)+1)$. Apparently, $K(n) \leq \sqrt{2n}$.

Let $k \leq n$. By Example C, we see the existence of minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ when $(k+1)n - k(k+1)/2 \leq N \leq (k+1)n - k(k-1)/2$. If $k-1 > 0$, applying Lemma 2.2.1 with $\kappa_0 = k-1$ and $\tau = 0, ..., k-1$, we see the existence of minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $(k+1)n - k(k-1)/2 \leq N \leq (k+1)n - (k-1)(k-2)/2 - 1$. Again, applying Lemma 2.2.1 with $\kappa_0 = k-2$ (if $k-2 > 0$) and $\tau = 0, ..., k-2$, we see the existence of minimum proper monomial maps from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $(k+1)n - (k-1)(k-2)/2 - 1 \leq N \leq (k+1)n - (k-2)(k-3)/2 - 1$. By an inductive use of Lemma 2.2.1, we see the existence of the required maps for $N$ with $(k+1)n - k(k+1)/2 \leq N \leq (k+1)n$ for $k \leq n$.

Next, letting $k = n+1$ in Lemma 2.2.1 and inductively applying Lemma 2.2.1 with $\kappa_0 = n, n-1, ..., n$, we conclude the existence of the required maps when $(n+2)n - n(n+1)/2 - 1 \leq N \leq (n+2)n$. In particular, this would give the existence of the required maps when $(n+1)n \leq N \leq (n+2)n$. Applying an induction argument, we easily conclude the existence of the required maps for any $N \geq (n+1)n$. □
2.3 Some Important Maps

Let us survey some important maps.

\[ N = n \geq 2, \text{ Alexander's theorem. } [A77], \text{ Prop} \_2(\mathbb{B}^n, \mathbb{B}^n) = \text{Aut}(\mathbb{B}^n). \]

\[ n < N < 2n - 1, \text{ the first gap theorem, the linear map.} \]

\[ N = 2n - 1 \text{ with } n \geq 3, \text{ Huang and Ji (2001) } [HJ01], F \approx \text{Id}, \text{ or } F \approx \text{Whitney map.} \]

Here \textit{Whitney map} is the map as in Example A:

\[ W_{n,1} = (z', z_n z) \text{ where } z = (z', w) \in \mathbb{C}^n. \]

\[ N = 2n - 1 = 3 \text{ with } n = 2, \text{ Faran (1982) } [Fa82], \text{ four equivalent classes of maps:} \]

\[ (z, w, 0); \ (z, zw, w^2), \ (z^2, \sqrt{2}zw, w^2); \ (z^3, \sqrt{3}zww, w^3). \]

\[ N = 2n, \text{ D'Angelo maps } [DA88]. \]

\[ F_\theta = (z, wcos\theta, z_1wsin\theta, ..., z_{n-1}wsin\theta, w^2sin\theta), \text{ with } 0 \leq \theta \leq \frac{\pi}{2}, \]

is a monomial map from \( \mathbb{B}^n \) into \( \mathbb{B}^{2n} \). In other words, such \( F_\theta = W_{n,1}(\theta, z) \) as in Example C. It is called the \textit{D’Angelo family}.

\[ 2n < N < 3n - 3 \text{ with } n \geq 4, \text{ the second gap theorem, } [HJX06] \text{ Any } F \in \text{Prop}_3(\mathbb{B}^n, \mathbb{B}^N) \text{ is equivalent to a map } (W_{n,1}(\lambda, z), 0) \text{ where } \lambda \in [0, 1]. \]
2.4 Rational and Polynomial Map

All examples above are polynomial maps. Nevertheless, not every map in \( \text{Rat}(\mathbb{B}^n, \mathbb{B}^N) \) can be equivalent to a polynomial map.

Let us introduce a criterion which tells whether or not a rational map can be equivalent to a polynomial one as follows.

Let \( F = P = (P_1, \ldots, P_N) \) be a non-constant rational holomorphic map from \( \mathbb{B}^n \subset \mathbb{C}^n \) into \( \mathbb{B}^N \subset \mathbb{C}^N \), where \((P_j)_{j=1}^N, q\) are holomorphic polynomial functions and \((P_1, \ldots, P_N, q) = 1\). We define \( \deg(F) = \max\{\deg(P_j)_{j=1}^N, \deg(q)\} \). Then \( F \) induces a rational map from \( \mathbb{C}P^n \) into \( \mathbb{C}P^N \) given by

\[
\hat{F}([z_1 : \ldots : z_n : t]) = \left[ t^k P(\frac{\tilde{z}}{t}) : t^k q(\frac{\tilde{z}}{t}) \right]
\]

where \( z = (z_1, \ldots, z_n) \in \mathbb{C}^n \) and \( \deg(F) = k > 0 \). \( \hat{F} \) may not be holomorphic in general. Denote by \( \text{Sing}(\hat{F}) \) the singular set of \( \hat{F} \), namely, the collection of points where \( \hat{F} \) fails to be (or fails to extend to be) holomorphic. Then \( \text{Sing}(\hat{F}) \) is an algebraic subvariety of codimension two or more in \( \mathbb{C}P^n \).

**Theorem 2.4.1** [FHJZ2010] Let \( F \) be a non-constant rational holomorphic map from \( \mathbb{B}^n \) into \( \mathbb{B}^N \) with \( N, n \geq 1 \). Then \( F \) is equivalent to a holomorphic polynomial map from \( \mathbb{B}^n \) into \( \mathbb{B}^N \), namely, there are \( \sigma \in \text{Aut}(\mathbb{B}^n) \) and \( \tau \in \text{Aut}(\mathbb{B}^N) \) such that \( \tau \circ F \circ \sigma \) is a holomorphic polynomial map from \( \mathbb{B}^n \) into \( \mathbb{B}^N \), if and only if there exist (complex) hyperplanes \( H \subset \mathbb{C}P^n \) and \( H' \subset \mathbb{C}P^N \) such that \( H \cap \mathbb{B}^n_1 = \emptyset \), \( H' \cap \mathbb{B}^N_1 = \emptyset \) and

\[
\hat{F}(\mathbb{C}P^n \setminus (H \cup \text{Sing}(\hat{F}))) \subset \mathbb{C}P^N \setminus H'.
\]

**Example D**[FHJZ2010] Let \( G(z, w) = \left( z^2, \sqrt{2}zw, w^2(\frac{z-a}{1-az^2}, \frac{\sqrt{1-|a|^2w}}{1-az}) \right) \), \( |a| < 1 \), be a map in \( \text{Rat}(\mathbb{B}^2, \mathbb{B}^4) \). \( G \) is equivalent to a proper holomorphic polynomial map in \( \text{Poly}(\mathbb{B}^2, \mathbb{B}^4) \) if and only if \( a = 0 \).

In fact, we have

\[
\hat{G} = \left[ (t - \pi z)z^2 : (t - \pi z)\sqrt{2}zw : w^2(z - at) : w^2\sqrt{1-|a|^2w} : (t^3 - \pi t^2z) \right].
\]
2.4. RATIONAL AND POLYNOMIAL MAP

Suppose there exist hyperplanes $H = \{\mu_1 z_1 + \mu_2 w + \mu_0 t = 0\} \subset \mathbb{CP}^2$ and $H' = \{\sum_{j=1}^{4} \lambda_j z_j' + \lambda_0 t' = 0\} \subset \mathbb{CP}^4$ such that

$$H \cap \mathbb{B}_1^2 = \emptyset, \quad H' \cap \mathbb{B}_1^4 = \emptyset, \quad \hat{G}(H \setminus \text{Sing}(\hat{G})) \subset H', \quad \hat{G} \left( \mathbb{CP}^2 \setminus (H \cup \text{Sing}(\hat{G})) \right) \subset \mathbb{CP}^4 \setminus H'.$$

Then

$$\lambda_1 (t - az) z^2 + \lambda_2 (t - az) \sqrt{2} zw + \lambda_3 w^2 (z - at) + \lambda_4 w^2 \sqrt{1 - |a|^2} w$$

$$+ \lambda_0 (t^3 - at^2 z) = (\mu_1 z + \mu_2 w + \mu_0 t)^3 \quad \forall [z : w : t] \in \mathbb{CP}^2.$$

Apparently $\lambda_0 \neq 0$. Hence we can assume that $\lambda_0 = 1, \mu_0 = 1$. By comparing the coefficient of $z^3, w^3, wt^2, zt^2, zwt, z^2 w, zw^2, w^2 t$, respectively, in the above equation, we get

$$\mu_1^3 = -\bar{a} \lambda_1, \quad \mu_2^3 = \lambda_4 \sqrt{1 - |a|^2}, \quad 3 \mu_2 = 0, \quad 3 \mu_1 = -\bar{a}, \quad 3 \mu_2^2 = \lambda_1, \quad 6 \mu_1 \mu_2 = \sqrt{2} \lambda_2, \quad 3 \mu_1^2 \mu_2 = -\sqrt{2} \lambda_2 \bar{a}, \quad 3 \mu_1 \mu_2^2 = \lambda_3, \quad 3 \mu_2^2 = -a \lambda_3.$$

We then have $\lambda_2 = \lambda_3 = \lambda_4 = \mu_2 = 0$. If $a \neq 0$, then $\mu_1, \lambda_1 \neq 0$. From $\mu_1^3 = -\bar{a} \lambda_1$ and $3 \mu_1^2 = \lambda_1$, we get $\mu_1 = -3 \bar{a}$. Since $3 \mu_1 = -\bar{a}$, we get $\bar{a} = 0$. This is a contradiction. Notice that when $a = 0$, $F$ is a polynomial. By Theorem 2.4.1, we see the conclusion. \[ \Box \]

**Example E**[FHZJ2010] Let $F(z', w) = \left( z', w, z^2 (\frac{\sqrt{1-|a|^2} z'}{1-\bar{a}w}, \frac{w-a}{1-\bar{a}w}) \right)$ with $|a| < 1$ be a map in $\text{Rat}(\mathbb{B}^n, \mathbb{B}^{3n-2})$. $F$ is equivalent to a proper polynomial map in $\text{Poly}(\mathbb{B}^n, \mathbb{B}^{3n-2})$ if and only if $a = 0$.

By the criterion in Theorem 2.4.1, it is also proved that

**Theorem 2.4.2**[FHZJ2010] A map $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$ of degree two is equivalent to a polynomial proper holomorphic map in $\text{Poly}(\mathbb{B}^2, \mathbb{B}^N)$.

Recently, J. Lebl claimed in a preprint ([Le09], theorem 1.5):

**Theorem 2.4.3** Let $F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ with $n \geq 3$ and $\deg(F) = 2$. Then $F$ is equivalent to a monomial map.
[Example F][FHJZ2010] Let $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^5)$ be a rational mapping given by $F = (f, \phi_1, \phi_2, \phi_3, g)$ defined as follows:

$$
\begin{align*}
    f(z, w) &= \frac{z + \left(\frac{1}{2} - i\right)zw}{1 - iw - \frac{1}{3}w^2}, \\
    \phi_1(z, w) &= \frac{z^2}{1 - iw - \frac{1}{3}w^2}, \\
    \phi_2(z, w) &= \frac{\sqrt{\frac{13}{12}}zw}{1 - iw - \frac{1}{3}w^2}, \\
    \phi_3(z, w) &= \frac{\sqrt{\frac{3}{3}}w^2}{1 - iw - \frac{1}{3}w^2}, \\
    g(z, w) &= \frac{w - iw^2}{1 - iw - \frac{1}{3}w^2}.
\end{align*}
$$

Then this mapping $F$ is indeed equivalent to the polynomial map

$$
G(z, w) = \left(\frac{\sqrt{3}}{9}(-2 + 4z + z^2), -\frac{\sqrt{6}}{9}(1 + z + z^2), \frac{\sqrt{3}}{12}(5 + 3z)w, \frac{\sqrt{6}}{6}w^2, \frac{\sqrt{13}}{12}(1 - z)w\right).
$$

### 2.5 Classification of Maps from $\mathbb{B}^2$ to $\mathbb{B}^3$

Theorem 2.4.2 is proved based on the classification of maps of $\text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2 (see Theorem 2.6.1).

To illustrate techniques used to study the classification problem, we first give a proof for the following Faran’s theorem [Fa82]: Any map $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^3)$ must be equivalent to one of the following maps:

$$
\begin{align*}
    \text{degree 1} : &\ (z, w, 0); \\
    \text{degree 2} : &\ (z, zw, w^2), \text{ and } (z^2, \sqrt{2}zw, w^2); \\
    \text{degree 3} : &\ (z^3, \sqrt{3}zw, w^3).
\end{align*}
$$

The proof here is given in [J09] which is different from Faran’s original Proof. The difficulty to study $\text{Rat}(\mathbb{B}^2, \mathbb{B}^3)$, comparing study $\text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ with high $n$ and $N$, is that we have less numbers of equations.

It can be shown (by a similar argument as in the proof of Theorem 2.11.3) that $\text{deg}(F) \leq 3$. Since maps in $\text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$ with degree $\leq 2$ can be classified (see Theorem 2.6.1), it suffices to show: there exists exactly one map $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^3)$ with degree 3.
The normal form $F^{**}$ of $F$, still denoted as $(f, \phi, g)$, becomes

$$f = \frac{z - 2ib_{11}z^2 + (ie_1 + i/2)zw - 4b_{02}z^3 + E_{11}z^2w + A_{12}zw^2 + A_{03}w^3}{1 - 2ib_{11}z + ie_1w - 4b_{02}z^2 + E_{11}zw + E_{02}w^2 + E_{21}z^2w + E_{12}zw^2 + E_{03}w^3},$$

$$\phi = \frac{z^2 + b_{11}zw + b_{02}w^2 + B_{21}z^2w + B_{12}zw^2 + B_{03}w^3}{1 - 2ib_{11}z + ie_1w - 4b_{02}z^2 + E_{11}zw + E_{02}w^2 + E_{21}z^2w + E_{12}zw^2 + E_{03}w^3},$$

$$g = \frac{w - 2ib_{11}zw + ie_1w^2 - 4b_{02}z^2w + E_{11}zw^2 + C_{03}w^3}{1 - 2ib_{11}z + ie_1w - 4b_{02}z^2 + E_{11}zw + E_{02}w^2 + E_{21}z^2w + E_{12}zw^2 + E_{03}w^3},$$

with $b_{02} > 0$ and $e_1 \in \mathbb{R}$.

Consider the basic equation: $\text{Im}(g) = |f|^2 + |\phi|^2$, $\forall \text{Im}(w) = |z|^2$, we obtain all algebraic equations about the parameters. Among these equations, we find

$$e_1 \text{Im}(b_{11}^2) = 0. \quad (2.1)$$

By (2.1), we consider

$$\begin{cases} 
\text{Case } A : e_1 \neq 0 \Rightarrow \\
\text{Case } A_1 : \text{Im}(b_{11}) = 0; \\
\text{Case } A_2 : \text{Re}(b_{11}) = 0. 
\end{cases}$$

In Case $A_1$, we list all the equations about the parameters:

- $A_{12} = E_{02} - \frac{1}{8} - \frac{5}{4}e_1 - \frac{1}{2}b^2$, $b_{11} = b$ is a real parameter,
- $b_{02}$ determined by $\frac{1}{2}e_1 + 4e_1b^2 + e_1^2 + 12b_{02}^2 + 4b_{02}b^2 = 0$,
- $B_{21} = i(\frac{1}{4} + \frac{3}{2}e_1 + b^2)$, $B_{12} = i(\frac{1}{4}b + \frac{3}{2}be + b^3)$,
- $B_{03} = ib_{02}(\frac{1}{4} + \frac{3}{2}e_1 + b^2)$, $C_{03} = E_{02} - \frac{e_1}{2}$, $e_1 \neq 0$ is a real parameter,
- $E_{11} = \frac{1}{2}b + e_1b + 2b^2 - 8bb_{02}$, $E_{12} = -i(eb + 2bb_{02})$,
- $E_{21} = -2ib_{02}$, $E_{02} = \frac{1}{16} + \frac{5}{4}e_1 + \frac{1}{2}b^2 + 2b_{02}^2 + 3e_1b^2 + \frac{5}{4}e_1^2 + b^4$,
- $E_{03} = i(\frac{1}{2}e_1^2 - |b_{02}|^2)$. 


From the equation for $b_{02}$ above, we obtain
\[
e_1 = -\left(\frac{1}{2} + 4b^2\right) \pm \sqrt{\left(\frac{1}{2} + 4b^2\right)^2 - 4(12b_{02}^2 + 4b_{02}b^2)}.
\]

Since $e_1$ is a real number, we must have
\[
\left(\frac{1}{2} + 4b^2\right)^2 - 4(12b_{02}^2 + 4b_{02}b^2) \geq 0,
\]
i.e.,
\[
\left(\frac{1}{2} + 4b^2\right)^2 + \frac{4}{3}b^4 \geq 48\left(b_{02}^2 + \frac{b^2}{6}\right)^2.
\]

Hence all of the coefficients of $F$ are bounded when $|b| = |b_{11}|$ is bounded.

Similar conclusion holds for Case $A_2$ and Case $B$.

Then we take a sequence $p_m \in \partial \mathbb{H}^2$ so that the associated map $F^{***}_{p_m}$ satisfies
\[
\lim_{m \to \infty} b^{***}_{11}(p_m) = \inf_p \{b^{***}_{11}\}.
\]

Then we show
\[
F \text{ is equivalent to } \tilde{F} = \lim_{m \to \infty} (F_{p_m})^{***}.
\]

Here we have to take care of the facts that $p_m$ could go to $\infty$: $[0 : a : b] \in \partial \mathbb{H}^2$ and the equivalence is not obvious. But this can be done because all other parameters are dominated by $|b_{11}|$.

The limit map $\tilde{F}$ has the minimum property for its parameter $b_{11}$, namely, if we denote by $b_{11}(p)$ the corresponding coefficient of the map $(\tilde{F}_p)^{***}$, we find
\[
|b_{11}(p)|^2 = |b_{11}|^2 - i(b_{11} + 2b_{11}e_1 + 12b_{11}b_{02} + 4b_{11}|b_{11}|^2)z_0 + i(b_{11} + 2b_{11}e_1 + 12b_{11}b_{02} + 4b_{11}|b_{11}|^2)\bar{z_0} + 32b_{02} Re(b_{11}) Im(b_{11}) u_0 + o(1).
\]

Since the critical point of the function $b_{11}(p)$ is zero by the minimum property, it gives the desired extra equation:
\[
Im(b_{11}) Re(b_{11}) = 0, \text{ and } b_{11} + 2e_1b_{11} + 4b_{11}|b_{11}|^2 + 12b_{02}b_{11} = 0.
\]

It leads us consider Case(C): $b_{11} = 0$ and Case(D): $b_{11} \neq 0$.

Finally we consider all cases:

<table>
<thead>
<tr>
<th>Case</th>
<th>Occur</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case A1 C</td>
<td>cannot occur</td>
</tr>
<tr>
<td>Case A2 C</td>
<td>cannot occur</td>
</tr>
<tr>
<td>Case B C</td>
<td>$\exists$ a unique map</td>
</tr>
<tr>
<td>Case A1 D</td>
<td>cannot occur</td>
</tr>
<tr>
<td>Case A2 D</td>
<td>cannot occur</td>
</tr>
<tr>
<td>Case B D</td>
<td>cannot occur</td>
</tr>
</tbody>
</table>
2.6. CLASSIFICATION OF MAPS FROM $\mathbb{B}^2$ WITH DEGREE TWO

The only map in $\text{Rat}(\mathbb{H}^2, \mathbb{H}^2)$ of degree 3 is of the normalized form $F = F^{**} = (f, \phi, g)$:

$$f = \frac{z + \frac{i}{2}zw - \frac{1}{16}zw^2}{1 + \frac{1}{16}w^2}, \quad \phi(z) = \frac{z^2 + \frac{i}{4}z^2w}{1 + \frac{1}{16}w^2}, \quad g = \frac{w + \frac{1}{16}w^3}{1 + \frac{1}{16}w^2}. \quad (2.3)$$

We notice that it is too complicated to find (2.3) directly by the definition of $F^{**}$.

2.6 Classification of Maps from $\mathbb{B}^2$ With Degree Two

The classification problem for maps in $\text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2 has been solved.

**Theorem 2.6.1** [JZ09] (i) Any nonlinear map in $\text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$ with degree 2 is equivalent to a map $(F, 0)$ where $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^5)$ is of one of the following forms:

(I): $F = (G_t, 0)$ where $G_t \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$G_t(z, w) = (z^2, \sqrt{1 + \cos^2 \theta}zw, (\cos \theta)zw, (\sin \theta)zw), \quad 0 \leq t < \pi/2. \quad (2.4)$$

(IIA): $F = (F_\theta, 0)$ where $F_\theta \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^4)$ is defined by

$$F_\theta(z, w) = (z, (\cos \theta)zw, (\sin \theta)zw, (\sin \theta)zw^2), \quad 0 < \theta \leq \frac{\pi}{2}. \quad (2.5)$$

(IIC): $F = F_{c_1, c_2, c_1, c_2} = \rho_5^{-1} \circ F \circ \rho_2 = (f, \phi_1, \phi_2, \phi_3, g) \in \text{Rat}(\mathbb{H}^2, \mathbb{H}^5)$ is of the form:

$$f = \frac{z + (\frac{i}{2} + ie_1)zw}{1 + ie_1w + e_2w^2}, \quad \phi_1 = \frac{z^2}{1 + ie_1w + e_2w^2},$$

$$\phi_2 = \frac{c_1zw}{1 + ie_1w + e_2w^2}, \quad \phi_3 = \frac{c_2w^2}{1 + ie_1w + e_2w^2}, \quad g = \frac{w + ie_1w^2}{1 + ie_1w + e_2w^2},$$

where $c_1, c_2 > 0, -e_1, -e_2 \geq 0, e_1e_2 = c_3^2, -e_1 - e_2 = \frac{1}{4} + c_4^2$, satisfying one of the following conditions: either

$$\begin{cases} 
    e_1 = \frac{(\frac{1}{4} + c_4^2) - \sqrt{(\frac{1}{4} + c_4^2)^2 - 4c_3^2}}{2}, 
    e_2 = \frac{(\frac{1}{4} + c_4^2) + \sqrt{(\frac{1}{4} + c_4^2)^2 - 4c_3^2}}{2}, 
    0 < 4c_3^2 \leq (\frac{1}{4} + c_4^2)^2, 
\end{cases} \quad (2.6)$$

or

$$\begin{cases} 
    e_1 = \frac{(\frac{1}{4} + c_4^2) + \sqrt{(\frac{1}{4} + c_4^2)^2 - 4c_3^2}}{2}, 
    e_2 = \frac{(\frac{1}{4} + c_4^2) - \sqrt{(\frac{1}{4} + c_4^2)^2 - 4c_3^2}}{2}, 
    \frac{1}{2}c_1^2 + c_4^4 \leq 4c_3^2 \leq (\frac{1}{4} + c_4^2)^2. 
\end{cases} \quad (2.7)$$

(ii) Any two maps in $\text{Rat}(\mathbb{B}^2, \mathbb{B}^5)$ in the form of types (I), (IIA), and (IIC) above are equivalent if and only if they are identical.
In Faran’s Theorem on $\text{Rat}(\mathbb{B}^2, \mathbb{B}^3)$, there are four maps, up to automorphisms, which are isolated. Nevertheless, for $\text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$ with $N > 3$, there exists a continuous family of maps, up to automorphism. For example, D’Angelo constructed $F_t = (z, w \cos t, (w \sin t)z) \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^{2n})$ with $t \in (0, \frac{\pi}{2})$ satisfies: $F_t$ is equivalent to $F_s$ if and only if $t = s$. To classify continuous family of maps, we have to use different technique.

### 2.7 Proof of Theorem 2.6.1 - Part 1

As a reduction in the proof of Theorem 2.6.1, Huang-Ji-Xu [HJX06] proved: Any map $F$ in $\text{Rat}(\mathbb{H}^2, \mathbb{H}^N)$ with $\deg(F) = 2$ is equivalent to a map $(G, 0)$ where $G = (f, \phi_1, \phi_2, \phi_3, g) \in \text{Rat}(\mathbb{H}^2, \mathbb{H}^5)$ is of the form (see also Lemma 2.3 below)

\[
\begin{align*}
    f(z, w) &= \frac{z - 2ibz^2 + (\frac{i}{2} + ie_1)zw}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\
    \phi_1(z, w) &= \frac{z^2 + bzw}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\
    \phi_2(z, w) &= \frac{c_2 w^2 + c_1 zw}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\
    \phi_3(z, w) &= \frac{c_3 w^2}{1 + ie_1 w + e_2 w^2 - 2ibz}, \\
    g(z, w) &= \frac{w + ie_1 w^2 - 2ibzw}{1 + ie_1 w + e_2 w^2 - 2ibz},
\end{align*}
\]

where $b, -e_1, -e_2, c_1, c_2, c_3$ are real non-negative numbers satisfying $e_1 e_2 = c_2^2 + c_3^2, -e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2, -b e_2 = c_1 c_2$, and $c_3 = 0$ if $c_1 = 0$.

Since $b$ and $c_2$ are determined by $c_1, c_3, e_1$ and $e_2$, a map in the above form is determined by $c_1, c_3, e_1$ and $e_2$. We denote a map of the above form, which is determined by $c_1, c_3, e_1$ and $e_2$, to be

$$F_{(c_1, c_3, e_1, e_2)} \in \mathcal{K}. \quad (2.8)$$

It was unclear which of the coefficients $e_1, e_2, c_1$ and $c_3$ of $F$ are independent parameters.

### 2.8 Proof of Theorem 2.6.1 - Part 2

In [CJX06], by obtaining an extra equation, we got a more clearer picture on the maps as above.
Let us describe how to obtain this extra equation.

For any \( F \in \text{Rat}(\mathbb{H}^2, \mathbb{H}^5) \) with \( \deg(F) = 2 \), \( F \) is equivalent to another map \( F^{***} \in \text{Rat}(\mathbb{H}^2, \mathbb{H}^5) \) of the above form. Also we can associate a family of maps \( F_p \in \text{Rat}(\mathbb{H}^2, \mathbb{H}^5) \) for any \( p \in \partial \mathbb{H}^2 \), as well as the associated maps \((F_p)^{***}\) that is of the above form.

We define a real analytic function

\[
W(F_p^{***}) = c_1(p)^2 - e_1(p) - e_2(p)
\]

where \( c_1(p), e_1(p) \) and \( e_2(p) \) are the coefficients of \( F_p^{***} \):

\[
f_p^{***}(z, w) = \frac{z - 2ib(p)z^2 + \left(\frac{1}{2} + ie_1(p)\right)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},
\]

\[
\phi_{1,p}^{***}(z, w) = \frac{z^2 + b(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},
\]

\[
\phi_{2,p}^{***}(z, w) = \frac{c_2(p)w^2 + c_1(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},
\]

\[
\phi_{3,p}^{***}(z, w) = \frac{c_3(p)w^2}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z},
\]

\[
g_p^{***}(z, w) = \frac{w + ie_1(p)w^2 - 2ib(p)zw}{1 + ie_1(p)w + e_2(p)w^2 - 2ib(p)z}.
\]

Here \( b(p), e_1(p), e_2(p), c_1(p), c_2(p), c_3(p) \) satisfy

\[
e_2(p)e_1(p) = c_2^2(p) + c_3^2(p), -e_2(p) = \frac{1}{4} + e_1(p) + b^2(p) + c_1^2(p),
\]

and \(-b(p)e_2(p) = c_1(p)c_2(p), c_3(p) = 0 \) if \( c_1(p) = 0 \), with

\[
c_1(p), c_2(p), b(p) \geq 0, e_2(p), e_1(p) \leq 0.
\]

We observe that as long as \( W(F_p^{***}) \) is bounded, all

\[
e_1(p_m), e_2(p_m), c_1(p_m), c_2(p_m), c_3(p_m) b(p_m)
\]

are uniformly bounded for all \( m \). In fact, since \( e_1(p_m), -e_1(p_m), -e_2(p_m) \) are non-negative, \( c_1(p_m), c_1(p_m), c_1(p_m) \) and \( e_2(p_m) \) are uniformly bounded for all \( m \). From \(-e_1(p_m) - e_2(p_m) = \frac{1}{4} + b^2(p_m) + c_2^2(p_m), b(p_m) \) is uniformly bounded for any \( m \). Finally, from \( e_1(p_m)e_2(p_m) = c_2^2(p_m) + c_3^2(p_m), c_2(p_m) \) and \( c_3(p_m) \) are uniformly bounded.
The desired extra equation is obtained by moving up $p$ to the extremal value as follows. We choose a sequence of $p_m \in \partial \mathbb{H}^2$ such that

$$p_m \to p_0 \in \partial \mathbb{H}^2 \quad \text{and} \quad \lim_{m} \mathcal{W}(F_{p_m}^{***}) = \inf_{p \in \partial \mathbb{H}^2 - \Xi_p} \{ \mathcal{W}(F_p^{***}) \}. \quad (2.14)$$

Then $F$ is equivalent to $F_{p_0}^{***}$ which is of the above form and with the minimum property $\mathcal{W}(F_{p_0}^{***}) = \inf_{p \in \partial \mathbb{H}^2 - \Xi_p} \mathcal{W}(F_p^{***})$.

A key lemma used to prove convergence of the limit map is the following result.

**Lemma 2.8.1** ([CJX06] lemma 2.5) Let $F \in \text{Rat}(\partial \mathbb{H}^2, \partial \mathbb{H}^5)$ with $F(0) = 0$ and $\deg(F) = 2$. Suppose that $p_m \in \partial \mathbb{H}^2$ is a sequence converging to 0, $F_{p_m}$ is of rank 1 at 0 for any $m$ and $F_{p_m}^{***}$ converges such that \( \frac{\partial^2 \phi_{1,m}^{***}}{\partial z \partial w} |_{0}, \frac{\partial^2 \phi_{2,m}^{***}}{\partial z \partial w} |_{0}, \frac{\partial^2 \phi_{3,m}^{***}}{\partial z \partial w} |_{0} \) and $\frac{\partial \phi_{1,m}^{***}}{\partial w} |_{0}$ are bounded for all $m$. Then

(i) $F$ is of geometric rank 1 at 0: $\deg_F(0) = 1$, and hence $F^{***}$ is well-defined.
(ii) $F_{p_m}^{***} \to F^{***}$.
(iii) If we write $F_{p_m}^{***} = \tilde{G}_{2,m} \circ \tau_{p_m} \circ F \circ \sigma_{p_m} \circ \tilde{G}_{1,m}$ where $\sigma_{p_m}$ and $\tau_{p_m}$ is as in [CJX06, (3)], $\tilde{G}_{1,m}$ and $\tilde{G}_{2,m}$ are as in [CJX06, (7)], then $\tilde{G}_{1,m}$ and $\tilde{G}_{2,m}$ are convergent to some $G_1 \in \text{Aut}_0(\partial \mathbb{H}^2)$ and $G_2 \in \text{Aut}_0(\partial \mathbb{H}^5)$ respectively.

The minimum property for $\mathcal{W}(F_{p}^{***})$ implies the vanishing of derivatives of the function $\mathcal{W}(F_{p}^{***})$ at $p_0$, which derives the extra equation.

In order to get this extra equation, we have to compute the first order derivatives of the function $\mathcal{W}(F_{p}^{***})$, which is done by the following lemma. The proof of this lemma used the differential formulas for $F_p$ and $F_{p}^{***}$ listed in Chapter 1. Although the computation is long, since every time it only counts for derivative at 0 so that lots of higher order terms can be dropped, the calculation is manageable.

**Lemma 2.8.2** ([CJX06], lemma 3.1) Let $F = F_{c_1,c_3,e_1,e_2}$ and $F_{p}^{***}$ be as above. Then for $p = (z_0, w_0) = (z_0, u_0 + i|z_0|^2) \in \partial \mathbb{H}^2$ near 0, we have real analytic functions

$$b^2(p) = b^2 - 4b(2c_1 + c_1^2)\Im(z_0) + o(1), \quad c_1^2(p) = c_1^2 + 4c_1(bc_1 + 2c_2)\Im(z_0) + o(1),$$

$$e_2(p) + e_1(p) = e_2 + e_1 + 8b(e_1 + e_2)\Im(z_0) + o(1),$$

$$c_1^2(p) - e_1(p) - e_2(p) = c_1^2 - e_1 - e_2 + \left(4c_1(bc_1 + 2c_2) - 8b(e_1 + e_2)\right)\Im(z_0) + o(1)$$

where we denote $o(k) = o(||(z_0, u_0)||^k)$.
2.8. PROOF OF THEOREM 2.6.1 - PART 2

If \( c_1 = 0 \), by the minimum property, it implies that the coefficient of \( \Im(z_0) \) must be zero. Then we obtain

\[-8b(e_1 + e_2) = 0.\]

Since \( -e_1 - e_2 = \frac{1}{4} + b^2 \neq 0 \), it implies \( b = 0 \).

If \( c_1 > 0 \), by the minimum property of \( F = F_0^{***} \), it implies that

\[4c_1(c_1b + 2c_2) - 8b(e_1 + e_2) = 0.\]

Since \( -e_1 - e_2 = \frac{1}{4} + b^2 + c_1^2 \neq 0 \) and \( c_1, b, c_2, -e_1, -e_2 \geq 0 \), it implies \( b = c_2 = 0 \).

To study \( F \), we distinguish two cases:

Case (I) \( c_1 = b = 0 \);
Case (II) \( c_1 \neq 0 \) and \( b = c_2 = 0 \).

It was proved in [CJX06] that \( F \) is equivalent to a new map \( F_{c_1, c_2, e_1, e_2} \) that is of the form in one of the following types (from Case (I), we obtain (I); from Case (II), we obtain (IIA)(IIB) and (IIC)):

(I) \( F_{0,0,e_1,e_2} = (f, \phi_1, \phi_2, \phi_3, g) \) is of the form

\[
f = \frac{z + (\frac{1}{2} + ie_1)zw}{1 + ie_1w}, \quad \phi_1 = \frac{z^2}{1 + ie_1w}, \quad \phi_2 = \frac{c_1zw}{1 + ie_1w}, \quad \phi_3 = 0, \quad g = w
\]  

(2.15)

where \( e_1e_2 = c_2^2 \) and \( -e_1 - e_2 = \frac{1}{4} \). Here \( e_2 \in [-\frac{1}{4}, 0) \) is a parameter. It then corresponds to the family \( \{G_{\theta}\}_{0 \leq \theta \leq \pi/2} \) in (2.4). When \( e_2 = \frac{1}{4} \), \( F_{0,0,e_1,e_2} \) corresponds to \( G_0 \), i.e., \((z, w) \mapsto (z^2, \sqrt{2}zw, w^2, 0)\); when \( e_2 \to 0 \), \( F_{0,0,e_1,e_2} \) goes to \( G_{\pi/2} = F_{\pi/2} \), i.e., \((Z, W) \mapsto (z, zw, w^2)\).

(IIA) \( F_{c_1,0,e_1,0} = (f, \phi_1, \phi_2, \phi_3, g) \) is of the form

\[
f = \frac{z + (\frac{1}{2} + ie_1)zw}{1 + ie_1w}, \quad \phi_1 = \frac{z^2}{1 + ie_1w}, \quad \phi_2 = \frac{c_1zw}{1 + ie_1w}, \quad \phi_3 = 0, \quad g = w
\]  

(2.16)

where \( -e_1 = \frac{1}{4} + c_1^2 \) and \( c_1 \in [0, \infty) \) is a parameter. It corresponds to the family \( \{F_{\theta}\}_{0 < \theta \leq \pi/2} \) in (2.5). When \( c_1 = 0 \), \( F_{c_1,0,e_1,0} \) corresponds to \( F_{\pi/2} \); when \( c_1 \to \infty \), \( F_{c_1,0,e_1,0} \) goes to the linear map, i.e., \((z, w) \mapsto (z, w, 0)\).

(IIB) \( F_{c_1,0,e_2} = (f, \phi_1, \phi_2, \phi_3, g) \) is of the form:

\[
f = \frac{z + \frac{1}{2}zw}{1 + e_2w^2}, \quad \phi_1 = \frac{z^2}{1 + e_2w^2}, \quad \phi_2 = \frac{c_1zw}{1 + e_2w^2}, \quad \phi_3 = 0, \quad g = \frac{w}{1 + e_2w^2}
\]  

(2.17)
where \(-e_2 = \frac{1}{4} + c_1^2\) and \(c_1 \in (0, \infty)\) is a parameter. Notice that when \(c_1 \to 0\), the map \(F_{c_1,0,e_2}\) goes to the map \(G_0\), i.e. the one in type (I) when \(e_2 = -\frac{1}{4}\).

**Case (IIC)** \(F_{c_1,c_3,e_1,e_2} = (f, \phi_1, \phi_2, \phi_3, g)\) is of the form:

\[
\begin{align*}
f & = \frac{z + (\frac{1}{2} + ie_1)w}{1 + ie_1 w + e_2 w^2}, \\
\phi_2 & = \frac{1}{1 + ie_1 w + e_2 w^2}, \\
\phi_3 & = \frac{c_3}{c_1 w^2}, \\
g & = \frac{w + ie_1 w^2}{1 + ie_1 w + e_2 w^2}.
\end{align*}
\]

(2.18)

where \(c_1, c_3 > 0, -e_1, -e_2 \geq 0\), \(e_1 e_2 = c_3^2\), \(-e_1 - e_2 = \frac{1}{4} + c_1^2\).

For any map \(F_{c_1,c_3,e_1,e_2}\) in one of these four types, we denote \(F_{c_1,c_3,e_1,e_2}\), or \((c_1, c_3, e_1, e_2)\), \(\in K_I, K_{IIA}, K_{IIB}, \text{and } K_{IIIC}\), respectively.

At this moment, it is not clear whether different such maps are not equivalent.

## 2.9 Proof of Theorem 2.6.1 - Part 3

It is proved by Ji-Zhang [JZ09] that the case (IIB) never occur.

We denote by \(K\) the collection of all such maps \(F_{c_1,c_3,e_1,e_2}\). We may identify a map \(F_{c_1,c_3,e_1,e_2}\) with a point \((c_1, c_3, e_1, e_2)\) in \(\mathbb{R}^4\).

The set \(K\) is equal to a disjoint union

\[
K = K_I \cup K_{II}
\]

where \(K_I = \{F_{c_1,c_3,e_1,e_2} \in K \mid F_{c_1,c_3,e_1,e_2} \text{ is of form } (I)\}, \text{etc. The set } K\text{ is also equal to a disjoint union}

\[
K = K_{I,II,1+4e_2+2c_1^2>0} \cup K_{I,II,1+4e_2+2c_1^2>0} \cup K_{I,II,1+4e_2+2c_1^2>0},
\]

where \(K_{I,II,1+4e_2+2c_1^2>0} = (K_I \cup K_{II}) \cap \{(c_1, c_3, e_1, e_2) \mid 1 + 4e_2 + 2c_1^2 > 0\}, \text{etc.}

**Lemma 2.9.1** ([JZ09], lemma 3.1)

(a) If \((c_1, c_3, e_1, e_2) \in K_{I,II,1+4e_2+2c_1^2>0}, \text{ then locally the function } \mathcal{W}(\langle F_{c_1,c_3,e_1,e_2}^{***}\rangle) \text{ is increasing as } p \text{ moves along any ray from } 0 \text{ in } \partial \mathbb{H}^2.\)

(b) If \((c_1, c_3, e_1, e_2) \in K_{I,II,1+4e_2+2c_1^2=0}, \text{ then locally the function } \mathcal{W}(\langle F_{c_1,c_3,e_1,e_2}^{***}\rangle) \text{ is constant as } p \text{ moves along any ray from } 0 \text{ in } \partial \mathbb{H}^2.\)

(c) If \((c_1, c_3, e_1, e_2) \in K_{I,II,1+4e_2+2c_1^2<0}, \text{ then locally the function } \mathcal{W}(\langle F_{c_1,c_3,e_1,e_2}^{***}\rangle) \text{ is decreasing as } p \text{ moves along any ray from } 0 \text{ in } \partial \mathbb{H}^2.\)
Lemma 2.9.2 ([JZ09], lemma 3.2) (i) $\mathcal{K}_{\text{II},e_1+e_2} \subseteq \mathcal{K}_{\text{I,II},1+e_2+2c_1^2} > 0$, and $\mathcal{K}_{\text{II},e_1} + e_2 \subseteq \mathcal{K}_{\text{I,II},1+e_2+2c_1^2} > 0$.

(ii) Let $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{\text{II},e_1}$. Then

(a) $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{\text{I,II},1+e_2+2c_1^2} > 0$ if and only if $\frac{1}{2}c_1^2 + c_1^4 < 4c_3^2 < (\frac{1}{2} + c_1^2)^2$ holds.
(b) $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{\text{I,II},1+e_2+2c_1^2} > 0$ if and only if $\frac{1}{2}c_1^2 + c_1^4 = 4c_3^2$ holds.
(c) $(c_1, c_3, e_1, e_2) \in \mathcal{K}_{\text{I,II},1+e_2+2c_1^2} < 0$ if and only if $0 \leq 4c_3^2 < \frac{1}{2}c_1^2 + c_1^4$ holds.

By last section, we can consider $F_{c_1,c_3,e_1,e_2}$ satisfying the minimum property (2.14). Such map $F_{c_1,c_3,e_1,e_2}$ will contradict with the statement in Lemma 2.9.1(c). Therefore, it follows:

Lemma 2.9.3 ([JZ09], lemma 3.4) Let $(c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{\text{II}}$. Then $F_{c_1,c_3,e_1,e_2}$ satisfies (2.14) if and only if $F_{c_1,c_3,e_1,e_2} \in \mathcal{K}^* := \mathcal{K}_I \cup \mathcal{K}_{\text{II}} - \mathcal{K}_{\text{I,II},1+e_2+2c_1^2} < 0$.

This proves the part (i) of Theorem 2.6.1. From the definition of $K$, $e_1$ and $e_2$ are determined by $c_1$ and $c_3$ through a quadratic equation. This show how we obtain the domain of the parameters $c_1$ and $c_3$ in Theorem 2.6.1.

We may outline the idea for the proof of Lemma 2.9.1 here. The monotonicity in Lemma 2.9.1 (a) means

$$
\frac{d\mathcal{W}(F_{\Gamma(t)}^*)}{dt} = \lim_{\Delta t \to 0} \frac{\mathcal{W}(F_{\Gamma(t)+\Delta t}^*) - \mathcal{W}(F_{\Gamma(t)}^*)}{\Delta t} \geq 0, \forall t \in [0, \delta].
$$

(2.19)

For any $0 < t < \delta$ and sufficiently small $\Delta t > 0$, if we can write

$$
F_{\Gamma(t+\Delta t)}^* = \left(F_{\Gamma(t)}^*ight)_{q(t, \Delta t)}^{***}
$$

(2.20)

for some differentiable map $q(t, \Delta t) \in \partial \mathbb{H}^2$, then from Lemma 2.8.2 we should have

$$
\mathcal{W}(F_{\Gamma(t+\Delta t)}^*) = \mathcal{W}(F_{\Gamma(t)}^*) + \left[4c_1(bc_1+2c_2) - 8b(e_1+e_2)\right](\Gamma(t))\mathbb{Z}(q_1(t))\Delta t + o(|\Delta t|),
$$

(2.21)

where we write $q(t, \Delta t) := (q_1(t), q_2(t))\Delta t + o(|\Delta t|)$. Notice that $[4c_1(bc_1+2c_2) - 8b(e_1+e_2)](\Gamma(t)) \geq 0$ always holds because $c_1, c_2, -e_1 - e_2 \geq 0$. Then (2.19) follows if $\mathbb{Z}(q_1(t)) \geq 0$ holds. In particular, if $[4c_1(bc_1+2c_2) - 8b(e_1+e_2)](\Gamma(t)) \neq 0$ for any fixed $t \in [0, \delta)$, and if the following condition is satisfied:

$$
\mathbb{Z}(q_1(t)) > 0, \quad \forall t \in [0, \delta],
$$

(2.22)

then the strict inequality (2.19) holds. To prove (2.19), it suffices to prove (2.22). (2.22) is proved by local calculation of $\mathbb{Z}(q_1(t))$. 

CHAPTER 2. CONSTRUCTION AND CLASSIFICATION OF RATIONAL MAPS

2.10 Proof of Theorem 2.6.1 - Part 4

As the final step to complete the proof of Theorem 2.6.1, it is proved by Ji-Zhang [JZ09] that the cases (I)(IA) and (IIC) indeed give a complete classification for mappings in \( \text{Rat}(\mathbb{B}^2, \mathbb{B}^N) \) with degree 2, up to equivalent classes.

To solve the classification problem, by Lemma 2.9.3, we need to show: for maps \( F_{c_1, c_3, e_1, e_2} \) and \( F_{c_1', c_3', e_1', e_2'} \) in \( \mathcal{K}^* \), we have

\[
F_{c_1, c_3, e_1, e_2} \text{ is equivalent to } F_{c_1', c_3', e_1', e_2'} \iff (c_1', c_3', e_1', e_2') = (c_1, c_3, e_1, e_2). \tag{2.23}
\]

We first prove a local version of (2.23).

Lemma 2.10.1 For any \( P^{(0)} = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \in \mathcal{K}^* \), there is a neighborhood \( U \) of \( P^{(0)} \) in \( \mathcal{K}^* \) and a constant \( c > 0 \) such that for any point \( (c_1', c_3', e_1', e_2') \in U \) with \( F_{c_1', c_3', e_1', e_2'}^{(0)}(0) = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \), we have

\[
(c_1', c_3', e_1', e_2') = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}). \tag{2.24}
\]

To prove this, we use the monotone property in Lemma 2.9.1 to show:

\[
\mathcal{W}(F_{c_1, c_3, e_1, e_2}) = \mathcal{W}(F_{c_1, c_3, e_1, e_2})_{\Gamma(0)}^{(**)} \leq \mathcal{W}(F_{c_1, c_3, e_1, e_2})_{\Gamma(t^*)}^{(**)} = \mathcal{W}(F_{c_1', c_3', e_1', e_2'}), \tag{2.25}
\]

and

\[
\mathcal{W}(F_{c_1', c_3', e_1', e_2'}) = \mathcal{W}(F_{c_1', c_3', e_1', e_2'})_{\Gamma(t^*)}^{(**)} \leq \mathcal{W}(F_{c_1', c_3', e_1', e_2'})_{\Gamma(t)}^{(**)} = \mathcal{W}(F_{c_1, c_3, e_1, e_2}^{(0)}), \tag{2.26}
\]

By (2.25) and (2.26), it follows that the function \( \mathcal{W}(F_{c_1, c_3, e_1, e_2})^{(**)} = \text{constant} \). Then it implies that \( (F_{c_1, c_3, e_1, e_2})^{(**)}_{\Gamma(t)} \) is constant. Since \( F_{c_1', c_3', e_1', e_2'}^{(0)} = (F_{c_1', c_3', e_1', e_2'})_{p^{***}}, \) Lemma 2.10.1 is proved.

Next, we prove the global version of (2.23). We need to show: if \( F_{c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}} \) and \( F_{c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}} \) in \( \mathcal{K}^* \) are equivalent, then

\[
(c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}). \tag{2.27}
\]

Let \( \mathcal{E} := \{(c_1, c_3, e_1, e_2) \in \mathcal{K}_I \cup \mathcal{K}_{II} \mid (F_{c_1, c_3, e_1, e_2})^{(**)}_p \equiv F_{c_1, c_3, e_1, e_2}, \ \forall p \in \partial \mathbb{H}^2 \text{ near } 0 \} \). We assume that \( (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \notin \mathcal{E} \); otherwise \( F_{c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}} \) and \( F_{c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}} \) cannot be equivalent.
Since \( F_{c_1, c_3}^{(0)}, c_1^{(0)}, c_2^{(0)} \) and \( F_{c_1, c_3}^{(0)}, c_1^{(0)}, c_2^{(0)} \) are equivalent,

\[
F_{c_1, c_3}^{(0)}, c_1^{(0)}, c_2^{(0)} = \Psi \circ F_{c_1, c_3}^{(0), 0} \circ \Theta
\]

(2.28)

where \( \Theta \in Aut(\mathbb{H}^2) \) and \( \Psi \in Aut(\mathbb{H}^5) \).

We take a real analytic curve \( L = L(s) \in K^* - \mathcal{E} \), \( 0 \leq s \leq 1 \), where \( \mathcal{E} \) is a such that \( L(0) = (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \). In fact, since \( (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \not\in \mathcal{E} \) and \( \mathcal{E} \) is closed, \( L \) could be taken in a neighborhood of \( (c_1^{(0)}, c_3^{(0)}, e_1^{(0)}, e_2^{(0)}) \).

We shall use some deformation. By using automorphisms of balls, we can take a real analytic curve \( L \in K^* - \mathcal{E} \). For any \( s \in (0, 1) \), \( \Theta_s(0) \not\in \mathcal{E} \), \( \Psi_s \circ F_{L(s)} \circ \Theta_s(0) = 0 \); when \( s = 1 \), \( \Theta_1 = Id, \Psi_1 = Id \). Then we define

\[
\tilde{L}_0(s) := \Psi_s \circ F_{L(s)} \circ \Theta_s \in Rat(\mathbb{H}^2, \mathbb{H}^5), \ 0 \leq s \leq 1,
\]

such that \( \tilde{L}_0(s)(0) = 0 \) for all \( s \), \( F_{L_0(0)} = \Psi \circ F_{L(0)} \circ \Theta \) and \( \tilde{L}_0(1) = L(1) \). Our goal is to show: \( \tilde{L}_0(s) = L(s), \forall s \in [0, 1] \), so that \( \tilde{L}_0(0) = L(0) \), i.e., (2.27) holds.

Even though \( (F_{L_0(s)})^{***} \) is in \( \mathcal{K} \) for any \( s \in (0, 1) \), it may not be in \( \mathcal{K}^* \) because the minimum property (2.14) may not be satisfied. We claim that \( (F_{L_0(s)})^{***} \) is equivalent to another map \( F_{L(s)} \in \mathcal{K}^* \). More precisely, we want to find \( q(s) \in \partial \mathbb{H}^2 \) so that

\[
F_{L(s)} := (F_{L_0(s)})^{***} \in \mathcal{K}^*, \ \forall s \in (0, 1].
\]

(2.29)

As points in \( \mathcal{K} \), we show

\[
dist(F_{L(s)}, F_{L_0(s)}) \to 0, \ as \ s \to 1,
\]

(2.30)
i.e.,

\[
dist(F_{L(s)}, F_{L(s)}) \to 0, \ as \ s \to 1.
\]

Since both \( F_{L(s)} \in \mathcal{K}^* \) and \( F_{L(s)} \in \mathcal{K}^* - \mathcal{E} \) where \( s \in (s_0, 1] \) for some \( s_0 > 0 \) such that \( 0 \leq 1 - s_0 \) is sufficiently small, by the local version of Theorem 2.6.1, we conclude

\[
F_{L(s)} = F_{L(s)}, \ \forall s \in (s_0, 1].
\]

Repeating this process. Finally by continuity \( F_{L(s)} = F_{L(s)}, \ \forall s \in [0, 1] \). When restricted at 0, \( F_{L_0(0)} = F_{L(0)} = F_{L(0)} \), so that (2.27) is proved.
2.11 Degree of Rational Maps between Balls

For any rational map $H \not\equiv 0$, write $H = \frac{(P_1, \cdots, P_m)}{R}$, where $P_j, R$ are holomorphic polynomials and $(P_1, \cdots, P_m, R) = 1$. We then define

$$\text{deg}(H) = \max(\text{deg}(P_j)_{j=1, \cdots, m}, \text{deg}(R)).$$

(When $H \equiv 0$, we set $\text{deg}(H) = -\infty$).

D’Angelo raised a conjecture [DKR 03]: For any $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^N)$, does it satisfy

$$\text{deg}(F) \leq \begin{cases} 2N - 3, & \text{if } n = 2, \\ \frac{N-1}{n-1}, & \text{if } n \geq 3. \end{cases} \tag{2.31}$$

Both of the above bounds are sharp. In fact, when $n = 2$, the degree bound $2N - 3$ is achieved (see p.173 and p. 189 in [DA93]) for the polynomial map $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^{2+r})$ defined by $F(z, w) = (z^{2r+1}, \ldots, c_s z^{2(r-s)+1}, w^s, \ldots, w^{2r+1})$ where $c_s$ are certain constants. When $n \geq 3$, we consider the Whitney map $h(z, w) = (z, w(z, w)) : \mathbb{B}^n \to \mathbb{B}^{2n-1}$ with degree 2. By letting $(z, w) \mapsto (z, wh)$, we get a proper polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N = 3n - 2$ of degree 3. Inductively, we can construct a proper polynomial map from $\mathbb{B}^n$ into $\mathbb{B}^N$ with $N = kn - (k - 1)$ of degree $k$. Hence $\frac{N-1}{n-1} = k$ so that the bound in (2.31) is sharp.

[Example] We can show that any $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^5)$ has degree $\text{deg}(F) \leq 7$. We have classified all degree 2 maps in $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^5)$. For higher degree maps, the situation should be very complicated. D’Angelo classified all monomial maps in $F \in \text{Rat}(\mathbb{B}^2, \mathbb{B}^5)$. He find out

$$\begin{cases} \text{degree 3} : & 31 \text{ isolated maps or continuous families;} \\ \text{degree 4} : & 47 \text{ isolated maps or continuous families;} \\ \text{degree 5} : & 24 \text{ isolated maps or continuous families;} \\ \text{degree 6} : & 5 \text{ isolated maps or continuous families;} \\ \text{degree 7} : & 3 \text{ isolated maps;} \end{cases}$$

For example, maps with degree 7 in $\text{Rat}(\mathbb{B}^2, \mathbb{B}^5)$ are

1. $(z^7, w^7, \sqrt[7]{\sqrt{2}} w z^5, \sqrt[7]{\sqrt{2}} w^5 z, \sqrt[7]{\sqrt{2}} w z)$
2. $(z^7, w^7, \sqrt[7]{4} w^2 z^3, \sqrt[7]{4} w^3 z)$
3. $(z^7, w^7, \sqrt[7]{7} w^3 z^3, \sqrt[7]{7} w^3 z)$
2.11. DEGREE OF RATIONAL MAPS BETWEEN BALLS

- Forstnerič proved that for any \( F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^N) \), its degree \( \deg(F) \leq N^2(N - n + 1) \) in [Fo86].

**Theorem 2.11.1** [HJX06] Let \( F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^N) \) with geometric rank \( \kappa_0 = 1 \) and \( n \geq 3 \). Then \( \deg(F) \leq \frac{N-1}{n-1} \).

**Proof:** For each \( N \geq n \geq 3 \), there is a unique positive integer \( k \) such that \( k(n-1)+1 \leq N \leq (k+1)(n-1) \). We use induction on \( k \). When \( k = 1 \), \( F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^{2n-2}) \), by the first gap theorem, so that \( \deg(F) = 1 \leq \frac{N-1}{n-1} \) holds. Assume \( \deg(F) \leq \frac{N-1}{n-1} \) holds for any \( k \). Consider \( k+1 \), by Theorem 3.5.1 below [HJX06], \( F \) is equivalent to \((z,wh)\) where \( h \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^{N-n+1}) \). Then by the assumption, \( \deg(F) \leq 1+\deg(h) \leq 1+\frac{(N-n+1)-1}{n-1} = \frac{N-1}{n-1} \). \( \square \)

To illustrate the idea how to deal with degree \( \deg(F) \), we present a lemma and a theorem below.

**Lemma 2.11.2** ([HJ01], lemma 5.4) Let \( H = (P_1, \cdots, P_m) \) be a rational map from \( \mathbb{C}^n \) into \( \mathbb{C}^m \), where \( P_j, R \) are holomorphic polynomials with \((P_1, \cdots, P_m, R) = 1 \) (\( m > n > 1 \)). Assume for each \( p \in \partial \mathcal{H}^n \) close to the origin,

\[
\deg(H|_{Q_p}) \leq k
\]

with \( k > 0 \) a fixed integer, where \( Q_{(\zeta,\eta)} = \{(z, w) | \frac{w-R}{2i} = \sum z_j \eta_j\} \) is the Segre family of \( \partial \mathcal{H}^n \). Then \( \deg(H) \leq k \).

**Theorem 2.11.3** ([HJ01], lemma 5.2) Let \( F \in \text{Prop}_2(\mathbb{B}^n, \mathbb{B}^{2n-1}) \) with \( n \geq 3 \). Then \( F \) is rational and \( \deg(F) \leq 2 \).

**Proof:** By Cayley transformation, we consider \( F \in \text{Prop}_2(\mathbb{H}^n, \mathbb{H}^{2n-1}) \). By Lemma 2.11.2, it suffices to prove that \( \deg(F|_{Q_{p_0}}) \leq 2 \) for any \( p_0 \in \partial \mathcal{H}^n \).

It is equivalent to show that for every \( p \in \partial \mathcal{H}^n \), we have

\[
F_p^{***}|_{Q_0} \leq 2.
\]

Here \( Q_0 = \{w = 0\} \).
CHAPTER 2. CONSTRUCTION AND CLASSIFICATION OF RATIONAL MAPS

By the normalization, for any \( F \in \text{Prop}_2(\mathbb{H}_n, \mathbb{H}_{2n-1}) \), we knew that \( F^{**} = (f, \phi, g) \) satisfies

\[
F^{**}(0, w) = (0, w),
\]
\[
f_1 = z_1 + \frac{i}{2} z_1 w + z_1 a^{(1)}(z) w + o_w(4),
\]
\[
f_l = z_l + o_w(4), \quad 2 \leq l \leq n - 1,
\]
\[
\phi_j = z_1 z_j + b_j z_1 w + b_j^{(3)}(z) + o_w(3), \quad 1 \leq j \leq n - 1,
\]
\[
g = w + o(|z, w|^3).
\]

(2.33)

\[
g(z, w) - g(\zeta, \eta) = \sum_{l=1}^{n-1} f_l(z, w) f_l(\zeta, \eta) + \sum_{l=1}^{n-1} \phi_l(z, w) \phi_l(\zeta, \eta).
\]

(2.34)

Applying \( \mathcal{L}_j \) and \( \mathcal{L}_1 \mathcal{L}_j \) to the above equation, using (2.33) and letting \((z, w) = (0, 0)\), we get

\[
\begin{pmatrix}
\overline{\zeta}_1 \\
\vdots \\
\overline{\zeta}_{n-1} \\
0
\end{pmatrix} = \begin{pmatrix}
I_{(n-1)\times(n-1)} & 0 \\
A_{(n-1)\times(n-1)} & B_{(n-1)\times(n-1)}
\end{pmatrix} \begin{pmatrix}
\overline{f}(\zeta, 0) \\
\overline{\phi}(\zeta, 0)
\end{pmatrix}.
\]

Here \( I_{(n-1)\times(n-1)} \) is the identical \( (n - 1) \times (n - 1) \) matrix, \( A_{(n-1)\times(n-1)} = A = \begin{pmatrix}
-2\overline{\zeta}_1 & 0 & \ldots & 0 \\
-\overline{\zeta}_2 & 0 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots \\
-\overline{\zeta}_{n-1} & \ldots & 0 & 0
\end{pmatrix} \) and \( B_{(n-1)\times(n-1)} = B = \begin{pmatrix}
2 + 4ib_1\overline{\zeta}_1 & 4ib_2\overline{\zeta}_1 & \ldots & 4ib_{n-1}\overline{\zeta}_1 \\
2ib_1\overline{\zeta}_2 & 1 + 2ib_2\overline{\zeta}_2 & \ldots & 2ib_{n-1}\overline{\zeta}_2 \\
\vdots & \ddots & \ddots & \ddots \\
2ib_1\overline{\zeta}_{n-1} & 2ib_2\overline{\zeta}_{n-1} & \ldots & 1 + 2ib_{n-1}\overline{\zeta}_{n-1}
\end{pmatrix} \).

This implies

\[
\overline{f}(z, 0) = \begin{pmatrix}
z_1 z \\
1 - 2i \sum_{j=1}^{n-1} b_j \overline{z}_j
\end{pmatrix}.
\]

(2.35)

Finally, by putting \( z = w = \eta = 0 \), we get \( g(\zeta, 0) = 0 \) by (2.33). Hence, it is clear that \( F(z, 0) \) can be written as the quotient of a vector-valued quadratic polynomial with a linear function. Hence (2.32) is proved. \( \square \)

By similar method, the following results are proved.
2.11. DEGREE OF RATIONAL MAPS BETWEEN BALLS

Theorem 2.11.4 (1) [JX04] Let $F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ with geometric rank $\kappa_0$, $1 \leq \kappa_0 \leq n-2$, and with $N = n + \frac{(2n-\kappa_0-1)\kappa_0}{2}$. Then $\deg(F) \leq \kappa_0 + 2$.

(2) [HJX05] Let $F \in \text{Rat}(\mathbb{B}^3, \mathbb{B}^6)$ with geometric rank $\kappa_0(F) = 2$. Then $\deg(F) \leq 4$. 
Chapter 3

More Analytic and Geometric Approaches

3.1 A Model Case for Further Generalization

Theorem 3.1.1 [HJ01] Let \( F \in \text{Prop}_2(\mathbb{H}^n, \mathbb{H}^{2n^1}) \). Then \( F \) is equivalent to a map that is either linear, or Whitney map: \( W_{n,1}(z, w) = (z, wz) \) where \((z, w) \in \mathbb{C}^{n^1} \times \mathbb{C}\).

Here is the main ingredient of the proof:

1. \( F^{**} \) can be further normalized into \( F^{***} = (f, \phi, g) \):

   \[
   \begin{align*}
   f_1 &= z_1 + \frac{i}{2} z_1 w + o_{wt}(3), \\
   f_j &= z_j + o_{wt}(3), \quad 2 \leq j \leq n1, \\
   \phi_j &= z_1 z_j + o_{wt}(2), \quad 2 \leq j \leq n1, \\
   g &= w + o_{wt}(4),
   \end{align*}
   \]

2. Show: The geometric rank \( \kappa_0 = 1 \).

3. Furthermore,

   \[
   \begin{align*}
   f_1 &= z_1 + \frac{i}{2} z_1 w + o_{wt}(3), \\
   f_j &= z_j, \quad 2 \leq j \leq n1, \\
   \phi_j &= z_1 z_j + o_{wt}(2), \quad 2 \leq j \leq n1, \\
   g &= w,
   \end{align*}
   \]
4. $F$ is equivalent to a map that satisfies

$$F = (z_1\tilde{f}_1, z_2, ..., z_{n_1}, z_1\tilde{\phi}_1, ..., z_1\tilde{\phi}_{n_1}, w).$$

Here $\Phi = (\tilde{f}_1, \tilde{\phi}_1, ..., \tilde{\phi}_{n_1})$ defines a biholomorphic map from $\mathbb{H}^n$ and $\mathbb{B}^n$.

5. In particular, the restriction $F|_{\{z_1=0\}}$ is linear fractional.

### 3.2 Generalization of Five Facts

The above five facts are generalized into the following results:

1. **Theorem 3.2.1** ([H03]) Let $F \in \text{Prop}_2(\mathbb{H}^n, \mathbb{H}^N)$. Then $F$ is equivalent to a map $F^{***} = (f^{***}, \phi^{***}, g^{***})$ of the following form:

$$
\begin{align*}
\begin{cases}
 f^{***} &= \sum_{l=1}^{\kappa_0} z_j f^*_j(z, w) = \delta^*_l + \frac{i\delta^*_l \mu_l}{2} w + O((|z, w|^2)), \ l \leq \kappa_0; \\
 f^*_j &= z_j + o_{wt}(3), \ \kappa_0 + 1 \leq j \leq n - 1; \\
 \phi^{***}_{lk,p} &= \mu_{lk} z_l z_k + o_{wt}(2), \ \forall (l, k) \in S; \\
 g &= w + o_{wt}(4),
\end{cases}
\end{align*}
$$

where

$$S_0 = \{(j, l) : 1 \leq j \leq \kappa_0, j \leq l, 1 \leq l \leq n - 1\}$$

is the index set for those $\phi^{***}_{lk,p}$ that have non-zero coefficients of the $z_l z_k$ terms,

$$S := S_0 \cup \left\{(j, l) \mid j = \kappa_0 + 1, \kappa_0 + 1 \leq l \leq N - n - \frac{(2n - \kappa_0 - 1)\kappa_0}{2}\right\}$$

is the index set for all $\phi^{***}_{lk,p}$, and

$$
\mu_{jl} = \begin{cases}
\sqrt{\mu_j + \mu_l}, & \text{for } j, l \leq \kappa_0, j \neq l, \\
\sqrt{\mu_j}, & \text{if } j \leq \kappa_0 \text{ and } l > \kappa_0 \text{ or if } j = l \leq \kappa_0.
\end{cases}
$$

2. Due to the existence of the non-zero $z_l z_k$ terms of $\phi^{***}_{lk,p}$ above, which “occupy the room” in $\partial \mathbb{B}^N$, as application of Theorem 3.2.1, we immediately obtain the following result, which generalizes the second fact of the above five ones.
Corollary 3.2.2 [H03] Let $F \in \text{Prop}_2(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$ with geometric rank $\kappa_0$. Then

$$N \geq n + \frac{\kappa_0(2n - \kappa_0 - 1)}{2}.$$ 

This inequality is sharp.

[Example] If $F \in \text{Prop}_2(\mathbb{B}^n, \mathbb{B}^{2n+1})$ with $n \geq 3$, then $\kappa_0 \leq 1$. In fact, this follows from the inequality $2n - 1 \geq n + \frac{\kappa_0(2n - \kappa_0 - 1)}{2}$. □

3. Theorem 3.2.3 ([H03] and [HJX06], theorem 3.1) Let $F \in \text{Prop}_3(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$ with geometric rank $\kappa_0 \leq n - 2$. Then $F$ is equivalent to a map $F^* = (f^*, \phi^*, g^*)$ of the following form:

\[
\begin{aligned}
&f_{l,p}^* = \sum_{j=1}^{\kappa_0} z_j f_{l}^j(z, w), \quad f_{l}^j(z, w) = \delta_l^j + \frac{i\delta_l^j}{2} w + O(|(z, w)|^2), \ l \leq \kappa_0; \\
f_{j,p}^* = z_j, \quad \kappa_0 + 1 \leq j \leq n - 1; \\
\phi_{l,k,p}^* = \mu_{lk} z_l z_k + \sum_{j=1}^{\kappa_0} z_j \phi_{l,k,j,p}^*, \quad \phi_{l,k,j,p}^*(z, w) = a_{\omega t}(2), \ \text{for} \ (l, k) \in S_0; \\
\phi_{l,j,p}^* = \sum_{j=1}^{\kappa_0} z_j \phi_{l,j,p}^* = O(|(z, w)|^3) \ \text{for} \ (l, k) \in S - S_0; \\
g = w;
\end{aligned}
\]

4. Theorem 3.2.4 [HJX06] Let $F \in \text{Prop}_3(\mathbb{B}^n, \mathbb{B}^N)$ with $3 \leq n \leq N$ and geometric rank $\kappa_0 \leq n - 2$. Then $F$ is equivalent to a proper holomorphic map of the form

$$H = (z_1, ..., z_{n - \kappa_0}, H_1, ..., H_{N - n + \kappa_0}),$$

where $H_j = \sum_{l=n - \kappa_0+1}^{n} z_l H_{j,l}$ with $H_{j,l}$ holomorphic over $\mathbb{H}^n$.

5. Theorem 3.2.5 Let $F \in \text{Prop}_3(\mathbb{H}^{n+1}, \mathbb{H}^{N+1})$ with geometric rank $\kappa_0 \leq n - 2$. The $\forall p \in \mathbb{H}^n$, $\exists$ affine $(n - \kappa_0)$-dimensional complex subspace $S^a_p$ containing $p$ such that

$$F|_{S^a_p}$$

is linear fractional.
3.3 How to Construct $F^{**}$?

Recall for any $F \in \text{Prop}_2(\mathbb{H}_n, \mathbb{H}_N)$, $F$ is equivalent to $F^{**} = (f^{**}, \phi^{**}, g^{**})$ such that

$$f^{**} = z + \frac{i}{2} a^{**}(1)(z)w + o_{wt}(3), \quad \phi^{**} = \phi^{**}(2)(z) + o_{wt}(2), \quad g^{**} = w + o_{wt}(4), \quad (3.2)$$

$$\langle (\tau, a^{**}(1))|z^2 = |\phi^{**}(2)(z)|^2. \quad (3.2)$$

We can further normalize this map to get more properties while it preserves the above properties of $F^{**}$.

How to define $F^{**}$ in Theorem 3.2.1 from the map $F^{**}$ preserving the property (3.2)? Consider $\sigma \in \text{Aut}_0(\mathbb{H}_n)$ and $\tau \in \text{Aut}_0(\mathbb{H}_N)$:

$$\sigma = \frac{(\lambda(z + aw) \cdot U, \lambda^2 w)}{q(z, w)}, \quad (3.3)$$

where $q(z, w) = 1 - 2i\langle \bar{a}, z \rangle + (r - i|a|^2)w, \lambda > 0, r \in \mathbb{R}, a$ is an $(n - 1)$-tuple and $U$ is an $(n - 1) \times (n - 1)$ unitary matrix. Let

$$\tau^*(z^*, w^*) = \frac{(\lambda^*(z^* + a^* w^*) \cdot U^*, \lambda^{*2} w^*)}{q^*(z^*, w^*)} \quad (3.4)$$

where $q^*(z^*, w^*) = 1 - 2i\langle \bar{a}, z \rangle + (r^* - i|a|^2)w^*, \lambda^* > 0, r^* \in \mathbb{R}, a^*$ is an $(N - 1)$-tuple and $U^*$ is an $(N - 1) \times (N - 1)$ unitary matrix.

**Theorem 3.3.1** [H03] (A) Let $F = (f, \phi, g)$ and $F^* = (f^*, \phi^*, g^*)$ be $C^2$-smooth CR map from a neighborhood of 0 in $\mathbb{H}_n$ into $\mathbb{H}_N$ $(N \geq n > 1)$, satisfies the condition (3.2). Suppose that $F^* = \tau^* \circ F \circ \sigma$ where $\sigma$ and $\tau^*$ are as in (3.4) and (3.4). Then it holds that

$$\lambda^* = \lambda^{-1}, \quad a^*_1 = -\lambda^{-1}a \cdot U, \quad a^*_2 = 0, \quad r^* = -\lambda^{-2}r, \quad U^* = \begin{pmatrix} U^{-1} & 0 \\ 0 & U_{22}^* \end{pmatrix} \quad (3.5)$$

where $a^* = (a^*_1, a^*_2)$ with $a^*_1$ its first $(n - 1)$ components, $U_{22}^*$ is an $(N - n) \times (N - n)$ unitary matrix. Conversely, suppose $\tau^*$ and $\sigma$, given as above, are related by (3.5). Suppose that $F$ satisfies the condition (3.2). Then $F^* := \tau^* \circ F \circ \sigma$ also satisfies the (3.2).

(B) Let $F$ and $F^* := \tau^* \circ F \circ \sigma$ both satisfy the condition (3.2). Let us denote

$$f(z, w) = z + \frac{i}{2} zA w + \frac{1}{2} \frac{\partial^2 f}{\partial w^2}|_0 w^2 + o(|z, w|^2),$$

$$f^*(z, w) = z + \frac{i}{2} zA^* w + \frac{1}{2} \frac{\partial^2 f^*}{\partial w^2}|_0 w^2 + o(|z, w|^2).$$
3.3. HOW TO CONSTRUCT $F^{**}$?

and

$$\phi(z, w) = \frac{1}{2} z(B^1, \ldots, B^{N-n}) z^t + zBw + \frac{1}{2} \frac{\partial^2 \phi}{\partial w^2} |w|^2 + o((|z, w|^2)),$$

$$\phi^*(z, w) = \frac{1}{2} z(B^*1, \ldots, B^{*N-n}) z^t + zB^*w + \frac{1}{2} \frac{\partial^2 \phi^*}{\partial w^2} |w|^2 + o((|z, w|^2)),$$

where

$$A = -2i \begin{pmatrix} \frac{\partial^2 f_1}{\partial z_1 \partial w} & \cdots & \frac{\partial^2 f_n}{\partial z_1 \partial w} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f_1}{\partial z_{n-1} \partial w} & \cdots & \frac{\partial^2 f_n}{\partial z_{n-1} \partial w} \end{pmatrix} |_{0}$$

is the $(n-1) \times (n-1)$ matrix,

$$B^k = \begin{pmatrix} \frac{\partial^2 \phi_k}{\partial z_1^2} & \cdots & \frac{\partial^2 \phi_k}{\partial z_{n-1} \partial z_n} \\ \frac{\partial^2 \phi_{k+1}}{\partial z_1 \partial z_n} & \cdots & \frac{\partial^2 \phi_{k+1}}{\partial z_{n-1} \partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \phi_{n-1}}{\partial z_1 \partial z_n} & \cdots & \frac{\partial^2 \phi_{n-1}}{\partial z_{n-1} \partial z_n} \end{pmatrix} |_{0}, \quad 1 \leq k \leq N-n,$$

are $(n-1) \times (n-1)$ matrices, and

$$B = \begin{pmatrix} \frac{\partial^2 \phi_1}{\partial z_1 \partial w} & \cdots & \frac{\partial^2 \phi_{(N-n)}}{\partial z_1 \partial w} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 \phi_{n-1}}{\partial z_{n-1} \partial w} & \cdots & \frac{\partial^2 \phi_{(N-n)}}{\partial z_{n-1} \partial w} \end{pmatrix} |_{0}$$

is an $(n-1) \times (N-n)$ matrix. $A^*, B^*, B^*$ are defined similarly. Then

$$A^* = \lambda^2 UAU^{-1},$$

$$\frac{\partial^2 f}{\partial w^2}(0) = i\lambda^2 aUAU^{-1} + \lambda^2 \frac{\partial^2 f}{\partial w^2}(0)U^{-1}.$$  

$$z(B^1, \ldots, B^{N-n}) z^t = \lambda z(B^1, \ldots, B^{N-n}) U^t z U^t_{22},$$

$$B^* = \lambda U(B^1, \ldots, B^{N-n}) U^t a U^t_{22} + \lambda^2 UBU^t_{22},$$

$$\frac{1}{2} \frac{\partial^2 \phi^*}{\partial w^2} |0 = \frac{1}{2} \lambda a U(B^1, \ldots, B^{N-n}) U^t a U^t_{22} + \lambda^2 a UBU^t_{22} + \frac{1}{2} \lambda^3 \frac{\partial^2 \phi}{\partial w^2} |0 U^t_{22},$$

(C) Let $F_1$ be a non-constant $C^2$ CR map from $M \subset \partial \mathbb{H}_n$ into $\partial \mathbb{H}_N$. Assume that $F_2 = \tau \circ F_1 \circ \sigma$ with $\sigma \in \text{Aut}(\mathbb{H}_n)$ and $\tau \in \text{Aut}(\mathbb{H}_N)$. Then

$$Rk_{F_2}(p) = Rk_{F_1(\sigma(p))}.$$

The normalization $F^{***}$ in Theorem 3.2.1 is constructed by $\tau^* \circ F \circ \sigma$ for appropriate choice of $\tau^*$ and $\sigma$. 
3.4 Where is the Condition $\kappa_0 \leq n - 2$ used?

In Theorem 3.2.3 above, a very crucial condition is $\kappa_0 \leq n - 2$. This condition indeed produces exact equations for the map $F$. In fact, by the normalization $F^{**}$, we have the curvature information:

$$\langle \overline{z}, a^{**}_{p}(1)(z) \rangle |z|^2 = |\phi^{**}_{p}(2)(z)|^2.$$

(3.7)

Write $a^{**}_{p}(1)(z) = zA_p$ where

$$A_p = -2i \left( \frac{\partial^2 f_{i,p}^{**}}{\partial z_j \partial w} \bigg|_0 \right)$$

is an $(n-1) \times (n-1)$ Hermitian matrix.

Remarks

• The matrix $A_p$ is semi-positive because of (3.7).

• (3.7) can be written as

$$zA_p \overline{z} |z|^2 = |\phi^{**}_{p}(2)(z)|^2.$$

Then for a non zero vector $z$, we have

$$|\phi^{**}_{p}(2)(z)|^2 = 0 \iff zA_p \overline{z} = 0$$

$$\iff zA_p = 0 \quad \text{(because } A_p \geq 0)$$

$$\iff \phi^{**}_{p}(2)(z) = 0$$

• We define a vector space $E_p := \{ \xi(p) \in \mathbb{C}^{n-1} | \xi(p) \cdot A_p = 0 \} \neq \emptyset$. Then

$$\xi(p) \in E_p \iff \phi^{**}_{p}(2)(\xi(p)) = 0$$

From these equations, it derives more equations by taking differentiation that make Theorem 3.2.3 possible.

3.5 Structure Theorem For Rank 1 Maps

As an application of Theorem 3.2.3, we have the following structure theorem on maps with geometric rank one. The key condition here is $\kappa_0 \leq n - 2$, which allows the maps have more rigidity property.
3.6. PROOF OF THE SECOND GAP THEOREM

Theorem 3.5.1 ([HJX06], theorem 1.2) Let \( F \in \text{Prop}_3(\mathbb{B}^n, \mathbb{B}^N) \) with \( 3 \leq n \leq N \) and geometric rank 1. Then \( F \) is equivalent to a proper holomorphic map of the form

\[
H := (z_1, \cdots, z_{n-1}, H_1, \cdots, H_{N-n+1}),
\]

where \((H_1, \cdots, H_{N-n+1}) = w \cdot h\) with \( h \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^{N-n+1}) \). Both \( H \) and \( h \) are affine linear maps along each hyperplane defined by \( w = \text{constant} \).

In fact, from Theorem 3.2.3, when \( \kappa_0 = 1 \), we have

\[
\begin{cases}
  f_{1,p}^{***} = z_1 f_1^*(z, w), & f_1^*(z, w) = 1 + \frac{i \mu_1}{2} w + O(|(z, w)|^2), \\
  f_{j,p}^{**} = z_j, & 2 \leq j \leq n-1; \\
  \phi_{1k,p}^{**} = \mu_{1k} z_k + z_1 \phi_{1k,p}^*, & \phi_{1k,p}^*(z, w) = o_{\text{wt}}(2), & \text{for } 1 \leq k \leq n-1; \\
  \phi_{2\ell,p}^{**} = \varepsilon_\ell \phi_{2\ell,p}^*, & O(|(z, w)|^3) & \text{for } 2 \leq \ell \leq N-2n+1; \\
  g = w.
\end{cases}
\]

By Cayley’s transformation to obtain a new map \( H : \mathbb{B}^n \to \mathbb{B}^N \):

\[
H = (H_1, z_2, \cdots, z_{n-1}, H_n, \cdots, H_{N-n}, w).
\]

We can make change on variables in the following way:

\[
\begin{align*}
  z_1 & \leftrightarrow z_n \\
  \{z_2, \cdots, z_{n-1}\} & \leftrightarrow \{z_1, \cdots, z_{n-2}\} \\
  w & \leftrightarrow z_{n-1}
\end{align*}
\]

so that

\[
H = (z_1, \cdots, z_{n-1}, H_1, H_2, \cdots, H_{N-n+1}).
\]

As an application, Theorem 3.5.1 is used in the proof of Theorem 2.11.1

3.6 Proof of the Second Gap Theorem

The second gap theorem can be restated as

Theorem 3.6.1 [HJX06] Let \( F \in \text{Prop}_3(\mathbb{B}^n, \mathbb{B}^N) \) with \( 4 \leq n \leq N \leq 3n - 4 \). Then \( F \) is equivalent to \((F_{\theta}, 0)\) where

\[
F_{\theta} = (z, w\cos \theta, z_1 w\sin \theta, \cdots, z_{n-1} w\sin \theta, w^2 \sin \theta)
\]

for some \( \theta \in [0, \frac{\pi}{2}] \).
CHAPTER 3. MORE ANALYTIC AND GEOMETRIC APPROACHES

- In 2005, Hamada proved that any $F \in Prop_3(B^n, B^{2n})$ is equivalent to $F_\theta$ for some $\theta \in [0, \frac{\pi}{2}]$.

- By the inequality $N \geq n + \frac{\kappa_0(2n-\kappa_0-1)}{2}$, under the condition $N \leq 3n - 4$, it implies that the geometric rank $\kappa_0$ of $F$ is $\leq 2$.

- Applying the structure theorem 3.5.1 for rank 1 maps, we can write
  \[ H := (z_1, \ldots, z_{n-1}, H_1, \ldots, H_{N-n+1}) \]
  where $(H_1, \ldots, H_{N-n+1}) = w \cdot h$ with $h \in Rat(B^n, B^{N-n+1})$. Here
  \[ N - n + 1 \leq 3n - 4 - n + 1 = 2n - 3. \]

Then we can apply the first gap theorem to implies $h$ is linear map.

### 3.7 Rationality Problem

In 1989, Forstnerič proved [Fo89] that if $F \in Prop_{N-n+1}(B^n, B^N)$, then $F$ must be a rational map with degree $\deg(F) \leq N^2(N - n + 1)$.

**Theorem 3.7.1 ([HJX05], Corollary 1.3)** If $F \in Prop_3(B^n, B^N)$ with either $\kappa_0 < n - 1$ or $N \leq \frac{n(n+1)}{2}$, then $F$ must be rational.

- In order to prove that $F$ is rational, by a theorem of Frostnerič, it suffices to prove that $F$ is smooth on $\partial \mathbb{H}_n$.

- Under the hypothesis, $F$ has partial $k$-linear property: for any point $Z \in B^n - E$ where $E$ is an affine subvariety, there is a unique $k$ dimensional complex subspace $S_Z$ on which $F$ is linear fractional.

- Assume that $0 \in B^n - E$ and $S_0 = \{ z \mid z_{k+1} = \ldots = z_n = 0 \}$.

- Construct a holomorphic map $\Psi$ from a neighborhood of a rectangle $(-1 - \epsilon, 1 + \epsilon) \times (-\epsilon, \epsilon)$ in $\mathbb{C}^k \times \mathbb{C}^{n-k}$ to a neighborhood of $(-1 - \epsilon, 1 + \epsilon) \times \{ 0 \}$ in $\mathbb{C}^k \times \mathbb{C}^{n-k}$ such that
3.7. RATIONALITY PROBLEM

- $\Psi|_{S_0} \equiv Id$. ($\implies$ $\Psi$ is locally biholomorphic when $\epsilon$ is small)
- For each line segment $L_{(t,\tau)}$ that (i) passes through the point $(t,\tau)$ and (ii) $L_{(t,\tau)}$ and $S_0$ are parallel, we have

\[ \Psi(L_{(t,\tau)}) \subset S_{(0,\tau)}. \]
For each fixed $\tau$, since 
\[ F|_{S(0, \tau)} = \text{linear fractional}, \]
we have
\[ F \circ \Psi(t, \tau) = \frac{F(\tau) + \sum_{j=1}^{k} A_j(\tau)t_j}{1 + \sum_{j=1}^{k} b_j(\tau)t_j}. \]

On the other hand, we take a power series at the origin:
\[ F \circ \Psi(t, \tau) = \sum_{\alpha} C_\alpha(\tau)t^\alpha \text{ is holomorphic near } (0, 0). \]

$C_\alpha(\tau)$ is holomorphic $\implies A_j(\tau), b_j(\tau)$ and $F(\tau)$ are holomorphic of $\tau$ near 0.

- $F \circ \Psi(t, \tau)$ is holomorphic of $(t, \tau)$ whenever $\tau \sim 0$ and for any $t$.
- By the construction, $F \circ \Psi(t, \tau)$ is holomorphic is holomorphic of $(t, \tau)$ whenever $(t, \tau)$ in the rectangle $(-1 - \epsilon, 1 + \epsilon) \times (\epsilon, \epsilon)$.
- Choose $Z_0$ in the rectangle such that $F(Z_0) \in \partial B^n$. Then 
\[ F = (F \circ \Psi) \circ (\Psi^{-1}) \]
is holomorphic near $F(Z_0)$.
- $F$ is $C^\infty$ near $F(Z_0)$, so is on $\partial B^n$.
- By Forstnerič Theorem, $F$ is rational.

### 3.8 Flatness of CR Submanifolds

In Euclidean geometry, for a real submanifold $M^n \subset \mathbb{E}^{n+a}$, $M$ is a piece of $\mathbb{E}^n$ if and only if its second fundamental form $II_M \equiv 0$.

In projective geometry, for a complex submanifold $M^n \subset \mathbb{CP}^{n+a}$, $M$ is a piece of $\mathbb{CP}^n$ if and only if its projective second fundamental form $II_M \equiv 0$ (c.f. [IL03], p.81).

In CR geometry, we prove the CR analogue of this fact in this paper as follows:
3.9 Definition A, The CR Second Fundamental Form

Theorem 3.8.1 (Ji-Yuan [JY09]) Let $H : M' \to \partial B^{n+1}$ be a smooth CR-embedding of a strictly pseudoconvex CR real hypersurface $M' \subset \mathbb{C}^{n+1}$. Denote $M := H(M')$. If its CR second fundamental form $II_M \equiv 0$, then $M \subset F(\partial B^{n+1}) \subset \partial B^{n+1}$ where $F : \mathbb{B}^{n+1} \to \mathbb{B}^{n+1}$ is a certain linear fractional proper holomorphic map.

It was proved by P. Ebenfelt, X. Huang and D. Zaitsev ([EHZ04], corollary 5.5), under the above same hypothesis, that $M'$ and hence $M$ are locally CR-equivalent to the unit sphere $\partial \mathbb{B}^{n+1}$ in $\mathbb{C}^{n+1}$. This result allows us to consider

$$G = SU(N + 1, 1)$$

$$F : \partial \mathbb{H}^{n+1} \to M = F(\partial \mathbb{H}^{n+1}) \subset \partial \mathbb{B}^{n+1} = G/H$$

There are several definitions of the CR second fundamental forms $II_M$ of $M$. We have to prove that the above theorem is true for all of these definitions.

- Definition A, intrinsic one (Webster).
- Definition B, extrinsic one (cf. Ebenfelt-Huang-Zaitsev(2004)).
- Definition C, Cartan moving frame theory, with the group $G = GL^Q(CN + 2)$.
- Definition D, Cartan moving frame theory, with the group $G = SU(N + 1, 1)$.

3.9 Definition A, the CR Second Fundamental Form

Let $(M, \theta)$ be a strictly pseudoconvex pseudohermitian manifold where $\theta$ is a contact form. Associated with a contact form $\theta$ one has the Reeb vector field $R_\theta$, defined by the equations:

(i) $d\theta(R_\theta, \cdot) \equiv 0$, (ii) $\theta(R_\theta) \equiv 1$.

If there are $n$ complex 1-forms $\theta^\alpha$ so that $\{\theta^1, ..., \theta^n\}$ forms a local basis for holomorphic cotangent bundle and

$$d\theta = i \sum_{\alpha, \beta=1}^n h_{\alpha\beta} \theta^\alpha \wedge \overline{\theta^\beta}$$ (3.8)

where $(h_{\alpha\beta})$, called the Levi form matrix, is positive definite. Such $\theta^\alpha$ may not be unique. Following Webster (1978), a coframe $(\theta, \theta^\alpha)$ is called admissible if (3.8) holds.
Theorem 3.9.1 (Webster, 1978) Let \((M^{2n+1}, \theta)\) be a strictly pseudoconvex pseudohermitian manifold and let \(\theta^j\) be as in (3.8). Then there are unique way to write

\[d\theta^\alpha = \sum_{\gamma=1}^{n} \theta^\gamma \wedge \omega_\gamma^\alpha + \theta \wedge \tau^\alpha,\]

(3.9)

where \(\tau^\alpha\) are \((0,1)\)-forms over \(M\) that are linear combination of \(\theta^\alpha = \theta^\gamma_\alpha\), and \(\omega_\alpha^\beta\) are 1-forms over \(M\) such that

\[0 = dh_{\alpha\beta} - h_{\gamma\beta} \omega_\gamma^\alpha - h_{\alpha\gamma} \omega_\gamma^\beta,\]

(3.10)

We may denote \(\omega_{\alpha\beta} = h_{\gamma\beta} \omega_\gamma^\alpha\) and \(\omega_{\beta\alpha} = h_{\alpha\gamma} \omega_\gamma^\beta\). In particular, if \(h_{\alpha\beta} = \delta_{\alpha\beta}\), the identity in (3.10) becomes

\[0 = -\omega_{\alpha\beta} - \omega_{\beta\alpha},\] i.e.,

\[0 = \omega_\alpha^\beta + \omega_\beta^\alpha,\]

(3.12)

Lemma 3.9.2 ([EHZ04], corollary 4.2) Let \(M\) and \(\tilde{M}\) be strictly pseudoconvex CR manifolds of dimensions \(2n + 1\) and \(2\tilde{n} + 1\) respectively, and of CR dimensions \(n\) and \(\tilde{n}\) respectively. Let \(F : M \rightarrow \tilde{M}\) be a smooth CR-embedding. If \((\theta, \theta^\alpha)\) is a admissible coframe on \(M\), then in a neighborhood of a point \(\tilde{p} \in F(M)\) in \(\tilde{M}\) there exists an admissible coframe \((\tilde{\theta}, \tilde{\theta}^A) = (\tilde{\theta}, \tilde{\theta}^\alpha, \tilde{\theta}^\mu)\) on \(\tilde{M}\) with \(F^*(\tilde{\theta}, \tilde{\theta}^\alpha, \tilde{\theta}^\mu) = (\theta, \theta^\alpha, 0)\). In particular, the Reeb vector field \(\tilde{R}\) is tangent to \(F(M)\). If we choose the Levi form matrix of \(M\) such that the functions \(h_{\alpha\beta}\) in (3.8) with respect to \((\theta, \theta^\alpha)\) to be \((\delta_{\alpha\beta})\), then \((\tilde{\theta}, \tilde{\theta}^A)\) can be chosen such that the Levi form matrix of \(\tilde{M}\) relative to it is also \((\delta_{AB})\). With this additional property, the coframe \((\tilde{\theta}, \tilde{\theta}^A)\) is uniquely determined along \(M\) up to unitary transformations in \(U(n) \times U(\tilde{n} - n)\).

If \((\theta, \theta^\alpha)\) and \((\tilde{\theta}, \tilde{\theta}^A)\) are as above such that the condition on the Levi form matrices in Lemma 3.9.2 are satisfied, we say that the coframe \((\tilde{\theta}, \tilde{\theta}^A)\) is adapted to the coframe \((\theta, \theta^\alpha)\). In this case, by (3.12), we have \(\theta = F^*\tilde{\theta}, \theta^\alpha = F^*\tilde{\theta}^\alpha\), and

\[d\theta^\alpha = \sum_{\gamma=1}^{n} \theta^\gamma \wedge \omega_\gamma^\alpha + \theta \wedge \tau^\alpha, \quad 0 = \omega_\alpha^\beta + \omega_\beta^\alpha, \quad \forall 1 \leq \alpha, \beta \leq n,\]

and

\[d\tilde{\theta}^A = \sum_{B=1}^{\tilde{n}} \tilde{\theta}^C \wedge \tilde{\omega}_C^A + \tilde{\theta} \wedge \tilde{\tau}^A, \quad 0 = \tilde{\omega}_A^B + \tilde{\omega}_B^A, \quad \forall 1 \leq A, B \leq N.\]
3.10. Definition B, the CR Second Fundamental Form

For simplicity, we may denote $F^*\tilde{\omega}^B_A$ by $\omega^A_B$. We also denote $F^*\tilde{\omega}^\bar{A}\bar{B}$ by $\omega^\bar{A}\bar{B}$ where $\omega^\bar{A}\bar{B} = \omega^A_B$.

Write $\omega^\mu_{\alpha \beta} = \omega^\mu_{\alpha \beta \gamma}$. The matrix of $(\omega^\mu_{\alpha \beta})$, $1 \leq \alpha, \beta \leq n$, $n+1 \leq \mu \leq \hat{n}$, defines the CR second fundamental form of $M$. It was used in [W79] and [Fa90].

3.10 Definition B, the CR Second Fundamental Form

Let $F : M \to \tilde{M}$ be a smooth CR-embedding between $M \subset \mathbb{C}^{n+1}$ and $\tilde{M} \subset \mathbb{C}^{N+1}$ where $M$ and $\tilde{M}$ are real strictly pseudoconvex hypersurfaces of dimensions $2n+1$ and $2\hat{n}+1$, and CR dimensions $n$ and $\hat{n}$, respectively. Let $p \in M$ and $\tilde{p} = F(p) \in \tilde{M}$ be points. Let $\tilde{\rho}$ be a local defining function for $\tilde{M}$ near the point $\tilde{p}$. Let

$$E_k(p) := \text{span}_\mathbb{C}\{L^J(\tilde{\rho}'Z^\prime \circ F)(p) | J \in (Z^\prime)^n, 0 \leq |J| \leq k\} \subset T^1_\tilde{p}\mathbb{C}^{N+1},$$

where $\tilde{\rho}'Z^\prime := \partial \tilde{\rho}$ is the complex gradient (i.e., represented by vectors in $\mathbb{C}^{N+1}$ in some local coordinate system $Z'$ near $\tilde{p}$). $E_k(p)$ is independent of the choice of local defining function $\tilde{\rho}$, coordinates $Z'$ and the choice of basis of the CR vector fields $L_1, \ldots, L_n$.

The CR second fundamental form $I_H M$ of $M$ is defined by (cf. [EHZ04], §2)

$$II_M(X_p,Y_p) := \pi(XY(\tilde{\rho}'Z^\prime \circ f)(p)) \in T^1_\tilde{p}M/E_1(p) \quad (3.13)$$

where $\tilde{\rho}'Z^\prime = \partial \tilde{\rho}$ is represented by vectors in $\mathbb{C}^{N+1}$ in some local coordinate system $Z'$ near $\tilde{p}$, $X,Y$ are any $(1,0)$ vector fields on $M$ extending given vectors $X_p,Y_p \in T_{1,0}(M)$, and $\pi : T^1_\tilde{p}M \to T^1_\tilde{p}M/E_1(p)$ is the projection map.

3.11 Klein’s Erlanger Program

Let $G$ be a Lie group, and $H \subset G$ a closed Lie subgroup. Let $X := G/H$, the set of left cosets of $H$, is a homogeneous space with the induced differential structure from the quotient map.

Klein’s Erlanger Program We’ll study geometry of submanifolds $M \subset X = G/H$, where two submanifolds $M,M' \subset X = G/H$ is equivalent if there is some $g \in G$ such that $g(M) = M'$.

$$\begin{array}{ccc}
G & \xrightarrow{s} & \pi \\
M & \xrightarrow{\pi} & X = G/H
\end{array}$$
To illustrate Cartan’s moving frame theory, we study real surfaces $S \hookrightarrow \mathbb{E}^3$:

$$G = ASO(3)$$

$$F \rightarrow \pi$$

$$S \hookrightarrow \mathbb{E}^3 = G/H$$

Here

$G = ASO(3)$

$F$ = A first-order adapted lift

$H = SO(3)$

$X = \mathbb{E}^{n+s} = ASO(n+s)/SO(n+s)$.

For high dimensional situation, we consider

$$G = ASO(n + s) = \left\{ M = \begin{pmatrix} 1 & 0 \\ t & B \end{pmatrix}, t \in \mathbb{R}^{n+s}, B \in SO(n + s) \right\}$$

which is the group of Euclidean motions,

$H = SO(n + s)$,

which is the group of rotation and

$$X = \mathbb{E}^{n+s} = ASO(n+s)/SO(n+s).$$

Let $M \subset \mathbb{E}^{n+s}$ be an $n$-dimensional submanifold.

A map

$$s = (x, e_j, e_b) : M \rightarrow G$$

is called a first-order adapted lift if $x \in M$, $\text{span}\{e_j(x)\} = T_xM$ and $e_b(x)$ are normal to $M$.

Consequently,

$$s^*dx \equiv 0 \mod \{x, e_j\}. \quad (3.15)$$
3.12 Definition C, the CR Second Fundamental Form

The Maurer-Cartan form on $ASO(n + s)$ is defined to be $\omega = s^{-1}ds$, which is indeed independent of choice of the first-order adapted lifts. $\omega$ is of the form

$$\omega = \begin{pmatrix} 0 & 0 & 0 \\ \omega^i & \omega^j & \omega^b \\ \omega^a & \omega^a_j & \omega^a_b \end{pmatrix}$$

Then $ds = s\omega$ with (3.16) and (3.14), we have

$$dx = e_j\omega^j + e_a\omega^a.$$  

Then pulling back by $s$, by (3.15), we obtain

$$s^*\omega^a = 0$$  

and by (3.17), we obtain

$$-s^*(\omega^a_j \wedge \omega^j) = 0.$$  

By Cartan’s lemma $^1$, we write

$$s^*\omega^a = h^a_{ij} s^*\omega^j$$  

where $h^a_{ij} = h^a_{ji}$. This defines the second fundamental form of $M$

$$\Pi_M := h^a_{ij} s^*\omega^i s^*\omega^j \otimes e_a \in \Gamma(M, S^2T^*M \otimes NM)$$

where $NM$ denotes the normal bundle of $M$.

3.12 Definition C, the CR Second Fundamental Form

We consider a real hypersurface $Q$ in $C^{N+2}$ defined by the homogeneous equation

$$\langle Z, Z \rangle := \sum_{A} Z^A \overline{Z}^A + \frac{i}{2}(\overline{Z}^a Z^{N+1} - Z^0 \overline{Z}^{N+1}) = 0, \quad (3.19)$$

$^1$Let $v_1, ..., v_k$ be linearly independent elements of a vector space $V$ and let $w_1, ..., w_k$ be elements of $V$ such that $w_1 \wedge v_1 + ... + w_k \wedge v_k = 0$. Then there exist scalars $h_{ij} = h_{ji}$, $1 \leq i, j \leq k$, such that $w_i = \sum_j h_{ij} v_j$. 
where $Z = (Z^0, Z^A, Z^{N+1})^t \in \mathbb{C}^{N+2}$. Let
\begin{align*}
\pi_0 : \mathbb{C}^{N+2} - \{0\} &\to \mathbb{CP}^{N+1}, \quad (z_0, \ldots, z_{N+1}) \mapsto [z_0 : \ldots : z_{N+1}], \quad (3.20)
\end{align*}
be the standard projection. For any point $x \in \mathbb{CP}^{N+1}$, $\pi_0^{-1}(x)$ is a complex line in $\mathbb{C}^{N+2} - \{0\}$. For any point $v \in \mathbb{C}^{N+2} - \{0\}$, $\pi_0(v) \in \mathbb{CP}^{N+1}$ is a point. The image $\pi_0(Q - \{0\})$ is the Heisenberg hypersurface $\partial \mathbb{H}^{N+1} \subset \mathbb{CP}^{N+1}$.

For any element $A \in GL(\mathbb{C}^{N+2})$:
\begin{equation}
A = (a_0, \ldots, a_{N+1}) = \begin{bmatrix}
a_0^{(0)} & a_1^{(0)} & \cdots & a_{N+1}^{(0)} \\
a_0^{(1)} & a_1^{(1)} & \cdots & a_{N+1}^{(1)} \\
\vdots & \vdots & \ddots & \vdots \\
a_0^{(N+1)} & a_1^{(N+1)} & \cdots & a_{N+1}^{(N+1)}
\end{bmatrix} \in GL(\mathbb{C}^{N+2}), \quad (3.21)
\end{equation}
where each $a_j$ is a column vector in $\mathbb{C}^{N+2}$, $0 \leq j \leq N + 1$. This $A$ is associated to an automorphism $A^* \in Aut(\mathbb{CP}^{N+1})$ given by
\begin{equation}
A^*(\left[ z_0 : z_1 : \ldots : z_{N+1} \right]) = \left[ \sum_{j=0}^{N+1} a_j^{(0)} z_j : \sum_{j=0}^{N+1} a_j^{(1)} z_j : \cdots : \sum_{j=0}^{N+1} a_j^{(N+1)} z_j \right]. \quad (3.22)
\end{equation}

When $a_0^{(0)} \neq 0$, in terms of the non-homogeneous coordinates $(w_1, \ldots, w_{N+1})$, $A^*$ is a linear fractional from $\mathbb{C}^{N+1}$ which is holomorphic near $(0, \ldots, 0)$:
\begin{equation}
A^*(w_1, \ldots, w_{N+1}) = \left( \frac{\sum_{j=0}^{N+1} a_j^{(1)} w_j}{\sum_{j=0}^{N+1} a_j^{(0)} w_j}, \ldots, \frac{\sum_{j=0}^{N+1} a_j^{(N+1)} w_j}{\sum_{j=0}^{N+1} a_j^{(0)} w_j} \right), \quad \text{where } w_j = \frac{z_j}{z_0}. \quad (3.23)
\end{equation}

We denote $A \in GL^Q(\mathbb{C}^{N+2})$ if $A$ satisfies $A(Q) \subseteq Q$ where we regard $A$ as a linear transformation of $\mathbb{C}^{N+2}$. If $A \in GL^Q(\mathbb{C}^{N+2})$, we must have $A^*(\partial \mathbb{H}^{N+1}) \subseteq \partial \mathbb{H}^{N+1}$, so that $A^* \in Aut(\partial \mathbb{H}^{N+1})$. Conversely, if $A^* \in Aut(\partial \mathbb{H}^{N+1})$, then $A \in GL^Q(\mathbb{C}^{N+2})$.

We define a bundle map:
\begin{align*}
\pi : \quad GL(\mathbb{C}^{N+2}) &\to \mathbb{CP}^{N+1} \\
A = (a_0, a_1, \ldots, a_{N+1}) &\mapsto \pi_0(a_0).
\end{align*}
Then by (3.22), for any map $A \in GL(\mathbb{C}^{N+2})$, $A \in \pi^{-1}(\pi_0(a_0)) \iff A^*(\left[ 1 : 0 : \ldots : 0 \right]) = \pi_0(a_0)$. In particular, by the restriction, we consider a map
\begin{equation}
\pi : \quad GL^Q(\mathbb{C}^{N+2}) \to \partial \mathbb{H}^{N+1} \\
A = (a_0, a_1, \ldots, a_{N+1}) &\mapsto \pi_0(a_0). \quad (3.24)
\end{equation}
We get $\partial \mathbb{H}^{N+1} \simeq GL^Q(\mathbb{C}^{N+2})/P_1$ where $P_1$ is the isotropy subgroup of $GL^Q(\mathbb{C}^{N+2})$. Then by (3.22), for any map $A \in GL^Q(\mathbb{C}^{n+2})$,

$$A \in \pi^{-1}(\pi_0(a_0)) \iff A^*([1 : 0 : \ldots : 0]) = \pi_0(a_0).$$

(3.25)

**CR submanifolds of $\partial \mathbb{H}^{N+1}$**

Let $H : M' \to \partial \mathbb{H}^{N+1}$ be a CR smooth embedding where $M'$ is a strictly pseudoconvex smooth real hypersurface in $\mathbb{C}^{n+1}$. We denote $M = H(M')$.

Let $R_{M'}$ be the Reeb vector field of $M'$ with respect to a fixed contact form on $M'$. Then the real vector $R_{M'}$ generates a real line bundle over $M'$, denoted by $R_{M'}$. Since we can regard the rank $n$ complex vector bundle $T^{1,0}M'$ as the rank 2$n$ real vector bundle, over the real number field $\mathbb{R}$ we have:

$$TM' = T^cM' \oplus R_{M'} \simeq T^{1,0}M' \oplus R_{M'}.$$  

(3.26)

given by

$$(a_j \frac{\partial}{\partial x_j}, b_j \frac{\partial}{\partial y_j}) + cR_{M'} \mapsto (a_j + ib_j) \frac{\partial}{\partial z_j} + cR_{M'}, \quad \forall a_j, b_j, c \in \mathbb{R}. 
$$

(3.27)

Since $H$ is a CR embedding, we have

$$H_*(T^{1,0}M') = T^{1,0}M \subset T^{1,0}(\partial \mathbb{H}^{N+1}), TM \simeq H_*(T^{1,0}M') \oplus H_*(R_{M'}) \subset T(\partial \mathbb{H}^{N+1}).$$

(3.28)

**Lifts of the CR submanifolds**

Let $M = H(M') \subset \partial \mathbb{H}^{N+1}$ be as above. Consider the commutative diagram

$$GL^Q(\mathbb{C}^{N+2}) \xrightarrow{\pi} \partial \mathbb{H}^{N+1}$$

Any map $e$ satisfying $\pi \circ e = Id$ is called a lift of $M$ to $GL^Q(\mathbb{C}^{N+2})$.

In order to define a more specific lifts, we need to give some relationship between geometry on $\partial \mathbb{H}^{N+1}$ and on $\mathbb{C}^{N+2}$ as follows. For any subset $X \in \partial \mathbb{H}^{N+1}$, we denote $\hat{X} := \pi_0^{-1}(X)$ where $\pi_0 : \mathbb{C}^{N+2} - \{0\} \to \mathbb{CP}^{N+1}$ is the standard projection map (3.20). In particular, for any $x \in M$, $\hat{x}$ is a complex line and for the real submanifold $M^{2n+1}$, the real submanifold $\hat{M}^{2n+3}$ is of dimension $2n + 3$.

For any $x \in M$, we take $\hat{v} \in \hat{x} = \pi_0^{-1}(x) \subset \mathbb{C}^{N+2} - \{0\}$, and we define

$$\hat{T}_x M = T_{\hat{v}} \hat{M}, \quad \hat{T}^{1,0}_x M = T^{1,0}_{\hat{v}} \hat{M}, \quad \hat{R}_{M,\hat{v}} := R_{\hat{M},\hat{v}}$$

where $\hat{R}_{M} = \bigcup_{v \in \hat{M}} R_{\hat{M},v}$. These definitions are independent of choice of $v$. 

A lift \( e = (e_0, e_\alpha, e_\mu, e_{N+1}) \) of \( M \) into \( GL^Q(\mathbb{C}^{N+2}) \), where \( 1 \leq \alpha \leq n \) and \( n+1 \leq \mu \leq N \), is called a first-order adapted lift if it satisfies the conditions:

\[
e_0(x) \in \pi_0^{-1}(x), \quad \text{span}_\mathbb{C}(e_0, e_\alpha)(x) = \hat{T}_x^{1,0}M, \quad \text{span}(e_0, e_\alpha, e_{N+1})(x) = \hat{T}_x^{1,0}M \oplus \hat{R}_{M,x} \tag{3.29}
\]

where

\[
\text{span}(e_0, e_\alpha, e_{N+1})(x) := \{c_0e_0 + c_\alpha e_\alpha + c_{N+1}e_{N+1} \mid c_0, c_\alpha \in \mathbb{C}, \ c_{N+1} \in \mathbb{R}\}. \tag{3.30}
\]

Here we used (3.27) and the fact that the Reeb vector is real. Locally first-order adapted lifts always exist.

We have the restriction bundle \( \mathcal{F}_M^0 := GL^Q(\mathbb{C}^{N+2})|_M \) over \( M \). The subbundle \( \pi : \mathcal{F}_M^1 \to M \) of \( \mathcal{F}_M^0 \) is defined by

\[
\mathcal{F}_M^1 = \{(e_0, e_\mu, e_{N+1}) \in \mathcal{F}_M^0 \mid [e_0] \in M, \ (3.29) \text{ are satisfied}\}.
\]

Local sections of \( \mathcal{F}_M^1 \) are exactly all local first-order adapted lifts of \( M \).

For two first-order adapted lifts \( s = (e_0, e_\mu, e_{N+1}) \) and \( \tilde{s} = (\tilde{e}_0, \tilde{e}_j, \tilde{e}_\mu, \tilde{e}_{N+1}) \), by (3.29), we have

\[
\begin{align*}
\tilde{e}_0 &= g_0^0 e_0, \\
\tilde{e}_j &= g_0^j e_0 + g_\mu^j e_\mu, \\
\tilde{e}_\mu &= g_0^\mu e_0 + g_j^\mu e_j + g_\nu^\mu e_\nu + g_{N+1}^\mu e_{N+1}, \\
\tilde{e}_{N+1} &= g_0^{N+1} e_0 + g_j^{N+1} e_j + g_{N+1}^{N+1} e_{N+1},
\end{align*}
\tag{3.31}
\]

Notice that by (3.27), \( g_{N+1}^{N+1} \) is some real-valued function, while other are complex-valued functions. In other words, \( \tilde{s} = s \cdot g \) where

\[
g = (g_0, g_j, g_\mu, g_{N+1}) = \begin{pmatrix}
g_0^0 & g_0^j & g_0^\mu & g_0^{N+1} \\
g_0^j & g_j^j & g_j^\mu & g_j^{N+1} \\
g_0^\mu & g_\mu^j & g_\mu^\mu & g_\mu^{N+1} \\
g_0^{N+1} & g_j^{N+1} & g_\mu^{N+1} & g_{N+1}^{N+1}
\end{pmatrix}
\tag{3.32}
\]

is a smooth map from \( M \) into \( GL^Q(\mathbb{C}^{N+2}) \). Then the fiber of \( \pi : \mathcal{F}_M^1 \to M \) over a point is isomorphic to the group

\[
G_1 = \left\{ g = \begin{pmatrix}
g_0^0 & g_0^\beta & g_0^\mu & g_0^{N+1} \\
g_0^\beta & g_\beta^\beta & g_\beta^\mu & g_\beta^{N+1} \\
g_0^\mu & g_\mu^\beta & g_\mu^\mu & g_\mu^{N+1} \\
g_0^{N+1} & g_j^{N+1} & g_\mu^{N+1} & g_{N+1}^{N+1}
\end{pmatrix} \in GL^Q(\mathbb{C}^{N+2}) \right\}.
\]

where we use the index ranges $1 \leq \alpha, \beta \leq n$ and $n + 1 \leq \mu, \nu \leq N$.

We pull back the Maurer-Cartan form from $GL^Q(\mathbb{C}^{N+2})$ to $\mathcal{F}_M$ by a first-order adapted lift $e$ of $M$ as

$$
\omega = \begin{pmatrix}
\omega_0^0 & \omega_0^\alpha & \omega_0^\nu & \omega_{N+1}^0 \\
\omega_0^\alpha & \omega_\beta^\alpha & \omega_\nu^\alpha & \omega_{N+1}^\alpha \\
\omega_0^\nu & \omega_\beta^\nu & \omega_\nu^\nu & \omega_{N+1}^\nu \\
\omega_{N+1}^0 & \omega_{N+1}^\alpha & \omega_{N+1}^\nu & \omega_{N+1}^{N+1}
\end{pmatrix}.
$$

Since $\omega = e^{-1}de$, i.e., $e\omega = de$. Then we have

$$
de_0 = e_0\omega_0^0 + e_\alpha\omega_\alpha^0 + e_\nu\omega_\nu^0 + e_{N+1}\omega_{N+1}^0. \quad (3.33)
$$

On the other hand, we claim:

$$
de_0 = e_0\omega_0^0 + e_\alpha\omega_0^\alpha + e_{N+1}\omega_{N+1}^0. \quad (3.34)
$$

In fact, take local coordinates systems $(x_1, ..., x_{2n+1})$ for the real manifold $M$, and $(y_1, y_2, x_1, ..., x_{2n+1})$ for the real manifold $\hat{M}$ where $(y_1, y_2)$ is the coordinates for fibers. By the first condition in (3.29), fixing $x_1, ..., x_{j-1}, x_{j+1}, ..., x_{2n+1}, e_0(..., x_j, ...)$ is a curve into $M$ with parameter $x_j$. Then $\frac{\partial e_0}{\partial x_j} \in T\hat{M}$ is a tangent vector to this curve. Since $\text{span}(e_0, e_\alpha, e_{N+1})(x) = \hat{T}_{x}^{1,0}M \oplus \mathcal{R}_{M,x}$ in (3.29) and $T\hat{M} \cong T_{1,0}M \oplus \mathcal{R}_{M}$, we obtain

$$
\frac{\partial e_0}{\partial x_j} = b_0^j e_0 + b_\alpha^j e_\alpha + b_{N+1}^j e_{N+1}, \quad 1 \leq j \leq 2n + 1 \quad (3.35)
$$

for some functions $b_0^j, b_\alpha^j$ and $b_{N+1}^j$. We also have

$$
\frac{\partial e_0}{\partial y_i} = 0, \quad \text{for } i = 1, 2, \quad (3.36)
$$

because $(y_1, y_2)$ are the coordinates for fibers. From (3.35) and (3.36), we get

$$
de_0 = \frac{\partial e_0}{\partial y_1} dy_1 + \frac{\partial e_0}{\partial y_2} dy_2 + \sum_j \frac{\partial e_0}{\partial x_j} dx_j = \sum_j (b_0^j e_0 + b_\alpha^j e_\alpha + b_{N+1}^j e_{N+1}) dx_j
$$

$$
= (\sum_j b_0^j dx_j)e_0 + (\sum_j b_\alpha^j dx_j)e_\alpha + (\sum_j b_{N+1}^j dx_j)e_{N+1}. \quad (3.37)
$$

Since the 1-forms $\omega_0^0, \omega_\alpha^0, \omega_{N+1}^\alpha$ in (3.33) are unique, from (3.37), it proves Claim (3.34).
By (3.33) and (3.34), we conclude $\omega_0^\mu = 0$, $\forall \mu$. By the Maurer-Cartan equation $d\omega = -\omega \wedge \omega$, one gets $0 = d\omega^\mu = -\omega^\mu_\alpha \omega_0^\alpha - \omega^\mu_{N+1} \omega_0^{N+1}$, i.e., $0 = -\omega^\mu_\alpha \omega_0^\alpha$, $mod(\omega_0^{N+1})$. Then by Cartan’s lemma,

$$\omega^\nu_\beta = q^\nu_\alpha^\beta \omega_0^\alpha \ mod(\omega_0^{N+1}),$$

for some functions $q^\nu_\alpha^\beta = q^\nu_\beta^\alpha$.

**The CR second fundamental form**  
In order to define the CR second fundamental form $II_M = II_M = q^\mu_\beta \omega_0^\alpha \omega_0^\beta \otimes e_\mu$, $mod(\omega_0^{N+1})$, let us define $e_\mu$ as follows.  
For any first-order adapted lift $e = (e_0, e_\alpha, e_\nu, e_{N+1})$ with $\pi_0(e_0) = x$, we have $e_\alpha \in \hat{T}_x^{1,0} M$. Recall $T_{EG}(k, V) \simeq E^* \otimes (V/E)$ where $G(k, V)$ is the Grassmannian of $k$-planes that pass through the origin in a vector space $V$ over $\mathbb{R}$ or $\mathbb{C}$ and $E \in G(k, V)$ ([IL03], p.73). Then $T_x M \simeq (\hat{x})^* \otimes (\hat{T}_x M/\hat{x})$ and hence the vector $e_\alpha$ induces $e_\alpha \in T_x^{1,0} M$ by

$$e_\alpha = e_0 \otimes (e_\alpha \ mod(\omega_0^{N+1})).$$

where we denote by $(e^0, e^\alpha, e^\nu, e^{N+1})$ the dual basis of $(\mathbb{C}^{N+2})^*$. Similarly, we let

$$e_\mu = e_0 \otimes (e_\mu \ mod(\hat{T}_x^{1,0} M)) \in N_x^{1,0} M,$$

where $N^{1,0} M$ is the CR normal bundle of $M$ defined by $N_x^{1,0} M = T_x^{1,0} (\partial \mathbb{H}^{N+1})/T_x^{1,0} M$.

By direct computation, we obtain a tensor

$$II_M = II_M = q^\mu_\beta \omega_0^\alpha \omega_0^\beta \otimes e_\mu \in \Gamma(M, S^2 T_{\pi_0(e_0)}^{1,0} M \otimes N^{1,0}_{\pi_0(e_0)} M) \ mod(\omega_0^{N+1}).$$

(3.39)

The tensor $II_M$ is called the **CR second fundamental form** of $M$.

### 3.13 Definition D, the CR Second Fundamental Form

**Q-frames**  
We consider the real hypersurface $Q$ in $\mathbb{C}^{N+2}$ defined by the homogeneous equation

$$\langle Z, Z \rangle := \sum_A Z^A \overline{Z}^A + \frac{i}{2} (Z^{N+1} \overline{Z}^0 - Z^0 \overline{Z}^{N+1}) = 0,$$

(3.40)

where $Z = (Z^0, Z^A, Z^{N+1})^t \in \mathbb{C}^{N+2}$. This can be extended to the scalar product

$$\langle Z, Z' \rangle := \sum_A Z^A \overline{Z'}^A + \frac{i}{2} (Z^{N+1} \overline{Z'}^0 - Z^0 \overline{Z'}^{N+1}),$$

(3.41)
for any $Z = (Z^0, Z^A, Z^{N+1})^t$, $Z' = (Z'^0, Z'^A, Z'^{N+1})^t \in \mathbb{C}^{N+2}$. This product has the properties: $\langle Z, Z' \rangle$ is linear in $Z$ and anti-linear in $Z'$; $\langle Z, Z' \rangle = \langle Z', Z \rangle$; and $Q$ is defined by $\langle Z, Z \rangle = 0$.

Let $SU(N+1, 1)$ be the group of unimodular linear transformations of $\mathbb{C}^{N+2}$ that leave the form $\langle Z, Z \rangle$ invariant (cf. [CM74]).

By a $Q$-frame is meant an element $E = (E_0, E_A, E_{N+1}) \in GL(\mathbb{C}^{N+2})$ satisfying (cf. [CM74, (1.10)])

\begin{align}
\left\{ \begin{array}{l}
det(E) = 1, \\
\langle E_A, E_B \rangle = \delta_{AB}, \quad \langle E_0, E_{N+1} \rangle = -\langle E_{N+1}, E_0 \rangle = -\frac{i}{2},
\end{array} \right.
\end{align}

while all other products are zero.

There is exactly one transformation of $SU(N+1, 1)$ which maps a given $Q$-frame into another. By fixing one $Q$-frame as reference, the group $SU(N+1, 1)$ can be identified with the space of all $Q$-frames. Then $SU(N+1, 1) \subset GL^Q(\mathbb{C}^{N+1})$ is a subgroup with the composition operation.

We define a bundle map:

$$
\pi : GL(\mathbb{C}^{N+2}) \rightarrow \mathbb{CP}^{N+1},
$$

$$
A = (a_0, a_1, \ldots, a_{N+1}) \mapsto \pi_0(a_0),
$$

By taking restriction, we have the projection

$$
\pi : SU(N+1, 1) \rightarrow \partial \mathbb{H}^{N+1}, \quad (Z_0, Z_A, Z_{N+1}) \mapsto \text{span}(Z_0).
$$

which is called a $Q$-frames bundle. We get $\partial \mathbb{H}^{N+1} \cong SU(N+1, 1)/P_2$ where $P_2$ is the isotropy subgroup of $SU(N+1, 1)$. $SU(N+1, 1)$ acts on $\partial \mathbb{H}^{N+1}$ effectively.

The Maurer-Cartan Form over $SU(N+1, 1)$ Consider $E = (E_0, E_A, E_{N+1}) \in SU(N+1, 1)$ as a local lift. Then the Maurer-Cartan form $\Theta$ on $SU(N+1, 1)$ is defined by $dE = (dE_0, dE_A, dE_{N+1}) = E\Theta$, or $\Theta = E^{-1} \cdot dE$, i.e.,

$$
d(E_0 \ E_A \ E_{N+1}) = (E_0 \ E_B \ E_{N+1}) \left( \begin{array}{ccc}
\Theta_0^0 & \Theta_0^A & \Theta_0^{N+1} \\
\Theta_0^B & \Theta_0^B & \Theta_0^B \\
\Theta_0^{N+1} & \Theta_0^{N+1} & \Theta_0^{N+1}
\end{array} \right),
$$

where $\Theta_0^B$ are 1-forms on $SU(N+1, 1)$. By (3.42) and (3.44), the Maurer-Cartan form $(\Theta)$ satisfies

$$
\Theta_0^0 + \Theta_0^{N+1} = 0, \quad \Theta_0^{N+1} = -\Theta_0^0, \quad \Theta_0^A = \Theta_0^B = 0, \quad \Theta_0^A + \Theta_0^A + \Theta_0^{N+1} = 0,
$$

(3.45)
where $1 \leq A \leq N$. For example, from $\langle E_A, E_B \rangle = \delta_{AB}$, by taking differentiation, we obtain

$$\langle dE_A, E_B \rangle + \langle E_A, dE_B \rangle = 0.$$ 

By (3.44), we have

$$\begin{cases}
    dE_0 = E_0 \Theta_A^0 + E_B \Theta_B^0 + E_{N+1} \Theta_A^{N+1}, \\
    dE_A = E_0 \Theta_A^0 + E_B \Theta_B^A + E_{N+1} \Theta_A^{N+1}, \\
    dE_{N+1} = E_0 \Theta_A^{N+1} + E_B \Theta_B^{N+1} + E_{N+1} \Theta_A^{N+1}.
\end{cases}$$

Then

$$\langle E_0 \Theta_A^0 + E_B \Theta_A^C + E_{N+1} \Theta_A^{N+1}, E_B \rangle + \langle E_A, E_0 \Theta_B^0 + E_D \Theta_B^D + E_{N+1} \Theta_B^{N+1} \rangle = 0,$$

which implies $\Theta_B^A + \overline{\Theta_B^A} = 0$. In particular, from (3.45), $\Theta_A^0 = -2i \Theta_A^{N+1}$. $\Theta$ satisfies

$$d\Theta = -\Theta \wedge \Theta. \quad (3.46)$$

Let $M \hookrightarrow \partial \mathbb{H}^{N+1}$ be the image of $H : M' \to \partial \mathbb{H}^{N+1}$ where $M' \subset \mathbb{C}^{n+1}$ is a CR strictly pseudoconvex smooth hypersurface. Consider the inclusion map $M \hookrightarrow \partial \mathbb{H}^{N+1}$ and a lift $e = (e_0, e_1, ..., e_{N+1}) = (e_0, e_\alpha, e_\nu, e_{N+1})$ of $M$ where $1 \leq \alpha \leq n$ and $n+1 \leq \nu \leq N$.

$$SU(N+1,1)$$

$$\pi_0(e_0(x)) = x, \text{ span}_\mathbb{C}(e_0, e_\alpha)(x) = \hat{T}_x^{1,0} M, \text{ span}(e_0, e_\alpha, e_{N+1})(x) = \hat{T}_x^{1,0} M \oplus \hat{R}_{M,x}. \quad (3.47)$$

Locally first-order adapted lifts always exist. We have the restriction bundle $\mathcal{F}_M^1 := SU(N+1,1)|_M$ over $M$. The subbundle $\pi : \mathcal{F}_M^1 \to M$ of $\mathcal{F}_M^0$ is defined by

$$\mathcal{F}_M^1 = \{(e_0, e_j, e_\mu, e_{N+1}) \in \mathcal{F}_M^0 \mid [e_0] \in M, (3.47) \text{ are satisfied}\}.$$

Local sections of $\mathcal{F}_M^1$ are exactly all local first-order adapted lifts of $M$. The fiber of $\pi : \mathcal{F}_M^1 \to M$ over a point is isomorphic to the group

$$G_1 = \left\{ g = \begin{pmatrix}
    g_0^0 & g_0^1 & g_0^2 & g_0^{N+1} \\
    0 & g_1^0 & g_1^2 & g_1^{N+1} \\
    0 & 0 & g_2^0 & g_2^{N+1} \\
    0 & 0 & 0 & g_{N+1}^{N+1}
\end{pmatrix} \in SU(N+1,1) \right\},$$
where we use the index ranges $1 \leq \alpha, \beta \leq n$ and $n + 1 \leq \mu, \nu \leq N$.

By the remark below (3.31), $g^{N+1}_N$ is real-valued. By (3.42), we have $\langle g_0, g_{N+1} \rangle = -\frac{i}{2}$, it implies $g^0_0 \cdot g^{N+1}_N = 1$. In particular, both $g^{N+1}_N$ and $g^0_0$ are real. Since $\langle g_0, g_\mu \rangle = 0$ and $g^0_0 \neq 0$, it implies $g^{N+1}_\mu = 0$. Since $\langle g_\alpha, g_\beta \rangle = \delta_{\alpha \beta}$, it implies that the matrix $(g^{\beta}_\alpha)$ is unitary.

By considering all first-order adapted lifts from $M$ into $SU(N+1,1)$, as the definition of $II_M$ in Definition 3, we can defined CR second fundamental form $II_M$ as in (3.39):

$$II_M = II^e_M = \omega_0^\beta \omega^\beta_0 \otimes \mathcal{L}_\mu \in \Gamma(M, S^2T^{1,0}\pi_0^{*}M \otimes N^{1,0}_{\pi_0(e_0)}M), \text{ mod}(\omega^{N+1}_N),$$

which is a well-defined tensor, and is called the CR second fundamental form of $M$.

We remark that the notion of $II_M$ in Definition 4 was introduced in a paper by S.H. Wang [Wa06].

### 3.14 Geometric Rank And The Second Fundamental Form

**Geometric Rank and $II_M$**

**Lemma 3.14.1** (i) ([JY09], theorem 7.1) Let $F \in Prop_k(\partial \mathbb{H}^{n+1}, \partial \mathbb{H}^{N+1})$ with $k \geq 2$ and $F(0) = 0$. Then there exists a neighborhood of $0$ in $M := F(\partial \mathbb{H}^{n+1})$ and a $C^{k-1}$-smooth first-order adapted lift $e : U \rightarrow SU(N+1,1)$

$$e = (e_0, e_j, e_b, e_{N+1}) \in SU(N+1,1), \quad 1 \leq j \leq n, \quad n + 1 \leq b \leq N - 1. \quad (3.49)$$

(ii) ([JY09], Step 3 of the proof of Theorem 1.1) Let $F = F^{***} = (f, \phi, g)$, the induced first-order adapted lift $s$, and notation be as in Theorem 3.14.1. Then

$$h^\mu_{j,k}|_0 = \frac{\partial^2 \phi_\mu}{\partial z_j \partial z_k}|_0, \quad j, k \in \{1, 2, ..., n, N + 1\} \quad (3.50)$$

where $II_M = h^\mu_{j,k} \omega^j \omega^k \otimes e_\mu$ is the CR second fundamental form.
Theorem 3.14.2 ([HJ09]) Let $F \in \text{Prop}_2(\partial \mathbb{H}^{n+1}, \partial \mathbb{H}^{N+1})$. Then its geometric rank $\kappa_0$ equals to

$$
\kappa_0 = \sup_{p \in \partial \mathbb{H}^{n+1}} \left[ n - \dim_{\mathbb{C}} \{ \nu \mid II_{M,F(p)}(\nu, \nu) = 0 \} \right]
$$

where $II_{M,F(p)}$ is the CR second fundamental form of the submanifold $M$ at the point $F(p)$. Here $\{ \nu \mid II_{X,F(p)}(\nu, \nu) = 0 \}$ is a vector space over $\mathbb{C}$.

Corollary 3.14.3 Let $F \in \text{Prop}_2(\mathbb{H}^n, \mathbb{H})$. Then

$$
\kappa_0 = 0 \iff II_M = 0.
$$

Going back to Theorem 3.8.1. We have a lemma:

Lemma 3.14.4 Let $H : M' \to \partial \mathbb{H}^{N+1}$ be a CR smooth embedding where $M'$ is a strictly pseudoconvex smooth real hypersurface in $\mathbb{C}^{n+1}$. We denote $M = H(M')$. Then the following statements are equivalent:

(i) The CR second fundamental form $II_M$ by Definition A identically vanishes.
(ii) The CR second fundamental form $II_M$ by Definition B identically vanishes.
(iii) The CR second fundamental form $II_M$ by Definition C identically vanishes.
(iv) The CR second fundamental form $II_M$ by Definition D identically vanishes.

Lemma 3.14.5 (cf. [EHZ04], corollary 5.5) Let $H : M' \to \partial \mathbb{H}^{N+1}$ be a smooth CR embedding of a strictly pseudoconvex smooth real hypersurface $M \subset \mathbb{C}^{n+1}$. Denote by $(\omega_{\alpha \beta}^\mu)$ the CR second fundamental form matrix of $H$ relative to an admissible coframe $(\theta, \theta^A)$ on $\partial \mathbb{H}^{N+1}$ adapted to $M$. If $\omega_{\alpha \beta}^\mu \equiv 0$ for all $\alpha, \beta$ and $\mu$, then $M'$ is locally CR-equivalent to $\partial \mathbb{H}^{N+1}$.

To prove Theorem 3.8.1, we apply Lemma 3.14.4 and Lemma 3.14.5 and the hypothesis that the CR second fundamental form identically vanishes to know that $M$ is locally CR equivalent to $\partial \mathbb{H}^{n+1}$.

Then $M$ is the image of a local smooth CR map $F : U \subset \partial \mathbb{H}^{n+1} \to M \subset \partial \mathbb{H}^{N+1}$ where $U$ is a open set in $\partial \mathbb{H}^{n+1}$. By a result of Forstneric [Fo89], the map $F$ must be a rational map. It suffices to prove that $F$ is equivalent to a linear map. By the fact that $F$ is linear if and only if its geometric rank is zero, it is sufficient to prove that the geometric rank of $F$ is zero: $\kappa_0 = 0$. This can be done by applying Theorem 3.14.2.

The third gap theorem can be restated as
Theorem 3.14.6 (Huang-Ji-Yin, preprint, [HJY09]) If \( F \in Prop_3(\mathbb{B}^n, \mathbb{B}^{4n-7}) \) with \( n \geq 7 \), then \( F \) is equivalent to a map \((G, 0)\) where \( G \in Rat(\mathbb{B}^n, \mathbb{B}^{3n}) \).

The dimension \( 4n - 7 \) is the smallest integer that forces the geometric rank of \( F \) to satisfy \( \text{rank}(F) \leq 2 \). The condition \( n \geq 8 \) is necessary from the two target spaces \( \mathbb{B}^{4n-7} \) and \( \mathbb{B}^{2n} \): 
\[ 4n - 7 > 3n, \]
Also the target dimension \( 3n \) in Theorem above is sharp which is shown by the example: \( F \in Rat(\mathbb{B}^n, \mathbb{B}^{3n}) \) given by
\[
(z_1, z_3, z_4, ..., z_n, \sqrt{1 - c^2} z_2, cz_2 H(z))
\]
where \( |c| \leq 1 \) and \( H(z) = \begin{pmatrix} z_1, ..., z_{n-1}, (\cos \theta) z_n, (\sin \theta) z_1 z_n, ..., (\sin \theta) z_{n-1} z_n, (\sin \theta) z_n^2 \end{pmatrix} \in Rat(\mathbb{B}^n, \mathbb{B}^{2n}) \) is D’Angelo map with \( 0 < \theta \leq \frac{1}{2} \). \( F \) has geometric rank \( \kappa_0 = 2 \).

- For any \( F \in Prop_3(\mathbb{B}^n, \mathbb{B}^{4n-7}) \) with \( n \geq 7 \). Then the geometric rank of \( F \) is less than or equal to 2. Assume that \( F \) has geometric rank 2.

- To prove local existence of first-order adapted lift \( s : M \to SU(N + 1, 1) \).

- Since \( F \) has geometric rank 2, by normalization, \( F \) is equivalent to a new map in which the initial terms of its power series at 0 are simplified. By this, it calculates the second fundamental form of \( F(\partial \mathbb{H}^{n+1}) \subset \partial \mathbb{H}^{N+1} \) at 0 with respect to the first-order adapted lift \( e \).

- We generalize the above calculation from a point into an open set. What we do is to modify the lift \( s \).

- From the special CR second fundamental form, we determine other terms of the Maurer-Cartan form \( \omega \) from the formula \( d\omega = -\omega \wedge \omega \). In particular, vanishing of certain items are proved.

- From \( \omega = s^{-1} ds \), we have a complete differential system
\[
ds = s \omega.
\]

Due to certain terms vanishing, we replace \( s = (e_0, e_1, ..., e_{N+1}) \) by a new one \( \tilde{s} \) by removing some components and replace \( \omega \) by a smaller sized new matrix \( \tilde{\omega} \) by removing some terms such that
\[
d\tilde{s} = \tilde{s} \tilde{\omega}.
\]
is also a complete differential system.
By the size of $\tilde{\omega}$, it implies $\dim(F) \leq 3n$. $\square$
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