Projection and Perspective

For many artists and mathematicians the hardest concept to fully master is working in three-dimensional space. Though our eyes are accustomed to living in a world where everything before us is in three-dimensions, even a flat piece of paper, it is difficult to put what we might understand in sight onto that paper. For mathematicians, picturing three-dimensional shapes that one has not held in your hands or seen with your own eyes can be a struggle – for example, imagining what a dodecahedron looks like. Projective geometry, however, analyzed the problem of perspective in art and dimension in math to create a way for mathematicians to understand the third dimension.

I clearly remember the frustration of attempting to draw a tile floor in my elementary art class\(^1\). Even with a ruler, the lines did not meet at the place they met while I looked at an actual tile floor. It was a challenge trying to transfer the picture from my mind onto a two-dimensional surface. Now, there are plenty of online tutorials and pre-made graphics that offer hope to future artists who might find more success than I did.

The trouble, however, does not end when one decides to never draw a tile floor again. Unfortunately for the sanity of all of us trying to understand it, we live in a three-dimensional world. To record anything as a sketch, one needs to understand perspective. For instance, drawing those infamous cylinders proves to be a challenge for those who don’t understand how the bases look when printed on a two-dimensional surface. Many try and flatten the base, but that
would make a cylinder that (not only would no longer be a cylinder, but) would not be able to stand. The curve of the base should be equivalent to the curve of the outermost outline of the top. Imagine projecting a cylinder instead of drawing it and one will notice that with a focal point in the center of the cylinder, one should be able to create what looks like two cones meeting at the focal point, their tops touching (like an hour glass that the sand would not fall from).

The challenge of perspective does not only exist for those of us who cannot find the vanishing point (visited later), but also existed for the elite. For centuries, in fact, artists could not correctly create a three-dimensional piece that was accurate. For example, Lorenzetti’s piece offers some consolation to those of us who struggle. *Entry into Jerusalem*, depicting Jesus riding through the town, is flawed in many points. For one, the man knelt on the ground before Jesus is much smaller than the donkey that holds Jesus which is much
too long. Additionally, the men gathering palm leaves from the tree in the upper left are about the same size as the building to their right! One solution to the problem of perspective is projective geometry.

Projective geometry is credited to Girard Desargues, who based the mathematics branch on perspective art. (However, Lecture 26 informs that projective geometry as a mathematical field was established by Jean-Victor Poncelet, along with others.) Using the theory of perspective, mathematicians were able to describe how to project three-dimensional objects onto a two-dimensional surface, which introduced the idea that parallel lines can meet at some point in infinity. This is different from what is known as Euclidean geometry, where two parallel lines will never meet. Going back to perspective art, and Figure 3.1, we see that the parallel lines converge at the center point known as the vanishing point. The vanishing point is the point where the eye focuses, and as such, is the point where the lines used to create the three-dimensional perspective will meet.

Noticing the common errors in perspective, Blaise Pascal found a new hobby in reading geometry books. Pascal’s father, a friend of Desargues, gave Pascal a copy of Euclid’s Elements which led Pascal to write his Essay on Conics containing what became known as Pascal’s theorem. He showed that a hexagon inscribed in a conic section will have its lines meet in a straight line called the Pascal line, if the three pairs of lines continue (shown in Figure 3.5). Going back to Figure 3.2, we can now apply this and see that
when this is done for both bases, it creates the hourglass shape mentioned.

While perspective was something new and exciting in art, we have seen that it was not always done correctly. The vanishing point, and the idea that parallel lines will meet at this point, create the illusion that the objects in the piece are being projected onto a surface. Normally the math and art community don’t come together, but in this case the two worked together to create art that is realistic and three-dimensional mathematics that can be visualized and produced on a two-dimensional surface. This brings me to the next point of interest: camera lenses and perspective.

I can’t speak for the general population, but I don’t normally snap a photograph and start to take apart the camera to see how each of the parts work together to create realistic images. Catadioptric sensors are the instruments that cameras use to project the image, or increase a field of view. These systems combine catoptric and dioptric, or reflective and refractive, elements to focus light. This approach was discovered by the Greek geometer Diocles, and has been used in both cameras and telescopes. The mathematics behind the lenses cameras and telescopes use goes from geometry to linear algebra. The projection can be calculated using matrices, which is something that is often taught without real-world application. The lenses, if orthographic, will use an equation for orthographic projection:

\[
\begin{pmatrix}
U \\
V \\
W
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
X \\
Y \\
Z \\
T
\end{pmatrix}
\]
where X, Y, Z, and T are 3D points and U, V and W, are points in the image. It is important to note that lines will project to lines. That is, dissecting the image and projecting the lines onto a two-dimensional surface will not bring the lines “down a dimension” and create points, but rather lines on a 2D surface. The camera lens will find the focal point, or vanishing point, and project the lines onto the surface, which is what artists and mathematicians do when painting three-dimensional scenes on a two-dimensional surface or when examining three-dimensional objects in projective geometry to discover new theorems like Pascal’s.

The problem of perspective in art does not seem like one that will disappear for amateur artists like myself, but it is a problem that was solved using mathematics and then opened the door to a new way to think about geometry. Projective geometry analyzed and questioned things that Euclid’s and Descartes’ geometry ignored, and paved the way for technological advances, such as the inventions of the camera and the telescope. The illumination of images using mirrors and projection was an illumination of the necessity of mathematics to advance our world.

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5 Image from Lecture 26 cited as 4. No Title.