Lecture 4. Pythagoras’ Theorem and the Pythagoreans

Figure 4.1 Pythagoras was born on the Island of Samos

Pythagoras Pythagoras (born between 580 and 572 B.C., died about 497 B.C.) was born on the island of Samos, just off the coast of Asia Minor. He was the earliest important philosopher who made important contributions in mathematics, astronomy, and the theory of music.

Pythagoras is credited with introduction the words “philosophy,” meaning love of wisdom, and “mathematics" - the learned disciplines. Pythagoras is famous for establishing the theorem under his name: Pythagoras’ theorem, which is well known to every educated person. Pythagoras’ theorem was known to the Babylonians 1000 years earlier but Pythagoras may have been the first to prove it.

Pythagoras did spend some time with Thales in Miletus. Probably by the suggestion of Thales, Pythagoras traveled to other places, including Egypt and Babylon, where he may have learned some mathematics. He eventually settled in Groton, a Greek settlement in southern Italy. In fact, he spent most of his life in the Greek colonies in Sicily and

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1 Asia Minor is a region of Western Asia, comprising most of the modern Republic of Turkey.
southern Italy. In Groton, he founded a religious, scientific, and philosophical organization. It was a formal school, in some sense a brotherhood, and in some sense a monastery. In that organization, membership was limited and very secretive; members learned from leaders and received education in religion. Pythagoras' followers were commonly called “Pythagoreans.” The Pythagoreans were supposed to have mixed in politics; they allied themselves with the aristocratic faction and were driven out by the popular or democratic party. Pythagoras fled to nearby Metapontum and was murdered there around 497 B.C. His followers spread to other Greek centers and continued his teachings.

**Pythagoras’ teachers** Among his teachers, there were at least three philosophers who had great influence on Pythagoras. One was Pherekydes. The other two philosophers were Thales and his pupil Anaximander, both lived on Miletus. It is said that Pythagoras visited Thales in Miletus when he was between 18 and 20 years old. By this time Thales was an old man and although he created a strong impression on Pythagoras, he probably did not teach him a great deal. However, he did contribute to Pythagoras’ interest in mathematics and astronomy, and advised him to travel to Egypt to learn more of these subjects. Thales’ pupil, Anaximander, gave lectures on Miletus and Pythagoras attended these lectures. Anaximander was extremely interested in geometry and cosmology, and many of his ideas would influence Pythagoras’ own views.

**Figure 4.2** Anaximander (610-546 B.C.) was a pre-Socratic philosopher who lived in Meletus, a city of Ionia. He studied under Thales at the Milesian school and succeeded him as its second master. Pythagoras and Anaximanes were among his pupils. His work only survives in fragmentary form, but contributed greatly to astronomy, geography and geometry.

**Pythagoreans** Pythagoras’ followers were called “Pythagoreans,” and included, for example, Plato (see Chapter 7).

Pythagoras’ school’s motto was “All is number,” which means that all things in the universe obey the rules of numbers. By number the motto refers to whole numbers (positive integers) or ratios of whole numbers. The Pythagoreans tried to bring the realms of science,
religion, and philosophy all under the rule of numbers. Such a strong belief made the creation of Greek mathematics possible.

The school imposed a strict code of conduct on its members, which included secrecy, vegetarianism, and a curious taboo on the eating of beans. Such bean-eating prohibition could be explained by the resemblance of beans to genitals, or bean eating being compared to eating a living soul. Also mathematical results were considered to be the property of Pythagoreans’ school, and their individual discoveries were not allowed to be known to outsiders.

There are no written works by the Pythagoreans; we know about them through the writing of others, including Plato and Herodotus. In particular we are hazy about the personal life of Pythagoras and his followers. Katz wrote: From the surviving biographies, all written centuries after his death, we can infer that Pythagoras was probably more of a mystic than a rational thinker, but one who commanded great respect from his followers. We also are unable to make sure which work was done by Pythagoras personally and which work was done by his followers. As a result, when one speaks of the work of Pythagoras, one really refers to the work done by the group between 585 B.C and roughly 400 B.C.

If there is one theorem that is known to all mathematically educated people, it is surely the theorem of Pythagoras. Someone even suggested to use

\[ a^2 + b^2 = c^2 \]

with a right triangle to communicate with aliens in outer space.

At first sight, arithmetic and geometry seem to be completely unrelated realms. Arithmetic involves counting (discrete process), while geometry involves measures for line, curves and surfaces (continuous process). Pythagoras’ Theorem created the first deeper relationship between these two areas. Moreover, it has continued to hold a key position during the history of mathematics. For example, in Fourier series theory. A lesson we learned from Pythagoras’ Theorem is that new ideas often emerge from different areas, which resolves the conflict and allows previously irreconcilable ideas to interact fruitfully.

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5 Heroditos of Halicarnassus was a Greek historian who lived in the 5th century BC (c. 484 B.C. - c. 425 B.C.) and is regarded as the “Father of History” in Western culture.
Here is one proof of Pythagoras’ Theorem in Euclid’s *Elements*. Consider any right triangle. Let us denote by $c$ the length of its hypotenuse, and denote by $a, b$ the lengths of other two sides, respectively. In Figure 4.3, in each of the two big squares, we see four equal size triangles. Observe that from the left square,

$$\text{Area(Big square)} - 4 \text{Area(right triangle)} = a^2 + b^2,$$

and from the right square,

$$\text{Area(Big square)} - 4 \text{Area(right triangle)} = c^2,$$

Then we obtain $a^2 + b^2 = c^2$.

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Figure 4.3 The proof of Pythagoras’ Theorem by Euclid.

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Music, mathematics and the universe The Pythagoreans believed (but failed to prove) that the universe could be understood in terms of whole numbers or ratios of whole numbers. This belief came from observations of music, mathematics and astronomy.

Pythagoreans noticed that vibrating strings produce harmonious tones when the ratios of their lengths are whole numbers. According to legend, when one day he passed blacksmiths at work, Pythagoras thought that the sounds emanating from their anvils being hit were beautiful and harmonious and postulated that some scientific law may have caused this to happen. He went to the blacksmiths and took a look at their tools. He discovered that it was because the anvils were “simple ratios of each other, one was half the size of the first, another was 2/3 the size, and so on.” This could be the first attempt to express the universe in terms of whole numbers, and hence the very important idea that the world could be understood through mathematics was born.

Pythagorean were among the first to maintain that the Earth was spherical in shape and that the planes, the Sun, and the Moon have an independent motion of their own from west to east, in a direction opposite to the daily (apparent) rotation of the sphere of the fixed stars. Pythagoras believed that the planets and stars moved according to mathematical rules, which corresponded to musical notes and could thus produce a symphony.

Pythagorean were so pleased with their theory of law of music and harmony that they dared failed because of the enormous complexity of matter. Now, with the modern string theory in physics, physics and mathematics are going back to Pythagorean dream in some senses.

Creator of pure mathematics The recognition that mathematics deals with abstractions may with some confidence be attributed to the Pythagoreans. When the early Pythagoreans said that all objects were composed of (whole) numbers or that the numbers were essence of the universe, they meant it literally, because to them, numbers were like atoms are to us. Eudemus said:

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10Eudemus of Rhodes was an ancient Greek philosopher, and first historian of science who lived from 370 B.C. until 300 B.C. He was one of Aristotle’s most important pupils, editing his teacher’s work and making it more easily accessible.
“Pythagoras rose to higher principles (than had the Egyptians and Babylonians) and considered abstract problems for the pure intelligence. ....... Pythagoras was the creator of pure mathematics, which he made into a liberal art.”

**Pythagorean numbers**  The three sides of a right triangle are called *Pythagorean numbers*, e.g., \((3, 4, 5)\). The Pythagoreans may have discovered that when \(n\) is any positive integer and \(m > 1\) is an odd integer, then
\[
\left(2n + 1, \ 2n^2 + 2n, \ 2n^2 + 2n + 1\right), \ \left(m, \ \frac{1}{2}(m^2 - 1), \ \frac{1}{2}(m^2 + 1)\right)
\]
form Pythagorean numbers. Of course, these are not all Pythagorean numbers because some Pythagorean numbers may not be in these forms.

**Polygonal numbers**  Pythagoras thought that in order to understand the world it was important to indulge in the study of numbers. We know of his achievements only from his disciples, the Pythagoreans, who made important discoveries about number theory and geometry. The Pythagoreans represented whole numbers by using arrangements of dots or pebbles, and classified these numbers according to the shapes produced.\(^{12}\) The numbers 3, 6, 10, and so on were called *triangular numbers* because the pebbles could be arranged to form triangles. The numbers 4, 9, 16, and so on were called *square numbers* because the pebbles could be arranged as squares.

From these simple geometrical arrangements some properties of the whole numbers emerged. The Pythagoreans realized that the sums \(1, 1 + 2, 1 + 2 + 3, \ldots\) gave the triangular numbers and that
\[
1 + 2 + 3 + \ldots + n = \frac{n(n + 1)}{2}.
\]

\(^{12}\)The English word *calculation* is derived from the Greek word for stone or pebble.
They also concluded that the sum of two consecutive triangular numbers is always a square number, i.e.,

\[ \frac{n(n + 1)}{2} + \frac{(n + 1)(n + 2)}{2} = (n + 1)^2. \]

Pythagoreans also studied pentagonal numbers and hexagonal numbers, and polygonal numbers in general.

Lots of effort was made by Greeks on studying polygonal numbers. However, from a point of view today, the Greeks seemed to have been mistaken in attaching too much importance to polygonal numbers because there are no major theorems about them.

**Perfect numbers and amicable numbers** The mathematicians of Pythagoras’ school (500 B.C. to 300 B.C.) were interested in numbers for their mystical and numerological properties. They understood the idea of primality and were interested in perfect and amicable numbers.

A *perfect number* is one whose proper divisors sum to the number itself. e.g. The number 6 has proper divisors 1, 2 and 3 and \(1 + 2 + 3 = 6\), 28 has divisors 1, 2, 4, 7 and 14 and \(1 + 2 + 4 + 7 + 14 = 28\).

*Amicable numbers* are two different numbers such that the sum of the proper divisors of one number equals the other and vice versa. For example, \((220, 284)\) are amicable numbers for the proper divisors of 220 are 1, 2, 4, 5, 10, 11, 20, 22, 44, 55 and 110, of which the sum is 284; and the proper divisors of 284 are 1, 2, 4, 71, and 142, of which the sum is 220.

![Figure 4.6 Pythagoras](image)

**Incommensurable numbers** The Pythagoreans were in a strong belief that the universe can be understood by numbers, and that every number is either a whole number or a ratio of
whole numbers. In today’s language, this is equivalent to the statement that every number is a rational number.

However, Hippasus of Metapontum\textsuperscript{13}, a disciple of Pythagoras, made an astonishing discovery: $\sqrt{2}$ is an incommensurable ratio, i.e., $\sqrt{2}$ is an irrational number. We do not know what Hippasus’ proof was, but a proof of this fact is given in the box on the next page.

It gave a shock to the Pythagoreans. They could not believe this discovery was true but could not deny the powerful proof. Some of the previously proved mathematical theorems were based on the assumption that every number is rational, which had lost its basis now. It was said that all of this happened at sea and the Pythagoreans were so panicked that they threw Hippasus overboard into the water.

\begin{boxedminipage}{\textwidth}
\textbf{Fact:} $\sqrt{2} \neq \frac{p}{q}$ for any integers $p$ and $q$, $q \neq 0$.

\textbf{Proof:} Suppose $2 = \frac{p}{q}$ for some integers $p$ and $q \neq 0$. Assume that $p$ and $q$ have no common factors (otherwise it can be divided).

By taking square, we have $2 = \frac{p^2}{q^2}$, and hence $2q^2 = p^2$. We conclude that $p$ is an even number. Then we can write $p = 2m$. From above, we see $2q^2 = 4m^2$, i.e., $q^2 = 2m^2$. We also conclude that $q$ is an even number. Both $p$ and $q$ have common factor 2, which is a contradiction with the assumption that $p$ and $q$ have no common factor.

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The above method of proof method is a \textit{reductio ad absurdum}, i.e., the indirect method, which was included in old editions of Euclid’s \textit{Elements} as Proposition 117 of Book X. However, it was most likely not in Euclid’s original text and so is omitted in modern editions. For the impact of this discovery in the history of mathematics, which was huge, see Lecture 8. According to J. Katz, Aristotle may find the above proof: “Aristotle found a proof for the irrationality of the diagonal of a square by arguing that if the side and the diagonal are assumed commensurable, then it would imply that odd numbers are even numbers, which is a contradiction.”\textsuperscript{14}

\textsuperscript{13}Hippasus of Metapontum , 500 B.C. in Magna Graecia, was a Greek philosopher.

Without question, Pythagorean philosophy was the prime source of inspiration for Plato and Aristotle. The impact and influence of the Pythagorean philosophy in the history of mathematics is immeasurable.

**Figure 4.7** Proof of the Theorem of Pythagoras in Euclid’s De Arte Geometrica late 13th century.