Lecture 8. Eudoxus and the Avoidance of a Fundamental Conflict

Eudoxus of Cnidus  Eudoxus, 480 BC - 355 BC, was a Greek philosopher, mathematician and astronomer who contributed to Euclid’s *Elements*. His philosophy influenced Aristotle. In Mathematics, Eudoxus was the second only to Archimedes in all antiquity and was called “god-like” by a Greek mathematician and poet Eratosthenes\(^1\).

Eudoxus was born in Cnidus in Asia Minor (Asia Minor is a region of the Middle East, today Turkey). He studied geometry and medicine, and became a pupil of Plato. After learning Pythagorean astronomy at the newly founded academy, he traveled to Egypt where he remained for some time. Then Eudoxus founded a school in Asia Minor. About 368 B.C., he and his followers jointed Plato. Later he returned to Cnidus and died there.

![Figure 8.1 Cnidus where Eudoxus was born.](image)

\(^1\)Eratosthenes of Cyrene, 276 B.C. - 194 B.C., was a Greek mathematician, poet, athlete, geographer and astronomer. He made several discoveries and inventions including a system of latitude and longitude and calculating the circumference of the Earth (with remarkable accuracy).
Using theory of proportion in geometry to avoid irrational numbers  Recall in Lecture 4 that Hippasus of Metapontum, a Greek philosopher, discovered that $\sqrt{2}$ is irrational. Until Hippasus’ discovery, the Pythagoreans believed that all phenomenon can be reduced to whole numbers or their ratios. Many mathematical results were proved based on the assumption that all numbers are integers or ratios of integers. Hippasus’ proof showed that this was false and, perhaps with panic, the Pythagoreans initially treated it as a kind of religious heresy and murdered Hippasus.

The discovery of more and more irrationals (in-commensurable ratios) made it necessary for the Greeks to face reality. Before finding a solution, the Greeks hesitated to study numbers and equations for a period of time. It was a crisis in the history of mathematics.

What Eudoxus accomplished was to use geometry to avoid irrational numbers. Eudoxus introduced a new theory of proportion: one that does not involve numbers. Instead he studied geometrical objects such as line segments, angles, etc., while avoiding giving numerical values to lengths of line segments, sizes of angles, and other magnitudes.

His work on ratios formed the basis for Book V of Euclid’s *Elements*, and anticipated in a number of ways the notion of algebra, which is absent from ancient Greek mathematics.

![Figure 8.2](image)

**Figure 8.2** Eudoxus of Cnidos developed a system to bring some order to the complexities of planetary motion.

In his system, the motion of each planet (Mercury, Venus, Mars, Jupiter, and Saturn) was governed by a set of four nested concentric spheres —– one to govern its daily motion, one to govern its motion through the zodiac, and two to account for the looping appearance of its retrograde motion.

Although Eudoxus’ proportion theory avoided a crisis, it forced a sharp separation between number and geometry because only geometry can handle in-commensurable ratios.
Geometry thus became the basis for almost all rigorous mathematics for the next two thousand years. As a consequence, the study of algebra and number theory came to a halt. It was too bad, but it is a part of our history.

**Eudoxus’ theory of proportions** The theory of proportions is credited to Eudoxus and is expounded in Book V of Euclid’s *Elements*. The purpose of the theory is to enable lengths (and other geometric quantities) to be treated as precisely as numbers, which only admit the use of rational numbers. The Greeks could not accept irrational numbers, but they accepted irrational geometric quantities such as the diagonal of the unit square. This is because geometric quantities clearly exist. For example, \( \sqrt{2} \) is the geometric diagonal of a unit square, which is real and whose existence is obvious.

To simplify the exposition of the theory, we first fix a length, and we call a length *rational* if it has a rational multiple of the fixed length.

Eudoxus’ idea was to say that a length \( \lambda \) is determined by those rational lengths less than it and those rational lengths greater than it. To be precise, by the definition, we say that \( \lambda_1 = \lambda_2 \) if the following holds:

(i) any rational whose length < \( \lambda_1 \) is also < \( \lambda_2 \), and
(ii) any rational whose length < \( \lambda_2 \) is also < \( \lambda_1 \).

Also, by the definition, we say that \( \lambda_1 < \lambda_2 \) if there is a rational length > \( \lambda_1 \) but < \( \lambda_2 \). This definition uses the rationals to define irrational numbers. This process seemed to avoid using infinity, but actually it used an infinite process. Not only that, in fact, modern real number theory is based on this same idea.

To illustrate, let us consider a proposition in Eulcid’s *Elements* by using Eudoxus’ theory:
Proposition VI-I\(^2\) Triangles which have the same heights are to one another as their bases. In other words, for two triangles \(ABC\) and \(ADE\), the ratio of their bases is equal to the ratio of their areas:

\[
BC : DE = \text{area}(ABC) : \text{area}(ADE).
\]

Proof: Let us put points \(B_2, B_3, ..., B_m\) such that the length of the segment \(B_jC\) is \(j\) times the length of \(BC\), \(1 \leq j \leq m\). Thus the area of the triangle \(AB_jC\) is \(j\) times the area of the triangle \(ABC\).

Similarly we put points \(E_2, E_3, ..., E_n\) such that the length of the segment \(E_kD\) is \(k\) times the length of \(ED\), \(1 \leq k \leq n\). Thus the area of the triangle \(AE_kD\) is \(k\) times the area of the triangle \(AED\).

The triangles \(AB_mC\) and \(AE_nD\) have the same heights so that when the length of \(B_mC\) is greater than, equal to, or less than the length of \(E_nD\), the area of the triangle \(AB_mC\) is correspondingly greater than, equal to, or less than the area of the triangle \(AE_nD\).

In other words, when \(m\) times of the length of \(BC\) is greater than, equal to, or less than \(n\) times of the length of \(DE\), the area of the triangle \(AB_mC\) is correspondingly greater than, equal to, or less than the area of the triangle \(AE_nD\).

Then, by the definition of Eudoxus, the proposition is proved. \(\square\)

The theory of proportions was so successful that it delayed the development of theories for real numbers for 2000 years. As we said, the Greeks accepted \(\sqrt{2}\) as the diagonal of the unit square, but any arithmetic approach to \(\sqrt{2}\), whether by sequences, decimals, or continued fractions, is infinite and therefore, less intuitive. Until the nineteenth century this seemed a good reason for considering geometry to be a better foundation for mathematics than arithmetic.

\(^2\)See the textbook, p.76, the 3rd edition; or page 82, the 2nd edition.
Dedekind’s theory of real numbers  In 1872 Dedekind adapted Eudoxus’ ideal to develop a theory of real numbers. For example, $\sqrt{2}$ can be defined to be the pairs

$$L_{\sqrt{2}} = \{ r \text{ rational numbers} \mid r^2 < 2 \}, \quad U_{\sqrt{2}} = \{ r \text{ rational numbers} \mid r^2 > 2 \}.$$ 

In general, let any partition of the positive rationals into sets $L, U$ such that any member of $L$ is less than any member of $U$ be a positive real number. This idea, now known as a Dedekind cut, is more than just a twist of Eudoxus; it gives a complete and uniform construction of all real numbers, or points on the line, using just the discrete, finally resolving the fundamental conflict in Greek mathematics.

We also remark that the Eudoxian definition of proportionality uses the term “for every ...” to avoid mentioning the infinite process, which is the same as the modern $\epsilon - \delta$ definitions of limit and continuity today.

Method of exhaustion  The second accomplishment by Eudoxus is the method of exhaustion, the powerful Greek method of establishing the areas and volumes of curved
figures. It was really the first step into calculus but does not use an explicit theory of limits. It is based on earlier ideas of approximating the area of a circle by Antiphon where he\(^3\) took inscribed regular polygons with increasing numbers of sides. Eudoxus was able to make Antiphon’s theory into a rigorous one, applying his methods to give rigorous proofs of the theorems, first stated by Democritus, that the volume of a pyramid is one-third the volume of the prism having the same base and equal height; and the volume of a cone is one-third the volume of the cylinder having the same base and height. The proofs of these results are attributed to Eudoxus by Archimedes in his work on the sphere and cylinder and, of course, Archimedes went on to use Eudoxus’ method of exhaustion to prove a remarkable collection of theorems.

**Basis of axioms** There is no question that the work of Eudoxus established deductive organization on the basis of explicit axioms. The necessity for understanding and operating with in-commensurable ratios is undoubtedly the reason for this step. Since his approach successfully used the precise logical basis for these ratios, Eudoxus most likely saw the need to formulate axioms to study unfamiliar and troublesome problems. This also reinforced the earlier decision to rely on deductive reasoning for proof.

**Eudoxus and astronomy** Eudoxus elaborated a geocentric model composed of crystalline spheres, incorporating the Platonic ideal of uniform circular motion. Eudoxus was the first Greek to make a map of the stars.

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\(^3\) A Greek orator and statesman who took up rhetoric as a profession.