UPPER BOUNDARY POINTS OF THE GAP INTERVALS FOR RATIONAL MAPS BETWEEN BALLS

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Dedicated to Professor Ngaiming Mok on the occasion of his 60th birthday

Abstract. The paper focuses on the study of rational proper holomorphic maps from $\mathbb{B}^n$ to $\mathbb{B}^N$. We classify these maps when $N$ is the upper boundary point of the gap interval $I_k$, $k \leq n - 2$ and the geometric rank of the map is $k$.

Key words. Proper holomorphic maps, holomorphic classification, geometric rank, Chern-Moser equation.

Mathematics Subject Classification. 32H35.

1. Introduction. Let us denote by $\mathbb{B}^n$ the unit ball in $\mathbb{C}^n$ and $\text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ the set of all proper holomorphic rational maps $F$ from $\mathbb{B}^n$ to $\mathbb{B}^N$. We say that $f, g \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ are equivalent if there are $\sigma \in \text{Aut}(\mathbb{B}^n)$ and $\tau \in \text{Aut}(\mathbb{B}^N)$ such that $f = \tau \circ g \circ \sigma$. Let $K(n) := \max\{t \in \mathbb{Z}^+ \mid \frac{t(t+1)}{2} < n\}$. For any integer $k$ with $1 \leq k \leq K(n)$, we define the gap interval

$$I_k := (kn, (k+1)n - \frac{k(k+1)}{2}). \quad (1.1)$$

Let us recall the gap conjecture [HJY09]: Any proper holomorphic rational map $F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ ($n \geq 3$) is equivalent to a map of the form $(G, 0')$ where $G \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^{N'})$ where $N' < N$ if and only if $N \in I_k$ for some $1 \leq k \leq K(n)$. For the sake of simplicity, we call $F$ is equivalent to $G$.

Recently, P. Ebenfelt proposed a SOS conjecture (i.e., the Sums of Squares of Polynomial conjecture) [E16] and proved that if the SOS conjecture is true, then it implies the gap conjecture.

The “only if” part of the gap conjecture was proved in [HJY09]. For the “if” part, the cases for $I_1, I_2,$ and $I_3$ have been proved by Huang [Hu99], Huang-Ji[HJ01], Hamada [H05], Huang-Ji-Xu[HJX06] and Huang-Ji-Yin [HJY14].

Moreover, if $N$ is the boundary point of the interval $I_k$ for $k = 1$ and 2, maps in $\text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ have been determined, up to equivalence, as follows.

- When $N = n$ which is the lower boundary point of $I_1 = (n, 2n - 1)$, it was proved by Alexander [A77]: any map $F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^n)$ is an automorphism.

- When $N = 2n - 1$ which is the upper boundary point of $I_1$, it was proved by [HJ01]: any map $F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^{2n-1})$ is either the linear map, or the Whitney map $W_{n,1}$.

- When $N = 2n$ which is the lower boundary point of $I_2 = (2n, 3n - 3)$, it was proved by Hamada [Ha05] that any map $F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^{2n})$ is the linear map, or Whitney map, or in the D’Angelo map family.

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Recently, when \( N = 3n - 3 \) which is the upper boundary point of \( \mathcal{I}_2 \), it was proved by Andrews-Huang-Ji-Yin [AHJY16] that any map \( F \in Rat(\mathbb{B}^n, \mathbb{B}^{3n-3}) \) is linear, or Whitney map \( W_{n,1} \), or in the D’Angelo family, or a generalized Whitney map \( W_{n,2} \).

For any integer \( N \) in the closed interval \( \mathcal{I}_k \) and for any map \( F \in Rat(\mathbb{B}^n, \mathbb{B}^N) \), its geometric rank \( \kappa_0 \leq k \). In fact, for any \( F \in Rat(\mathbb{B}^n, \mathbb{B}^N) \) with the geometric rank \( \kappa_0 \), it is known [Hu03]

\[
N \geq n + \frac{(2n - \kappa_0 - 1)\kappa_0}{2} = (\kappa_0 + 1)n - \frac{\kappa_0(\kappa_0 + 1)}{2},
\]
(1.2)

which implies \( \kappa_0 \leq k \).

In this paper, we show that when \( N \) is the upper boundary point of \( \mathcal{I}_k \), we can determine maps in \( Rat(\mathbb{B}^n, \mathbb{B}^N) \) with geometric rank \( \kappa_0 = k \). This gives a new proof for the above mentioned result in [HJ01] when \( k = 1 \) and the above mentioned result in [AHJY16] when \( k = 2 \).

**Theorem 1.1.** Let \( F \in Rat(\mathbb{B}^n, \mathbb{B}^N) \) with \( n \geq 3 \). Let \( N \) be the upper boundary point of the gap interval \( \mathcal{I}_k \) and \( k \leq n - 2 \). Suppose that the geometric rank of \( F \) is \( \kappa_0 = k \). Then \( F \) is equivalent to the generalized Whitney map \( W_{n,k} \).

Notice that in above theorem, \( N = (k + 1)n - \frac{k(k+1)}{2} \) and that \( \kappa_0 = k \leq K(n) \) by the definition of \( \mathcal{I}_k \), i.e., \( \frac{\kappa_0(\kappa_0 + 1)}{2} < n \) holds. Here we recall [HJY09] that a map \( F \in Rat(\mathbb{B}^n, \mathbb{B}^N) \) is called the **generalized Whitney map** \( W_{n,k} \) if \( N = (k + 1)n - \frac{(k+1)k}{2} \) and

\[
W_{n,k}(z) = W_{n,k}(z_1, \ldots, z_n) = (z_1\psi_1, \ldots, z_k\psi_k, \psi_{k+1})
\]
(1.3)

where

\[
\begin{align*}
\psi_1 &= (z_1, \sqrt{2}z_2, \ldots, \sqrt{2}z_k, z_{k+1}, \ldots, z_n), \\
\psi_2 &= (z_2, z\sqrt{2}z_3, \ldots, \sqrt{2}z_k, z_{k+1}, \ldots, z_n), \\
&\quad \ldots, \\
\psi_{k-1} &= (z_{k-1}, \sqrt{2}z_k, z_{k+1}, \ldots, z_n), \\
\psi_k &= (z_k, z_{k+1}, \ldots, z_n), \\
\psi_{k+1} &= (z_{k+1}, \ldots, z_n).
\end{align*}
\]
(1.4)

We may have another interpretation for the above theorem. If \( N = n + \frac{(2n - \kappa_0 - 1)\kappa_0}{2} \), from (1.2), we say that \( N \) is the **minimum**. Based on the semi-linearity property, the following two problems are formulated in [JX04].

**Problem (A):** Study and classify maps in \( F \in Rat(\mathbb{B}^n, \mathbb{B}^N) \) with \( N \) minimum and \( 1 \leq \kappa_0 \leq n - 2 \).

**Problem (B):** Study and classify maps in \( F \in Rat(\mathbb{B}^n, \mathbb{B}^N) \) with \( N \) minimum and \( \kappa_0 = n - 1 \).

For Problem (B), when \( n = 2 \), it is \( Rat(\mathbb{B}^2, \mathbb{B}^3) \) which was solved by Faran’s theorem. The next case \( n = 3 \), \( Rat(\mathbb{B}^3, \mathbb{B}^6) \) is unsolved. In general, Problem (B) should be much more difficult than Problem (A) because it lacks of the semi-linearity property (see [Hu03]). For Problem (A), when \( \kappa_0 = 1 \), it is \( Rat(\mathbb{B}^n, \mathbb{B}^{2n-1}) \) which was solved by [HJ01] that \( F \) is the Whitney map \( W_{n,1} \); when \( \kappa_0 = 2 \), it is \( Rat(\mathbb{B}^n, \mathbb{B}^{3n-3}) \),
it is recently solved by [AHJY16] that the map $F$ must be the generalized Whitney map $W_{n,2}$. Then Theorem 1.1 covers the remaining part of Problem (A) under the condition $\kappa_{0(\kappa_{0}+1)} < n$.

**Theorem 1.2.** Let $F \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ be with geometric rank $\kappa_0 \leq n-2$, where $N = n + \frac{2n-n_0-1}{2}N_0$ is minimum and $\frac{\kappa_0(\kappa_0+1)}{2} < n$. Then $F$ is equivalent to the generalized Whitney map $W_{n,n_0}$.

Note that Theorem 1.1 and Theorem 1.2 are equivalent, and that the condition $\frac{\kappa_0(\kappa_0+1)}{2} < n$ in both of the above theorems will be used in Lemma 2.3 below.

### 2. Preliminaries.

**2a. The associated maps $F_p^{**}$.** Let $F = (f, \phi, g) = (\tilde{f}, g) = (f_1, \ldots, f_n, \phi_1, \ldots, \phi_{n-n}, g)$ be a non-constant rational CR map from an open subset $M$ of $\partial \mathbb{H}_n$ into $\partial \mathbb{H}_n$ with $F(0) = 0$. For each $p \in M$ close to 0, we write $\kappa_p^0 \in \text{Aut}(\mathbb{H}_n)$ for the map sending $(z, w)$ to $(z + z_0, w + w_0 + 2i(z, \overline{z_0})$ and $\tau_p^F \in \text{Aut}(\mathbb{H}_n)$ by defining

$$
\tau_p^F(z^*, w^*) = (z^* - \overline{f}(z_0, w_0), w^* - \overline{g(z_0, w_0) - 2i(z, \overline{z_0})}).
$$

Then $F$ is equivalent to $F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p)$. Notice that $F_0 = F$ and $F_p(0) = 0$. The following is fundamentally important for the understanding of the geometric properties of $F$. Let us denote $\text{Prop}(\mathbb{H}_n, \mathbb{H}_N) := \{\text{holomorphic proper maps from } \mathbb{H}_n \text{ into } \mathbb{H}_n\}$ and $\text{Prop}_p(\mathbb{H}_n, \mathbb{H}_N) := \text{Prop}(\mathbb{H}_n, \mathbb{H}_N) \cap C^k(\mathbb{H}_n)$.

**Lemma 2.1.** ([Hu99]) Let $F \in \text{Prop}_2(\mathbb{H}_n, \mathbb{H}_N)$ with $2 \leq n \leq N$. For each $p \in \partial \mathbb{H}_n$, there is an automorphism $\tau_p^{**} \in \text{Aut}_0(\mathbb{H}_n)$ such that $F_p^{**} := \tau_p^{**} \circ F_p$ satisfies the following normalization:

$$
f_p^{**} = z + \frac{i}{2} \phi_p^{**}(1)(z)w + o_{wt}(3), \quad \phi_p^{**} = \phi_p^{**}(2)(z) + o_{wt}(2), \quad g_p^{**} = w + o_{wt}(4),
$$

with $|\langle z, \phi_p^{**}(1)(z)w \rangle|^2 = |\phi_p^{**}(2)(z)|^2$.

**2b. Geometric rank.** Write $A(p) := -2i(\frac{\partial^2 f_p}{\partial z \partial w}^{**}|_{0})_{1 \leq j, l \leq (n-1)}$. We call the rank of the $(n-1) \times (n-1)$ matrix $A(p)$, which we denote by $Rk_F(p)$, the **geometric rank** of $F$ at $p$. $Rk_F(p)$ depends only on $p$ and $F$, and is a lower semi-continuous function on $p$, and is independent of the choice of $\tau_p^{**}$. ([Hu03]) Define the geometric rank of $F$ to be $\kappa_0(F) = \max_{p \in \partial \mathbb{H}_n} Rk_F(p)$. Notice that it always holds that $0 \leq \kappa_0 \leq n-1$. Define the geometric rank of $F \in \text{Prop}_2(\mathbb{B}^n, \mathbb{B}^N)$ to be the one for the map $\rho_{N}^{-1} \circ F \circ \rho_n \in \text{Prop}_2(\mathbb{H}_n, \mathbb{H}_N)$. ([Hu03]), $\kappa_0(F)$ depends only on the equivalence class of $F$ and when $N < \frac{n(n+1)}{2}$, the geometric rank $\kappa_0(F)$ of $F$ is precisely the $\kappa_0$ mentioned in the introduction.

Under the condition $1 \leq \kappa_0 \leq n - 2$, the following theorem was proved in [Hu03] and [HIJX06].

**Theorem 2.2 ([HIJX06]).** Suppose that $F \in \text{Prop}_3(\mathbb{H}_n, \mathbb{H}_N)$ has geometric rank $1 \leq \kappa_0 \leq n - 2$ with $F(0) = 0$. Then there are $\sigma \in \text{Aut}(\mathbb{H}_{n})$ and $\tau \in \text{Aut}(\mathbb{H}_N)$ such that $\tau \circ F \circ \sigma$ takes the following form, which is still denoted by $F = (f, \phi, g)$ for convenience of notation:
of its boundary. If necessary, we can assume that 0
dimensional complex subspaces in
Gr

there is a unique affine subspace
sufficiently small domain inside
form:
Let
F
that
the normalization condition as in Theorem 2.2, the restriction of
consisting of poles and the non-immers points of
linear. Write
V
such that
F
∈
Z

(2.1)

Here, for 1 ≤ κ₀ ≤ n − 2, we write \( S = S₀∪S₁ \), the index set for all components of \( φ \),
where \( S₀ = \{(j,l) : 1 ≤ j ≤ κ₀, 1 ≤ l ≤ n−1, j ≤ l \} \), \( S₁ = \{(j,l) : j = κ₀+1, κ₀+1 ≤ l ≤ κ₀ + N−n - \frac{2(n−κ₀−1)κ₀}{2} \} \), and

\[
μ_{jl} = \begin{cases} \sqrt{μ_j + μ_l} & \text{for } j < l ≤ κ₀; \\ \sqrt{μ_j} & \text{if } j ≤ κ₀ < l \text{ or if } j = l ≤ κ₀. \end{cases}
\]

(2.2)

2c. A family of affine hyperspaces \( L_ε \). Let us review some background
materials on the semi-linearity properties on \( Rat(\mathbb{B}^n, \mathbb{B}^N) \) (cf. [Hu03] and [HJX06]).
Let \( F ∈ Rat(\mathbb{B}^n, \mathbb{B}^N) \) with 1 ≤ κ₀ ≤ n − 2. Let \( E₀ \) be the proper complex variety
consisting of poles and the non-immers points of \( F \). We define

\( V_F := \{(Z,S_Z) ∈ (\mathbb{C}^n − E₀) × Gr_{n,k₀}(\mathbb{C})\} \), \( F \) is linear fractional when restricted to \( S_Z + Z \).

Here \( Gr_{n,k₀}(\mathbb{C}) \) is the Grassmannian manifold consisting of all \( k₀ := n−κ₀- \)
dimensional complex subspaces in \( \mathbb{C}^n \). Then \( V_F \) is a complex analytic variety with
the projection

\[
π : V_F → \mathbb{C}^n − E₀, \quad (Z,S_Z) ↦ Z
\]

is proper holomorphic. There is another proper complex variety \( E₁ ⊂ \mathbb{C}^n − E₀ \)
such that for any \( Z ∈ \mathbb{C}^n − E₀ ∪ E₁, π \) has a unique preimage in \( V_F \), i.e., for any
\( Z ∈ \mathbb{C}^n − E₀ ∪ E₁ \), there is a unique complex subspace \( S_Z \) of dimension \( k₀ \)
such that \( F \) is linear fractional when restricted to \( S_Z + Z \). In particular, if \( F \) satisfies
the normalization condition as in Theorem 2.2, the restriction of \( F \) on \( S_Z \) is affine
linear. Write \( V_F = \bigcup_j V_F^{(j)} \) for the irreducible decomposition of \( V_F \). Then there is
only one irreducible component, say \( V_F^{(1)} \), whose projection to \( \mathbb{C}^n − E₀ \) contains a
sufficiently small domain inside \( \mathbb{H}_n \) and has a small piece of \( ∂\mathbb{H}_n \) containing 0 as part
of its boundary. If necessary, we can assume that \( 0 ∉ E₁ \) and thus \( π \) is biholomorphic
near \((0,S₀) ∈ V_F \).

By [HJX06, p. 520], we can assume that for any \( ε = (ε₁,ε₂,...,ε_{κ₀})(∈ \mathbb{C}^{κ₀}) ≈ 0 \),
there is a unique affine subspace \( L_ε \) of codimension \( κ₀ \) defined by equations of the form:

\[
z_j = \sum_{i=κ₀+1}^{n−1} a_{ji}(ε)z_i + a_{jn}(ε)w + ε_j, \quad 1 ≤ j ≤ κ₀.
\]

(2.4)
such that \( F \) is a linear map on \( L_ε \), where \( a_{ji}(ε) \) are holomorphic functions in \( ε \) near
0 with \( a_{ji}(0,...,0) = 0 \) for all \( j \).
2d. Basic notation. Let $F = (f, \phi, g) \in \text{Rat}(\mathbb{B}^n, \mathbb{B}^N)$ be as in Theorem 2.2 with geometric rank $\kappa_0$. We have $N = \sharp(f) + \sharp(\phi) + \sharp(g)$ and $\sharp(f) = \sharp(S_0) + \sharp(S_1)$ where we denote by $\sharp(A)$ the number of elements of a set $A$, and $\sharp(f) = n - 1, \sharp(g) = 1$, $\sharp(S_0) = \frac{(2n-1-\kappa_0)\kappa_0}{2} = n\kappa_0 - \frac{1}{2}(\kappa_0+1)\kappa_0$, $\sharp(S_1) = N - n - \sharp(S_0)$.

We denote by $P^{(j,k)}(z,w)$ the polynomial of $(z,w)$ with degree $deg(z) = j$ and degree $deg(w) = k$ and denote $P^{(j,k)}(z)$ the coefficient of $w$. For example, $P^{(1,1)}(z,w) = \sum_{j=1}^\kappa a_j z^j w = P^{(1,1)}(z)w$, $P^{(1,1)}(z) = \sum_{j=1}^\kappa a_j z^j$.

For any rational holomorphic map $H = \frac{(P_1,...,P_m)}{Q}$ on $\mathbb{C}^n$, where $P_j, Q$ are holomorphic polynomials with $(P_1,...,P_m,Q) = 1$, the degree of $H$ is defined to be $\text{deg}(H) := \max\{\text{deg}(P_j), \text{deg}(Q), 1 \leq j \leq m\}$.

- The part $\phi$. Write $\phi = (\Phi_0, \Phi_1)$, $\Phi_0 = (\phi_{lk})(l,k) \in S_0$ and $\Phi_1 = (\phi_{lk})(l,k) \in S_1$. Here $\sharp(\Phi_0) = \sharp(S_0)$ and $\sharp(\Phi_1) = \sharp(S_1)$.

When $N = n + \frac{2n-\kappa_0-1}{2} \kappa_0$ and the geometric rank being $\kappa_0$, then

$$\sharp(\Phi_0) = (n-1) + \cdots + (n-\kappa_0) = \frac{2n-\kappa_0-1}{2} \kappa_0,$$

$$\sharp(\Phi_1) = N - n - \sharp(\Phi_0) = 0$$

Namely, there is no $\Phi$ term.

- The part $f_j^{(1,1)}(z)$. Write $f^{(1,1)}(z) = (f_1^{(1,1)}(z),...,f_{\kappa_0}^{(1,1)}(z),0,...,0)$. By Theorem 2.2, we have $f_j^{(1,1)}(z) = \frac{\mu_j}{2} z^j$, $\mu_j > 0$ for $1 \leq j \leq \kappa_0$.

- The part $\Phi_0^{(2,0)}(z)$. One important portion of $\Phi_0$ is the z-quadratic part (see Theorem 3.2): $\Phi_0^{(2,0)}(z) = \{\phi_{j,kl}(z) = \mu_{j,l} z_{j} z_l\}_{(j,l) \in S_0}$.

- The part $\Phi_0^{(1,1)}(z)$. Another portion of $\Phi_0$ is $\Phi_0^{(1,1)}(z)$ which are not mentioned in Theorem 2.2:

$$\Phi_0^{(1,1)}(z) = \sum_{j=1}^{\kappa_0} e_j z_j, \quad e_j \in \mathbb{C}^\sharp(S_0).$$

- The part $f^{(2,1)}(z)$. Write $f^{(2,1)}(z) = (f_1^{(2,1)}(z),...,f_{\kappa_0}^{(2,1)}(z),0,...,0)$. We see from ([HJY, (3.5)])) that

$$f_j^{(2,1)}(z) = -\xi_j.$$

Here

$$\xi_j = \Phi_0^{(2,0)} \cdot \mathbf{e}_j = \sum_{1 \leq k,l \leq \kappa_0} \phi_{kl}^{(2,0)} e_{j,kl} + \sum_{1 \leq k+\kappa_0 < a < n-1} \phi_{ka}^{(2,0)} e_{j,ka}.$$

2e. The components of $\Phi^{(3,0)}_1$.

**Lemma 2.3.** Let $\kappa_0 \geq 2$ and $(\kappa_0 + 1)n - \frac{\kappa_0(\kappa_0+1)}{2} \leq N \leq (\kappa_0 + 2)n - \kappa_0(\kappa_0 + 1) + \kappa_0 - 2$. Then

$$\Phi_1^{(3,0)}(z) = \left( \frac{2}{\sqrt{\mu_j} + \mu_l} \left( \sqrt{\frac{\mu_j}{\mu_l}} z_j \xi_l - \sqrt{\frac{\mu_l}{\mu_j}} z_l \xi_j \right), 0' \right)_{1 \leq j < l \leq \kappa_0}$$

$$|\phi^{(3,0)}(z)|^2 = 4 \left( \sum_{j=1}^{\kappa_0} \frac{1}{\mu_j} |\xi_j(z)|^2 \right) |z|^2.$$
The above result was proved in the third gap paper [HJY14], Corollary 3.4, under the condition \((\kappa_0 + 1)n - \kappa_0 \leq N \leq (\kappa_0 + 2)n - \kappa_0(\kappa_0 + 1) + \kappa_0 - 2\). By checking the proof in [HJY14], we find that this is still valid when the condition is replaced by

\[(\kappa_0 + 1)n - \frac{\kappa_0(\kappa_0 + 1)}{2} \leq N \leq (\kappa_0 + 2)n - \kappa_0(\kappa_0 + 1) + \kappa_0 - 2. \tag{2.9}\]

Geometrically we notice that the lower bound in (2.9) is the right end point of the gap interval \(I_{\kappa_0}\) and the upper bound in (2.9) is less than the left end point of the gap interval \(I_{\kappa_0+1}\).

When the condition in Theorem 1.1 or Theorem 1.2 is satisfied, the inequality \(N = (\kappa_0 + 1)n - \frac{\kappa_0(\kappa_0 + 1)}{2} \leq (\kappa_0 + 2)n - \kappa_0(\kappa_0 + 1) + \kappa_0 - 2\) holds because of the condition \(\frac{\kappa_0(\kappa_0 + 1)}{2} < n\). Lemma 2.3 can be applied to the maps in Theorem 1.1.

3. Properties of the semi-linear subspace. In this section, we will use the automorphisms of the balls to normalize the semi-linear subspace achieved in [Hu03]. More precisely, by (2.4), we will prove the following:

**Proposition 3.1.** Let \(F \in \text{Prop}_3(\mathbb{H}_n, \mathbb{H}_N)\) be as in Theorem 2.2, and the semi-linear subspace \(L_\epsilon\) be given by

\[z_j = \sum_{\alpha=\kappa_0+1}^{n-1} a_{j\alpha}(\epsilon)z_\alpha + a_{jn}(\epsilon)w + \epsilon_j, \quad 1 \leq j \leq \kappa_0. \tag{3.1}\]

Then there are automorphisms \(\sigma \in \text{Aut}(\mathbb{H}_n)\) and \(\tau \in \text{Aut}(\mathbb{H}_N)\) such that \(\hat{F} = \tau \circ F \circ \sigma\) still takes the form (2.1), and the semi-linear subspace still has the form (3.1). Moreover, we have

\[a_{1\alpha}^{(I_1)}(\epsilon) = 0 \text{ for } \kappa_0 + 1 \leq \alpha \leq n - 1 \text{ and } Re(a_{1n}^{(I_1)}(\epsilon)) = 0 \tag{3.2}\]

where we denote \(a_{jk}^{(1)}(\epsilon) = a_{jk}^{(I_1)}(\epsilon_1) + ... + a_{jk}^{(I_{\kappa_0})}(\epsilon_{\kappa_0})\).

**Proof.** Consider the image \(\hat{\sigma}_\epsilon(L_\epsilon) := \hat{\sigma}_\epsilon(L_\epsilon)\) given by

\[Z_j = \sum_{\alpha=\kappa_0+1}^{n-1} A_{j\alpha}(\epsilon)Z_\alpha + A_{jn}(\epsilon)W + \rho_j(\epsilon), \tag{3.3}\]

where the inverse of the the automorphism is given by

\[\hat{\sigma}_\epsilon^{-1}(Z, W) := \left(Z_1, ..., Z_{\kappa_0}, Z_{\kappa_0+1} + c_{\kappa_0+1}W, ..., Z_{n-1} + c_{n-1}W, W\right) \quad q_\epsilon \tag{3.4}\]

and \(q_\epsilon := 1 - 2i\vec{c} \cdot Z + (r - i|\vec{c}|^2)W\), where \(\vec{c} = (0, ..., 0, c_{\kappa_0+1}, ..., c_{n-1})\). Substituting (3.4) into (3.1), we obtain

\[Z_j = \sum_{\alpha=\kappa_0+1}^{n-1} a_{j\alpha}(\epsilon)(Z_\alpha + c_\alpha W) + a_{jn}(\epsilon)W + \epsilon_j q_\epsilon,\]
Combining this with (3.3), we get
\[ \sum_{\alpha=\kappa_0+1}^{n-1} A_{j\alpha}(\epsilon)Z_{\alpha} + A_{jn}(\epsilon)W + \rho_j(\epsilon) = \sum_{\alpha=\kappa_0+1}^{n-1} a_{j\alpha}(\epsilon)(Z_{\alpha} + c_{\alpha}W) + a_{jn}(\epsilon)W + \epsilon_j(1 - 2i\overline{c} \cdot Z + (r - i|c|^2)W). \] 

(3.5)

By considering the coefficients of $Z_j$, $\kappa_0 + 1 \leq j \leq n - 1$ and $W$ terms, we obtain
\[ A_{j\alpha}(\epsilon) = a_{j\alpha}(\epsilon) - \epsilon_j(2i\overline{c}_{\alpha}), \quad A_{jn}(\epsilon) = a_{jn}(\epsilon) + a_{jn}(\epsilon) + \epsilon_j(r - i|c|^2). \]

Thus we can choose $\overline{c}$ and $r$ such that (3.2) holds true. From [Hu03] Lemma 2.2, there is a corresponding $\tau \in \text{Aut}(\mathbb{H}_N)$ such that $\hat{F} = \tau \circ F \circ \sigma$ still takes the form (2.1).

Next, we give some applications of the semi-linear subspace defined as above, which will be used later:

Let $H$ be an affine linear function along $L_\epsilon$. Write
\[ H|_{L_\epsilon} = H\left( \sum_{\alpha=\kappa_0+1}^{n-1} a_{1\alpha}z_{\alpha} + a_{1n}w + \epsilon_1, \cdots, \sum_{\alpha=\kappa_0+1}^{n-1} a_{\kappa_0\alpha}z_{\alpha} + a_{\kappa_0n}w + \epsilon_{\kappa_0}, z_{\kappa_0+1}, \ldots, z_{n-1}, w \right). \]

Then we must have $\frac{\partial^2 H|_{L_\epsilon}}{\partial z_{\alpha}\partial w} \equiv 0$ and $\frac{\partial^2 H|_{L_\epsilon}}{\partial w^2} \equiv 0$ for $\kappa_0 + 1 \leq \alpha \leq n - 1$, from which we infer
\[ \frac{\partial^2 H|_{L_\epsilon}}{\partial z_{\alpha}\partial w} = \frac{\partial^2 H}{\partial z_{\alpha}\partial w} + \sum_{j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_j\partial w} a_{j\alpha} + \sum_{j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_j\partial z_{\alpha}} a_{jn} + \sum_{j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_j^2} a_{j\alpha}a_{jn} \]
\[ + \sum_{i<j, i,j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_i\partial z_j} (a_{i\alpha}a_{jn} + a_{j\alpha}a_{in}) = 0, \quad \text{at } (\epsilon, 0) \]

and
\[ \frac{\partial^2 H|_{L_\epsilon}}{\partial w^2} = \frac{\partial^2 H}{\partial w^2} + \sum_{j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_j\partial w} 2a_{jn} + \sum_{j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_j^2} a_{jn}^2 \]
\[ + \sum_{i<j, i,j=1}^{\kappa_0} \frac{\partial^2 H}{\partial z_i\partial z_j} 2a_{in}a_{jn} = 0, \quad \text{at } (\epsilon, 0). \]

Choosing $H = f_h$ for $1 \leq h \leq \kappa_0$ in (3.6), and collecting $\epsilon_j$, $1 \leq j \leq \kappa_0$ terms in the above equation, we get
\[ \frac{i}{2} \mu_h a_{h\alpha}(1) + \sum_{j=1}^{\kappa_0} f_h^{(I_1+I_0+I_n)}(\epsilon, 0) \epsilon_j = 0, \quad \text{at } (\epsilon, 0). \]

(3.8)

This, together with $h = 1$ in (3.2), gives $f_1^{(I_1+I_0+I_n)}(\epsilon, 0) = 0$. On the other hand, by (2.6), we have $f_1^{(2.1)}(z) = -\xi_1$. Recall that $\xi_1^{(I_1+I_n)} = \sqrt{\mu_1\epsilon_{1,1}}$ in (2.7). Thus we obtain $f_1^{(I_1+I_0+I_n)} = -\sqrt{\mu_1\epsilon_{1,1}}$. Hence
\[ \epsilon_{1,1} = 0 \text{ for } \kappa_0 + 1 \leq \alpha \leq n - 1. \]

(3.9)
Thus (4.1) implies
\[ i \mu_j a_j^{(1)}(\epsilon) + f_j^{(1,2)}(\epsilon, 0, \ldots, 0) = 0, \quad \phi^{(1,2)}(\epsilon, 0, \ldots, 0) + \sum_{j=1}^{\kappa_0} e_j a_j^{(1)}(\epsilon) = 0. \] (3.10)

4. Some applications of the Chern-Moser equation. Let \( F = (f, \phi, g) : \mathbb{H}_n \to \mathbb{H}_N \) with \( N = n + \frac{2n-\kappa_0-1}{2} \) and geometric rank \( \kappa_0 \). Moreover, \( F \) satisfies the normalization as in Theorem 2.2. We will derive some basic relations from the Chern-Moser equation, which is based on the calculations in Section 4 of [HJY14].

When \( N = n + \frac{2n-\kappa_0-1}{2} \kappa_0 \) and the geometric rank being \( \kappa_0 \), as shown in (2.5), we have \( \Phi_1 = 0 \). In particular, \( \Phi_1^{(3,0)} = 0 \), thus (2.8) gives
\[ \mu_j z_j \xi_l = \mu_l z_l \xi_j \quad \text{for} \ 1 \leq j, l \leq \kappa_0. \] (4.1)

Denote \( e_{i,jk} := e_{i,kj} \) and \( \phi_{j,k} := \phi_{k,j} \) when \( j > k \). Then for any \( j \) with \( 1 \leq j \leq \kappa_0 \),
\[ \xi_j = \Phi^{(2,0)} \cdot e_j = \sum_{(i,k) \in S_0, i \neq j, or, k \neq j} \phi_{ik}^{(2,0)} \cdot e_{ij,k} + \sum_{(i,k) \in S_0, i, k \neq j} \phi_{ik}^{(2,0)} \cdot e_{ij,k} \]
\[ = \sum_{j < k \leq n-1} \phi_{jk}^{(2,0)} \cdot e_{j,k} + \sum_{1 \leq i \leq j} \phi_{ij}^{(2,0)} \cdot e_{ij} + \sum_{(i,k) \in S_0, i, k \neq j} \phi_{ik}^{(2,0)} \cdot e_{ij,k} \]
\[ = \sum_{1 \leq i \leq j} \phi_{ij}^{(2,0)} \cdot e_{ij} + \sum_{(i,k) \in S_0, i, k \neq j} \phi_{ik}^{(2,0)} \cdot e_{ij,k}. \]

Observe that when \( j \neq l \), the terms \( z_l \sum_{(i,k) \in S_0, i, k \neq j} \phi_{ik}^{(2,0)} \cdot e_{ij,k} \) are not divided by \( z_j \). Thus (4.1) implies \( e_{ij,k} = 0 \) for \( (i, k) \in S_0, i, k \neq j \). Now (4.1) is of the form
\[ \mu_j z_j \sum_{1 \leq i \leq n-1} \phi_{il}^{(2,0)} e_{il} = \mu_l z_l \sum_{1 \leq i \leq n-1} \phi_{ij}^{(2,0)} e_{ij}. \] (4.2)

We can write the above identity as
\[ \mu_j z_j \sum_{1 \leq i \leq n-1} \phi_{il}^{(2,0)} e_{il} = \mu_l z_l \sum_{1 \leq i \leq n-1} \phi_{ij}^{(2,0)} e_{ij}. \]

Setting \( l = 1 \) and making use of (3.9), we obtain
\[ \mu_j z_j \sum_{1 \leq i \leq n-1} \phi_{ii}^{(2,0)} e_{i,i} = \mu_l z_l \sum_{1 \leq i \leq n-1} \phi_{ij}^{(2,0)} e_{ij}. \]

which implies \( e_{j,\alpha} = 0 \) for any \( \kappa_0 + 1 \leq \alpha \leq n - 1 \). Combining this with (2.6)-(2.7), we know \( f_h^{(I_1 + I_n + I_n)} = 0 \). Together with (3.8), we obtain
\[ a_{h,\alpha}^{(1)} = 0, \quad \forall 1 \leq h \leq \kappa_0, \ k_0 + 1 \leq \alpha \leq n - 1. \] (4.3)

The rest relations in (4.2) are
\[ \mu_j z_j \phi_{il}^{(2,0)} e_{il} = \mu_l z_l \phi_{ij}^{(2,0)} e_{ij} \quad \text{for} \ 1 \leq i, j, l \leq \kappa_0. \]

Namely, we obtain
\[ \mu_j \mu_i e_{i,l} = \mu_i \mu_j e_{j,i} \quad \text{for} \ 1 \leq i, j, l \leq \kappa_0. \] (4.4)
5. Proof of Theorem 1.1. This section is devoted to the proof of Theorem 1.1. The key point is to prove that the degree of the map in Theorem 1.1 is less than or equals to 2. Then we can apply Lebl’s Theorem [L11] to complete the proof of our main theorem.

**Lemma 5.1.** Keep the notations and assumptions in Theorem 1.1, then \( \deg(F) \leq 2 \).

**Proof.** From the basic Chern-Moser equation, we have

\[
\frac{g(z, w) - g(\overline{z}, \overline{w})}{2i} = f(z, w) \cdot \overline{f(z, w)} + \phi(z, w) \cdot \overline{\phi(z, w)}, \quad \forall \text{Im}(w) = |z|^2.
\]

By complexification, we write

\[
\frac{g(z, w) - g(\overline{z}, \overline{w})}{2i} = \sum_{l=1}^{n-1} f_l(z, w) \overline{f_l(z, w)} + \sum \phi_l(z, w) \overline{\phi_l(z, w)}, \quad \forall \frac{w - \eta}{2i} = z \cdot \chi.
\]

Applying \( \mathcal{L}_j := \frac{\partial}{\partial z_j} + 2i \chi_j \frac{\partial}{\partial w} \) for \( z = 0 \) and \( w = \eta = 0 \) to the both sides of the above identity, we obtain

\[
\frac{\mathcal{L}_j g(0, 0)}{2i} = \sum_{l=1}^{n-1} \mathcal{L}_j f_l(0, 0) \overline{f_l(0, 0)} + \sum \mathcal{L}_j \phi_l(0, 0) \overline{\phi_l(0, 0)}
\]

and

\[
\frac{\mathcal{L}_j \mathcal{L}_k g(0, 0)}{2i} = \sum_{l=1}^{n-1} \mathcal{L}_j \mathcal{L}_k f_l(0, 0) \overline{f_l(0, 0)} + \sum \mathcal{L}_j \mathcal{L}_k \phi_l(0, 0) \overline{\phi_l(0, 0)}.
\]

In terms of matrix, they take the form

\[
\begin{pmatrix}
\chi_1 \\
\vdots \\
\chi_{\kappa_0} \\
0 \\
\vdots \\
0
\end{pmatrix} = B
\begin{pmatrix}
f_1(\overline{\chi}, 0) \\
\vdots \\
\chi_{\kappa_0}(\overline{\chi}, 0) \\
\phi(\overline{\chi}, 0)
\end{pmatrix}
\]

(5.1)

where \( B(F) \) is a \( \mathbb{C}^{(2n - \kappa_0 + 1)} \times \mathbb{C}^{(2n - \kappa_0 + 1)} \) matrix:

\[
B(F) := \begin{pmatrix}
\mathcal{L}_j f_h & \mathcal{L}_j \phi_{hl} & \mathcal{L}_j \phi_{h\alpha} \\
\mathcal{L}_j \mathcal{L}_k f_h & \mathcal{L}_j \mathcal{L}_k \phi_{hl} & \mathcal{L}_j \mathcal{L}_k \phi_{h\alpha} \\
\mathcal{L}_j \mathcal{L}_\beta f_h & \mathcal{L}_j \mathcal{L}_\beta \phi_{hl} & \mathcal{L}_j \mathcal{L}_\beta \phi_{h\alpha}
\end{pmatrix}
\]

(5.2)

Write \( A_j = \frac{\mu_{j1} \varepsilon_{j1} j_1}{\mu_1} \), and let \( \overline{F} : \mathbb{C}^{n-1} \setminus \{1 - 2i \sum_{j=1}^{\kappa_0} A_j z_j = 0\} \to \mathbb{C}^{n-1} \) be defined as follows:

\[
\overline{f}_{\mu}(z) = z_\mu \text{ for } 1 \leq \mu \leq n - 1,
\]

\[
\overline{\phi}_{jj}(z) = \frac{\mu_{jj} z_j^2}{1 - 2i \sum_{j=1}^{\kappa_0} A_j z_j} \text{ for } 1 \leq j \leq \kappa_0.
\]

\[
\overline{\phi}_{jk}(z) = \frac{\mu_{jk} z_j z_k}{1 - 2i \sum_{j=1}^{\kappa_0} A_j z_j} \text{ for } 1 \leq j < k \leq \kappa_0.
\]

\[
\overline{\phi}_{j\alpha}(z) = \frac{\mu_{j\alpha} z_j z_\alpha}{1 - 2i \sum_{j=1}^{\kappa_0} A_j z_j} \text{ for } 1 \leq j \leq \kappa_0 < \alpha \leq n - 1.
\]

(5.3)
We claim that $F(z, 0) = \overline{F}$, which follows from the following identity:

\[
\begin{pmatrix}
\chi_1 \\
\vdots \\
\chi_{\kappa_0} \\
0 \\
\vdots \\
0
\end{pmatrix} = B \begin{pmatrix}
\overline{f_1(\overline{x})} \\
\vdots \\
\overline{f_{\kappa_0}(\overline{x})} \\
\overline{\phi(\overline{x})}
\end{pmatrix},
\tag{5.4}
\]

In fact, once (5.4) is achieved, we infer from (5.1) that

\[
B \begin{pmatrix}
\overline{f_1(\overline{x}, 0)} - \overline{f_1(\overline{x})} \\
\vdots \\
\overline{f_{\kappa_0}(\overline{x}, 0)} - \overline{f_{\kappa_0}(\overline{x})} \\
\overline{\phi(\overline{x}, 0)} - \overline{\phi(\overline{x})}
\end{pmatrix} = 0.
\]

Notice that

\[
B = \text{diag}(1, \cdots, 1, A_1, \cdots, A_{\kappa_0}, B_1, \cdots, B_{\kappa_0}) + O(|\chi|).
\]

Here

\[
A_j = (2\sqrt{\mu_j}, \sqrt{\mu_j + \mu_j + 1}, \cdots, \sqrt{\mu_j + \mu_{\kappa_0}}) \in \mathbb{C}^{\kappa_0 - j + 1},
\]

\[
B_j = (\sqrt{\mu_j}, \cdots, \sqrt{\mu_j}) \in \mathbb{C}^{n - \kappa_0}.
\]

Thus $B$ is nonsingular and we derive the claim $F(z, 0) = \overline{F}(z)$. Hence $\text{deg}(F(z, 0)) \leq 2$. Replacing $F$ by $F_p^{***}$ for any $p \in \partial\mathbb{H}_\eta$ near the origin, we can show $\text{deg}(F_p^{***}(z, 0)) \leq 2$ in a similar manner. By [HJ01, Section 5], we have that $\text{deg}(F) \leq 2$.

The identity (5.4) follows from the following direct computations:

- **Calculate** $(L_h H)(0, 0)$ with $1 \leq h \leq \kappa_0$ for $H = f_j, \phi_{jk}$. At the point $(0, 0)$, we have

  \[
  (L_h f_j)(0, 0) = \delta^j_h,
  \]

  \[
  (L_h \phi_{jk})(0, 0) = 0 \text{ for } (j, k) \in S_0.
  \]

  Then

  \[
  \sum_{j=1}^{\kappa_0} (L_h f_j)(0, 0) \cdot \overline{f_j(\overline{x})} + \sum_{(\mu, \nu) \in S_0} (L_h \phi_{\mu\nu})(0, 0) \cdot \overline{\phi_{\mu\nu}(\overline{x})} = \sum_{j=1}^{\kappa_0} \delta^j_h \cdot \chi_j = \chi_h. \tag{5.5}
  \]

- **Calculate** $(L_h^2 H)(0, 0)$ (1 \leq h \leq \kappa_0) for $H = f_j, \phi_{jk}$. A direct computation shows that $L_h^2 = \frac{\partial^2}{\partial z^2} + 4i\chi_h \frac{\partial^2}{\partial z \partial w} + (2i\chi_h)^2 \frac{\partial^2}{\partial w^2}$. At the point $(0, 0)$, we have

  \[
  (L_h^2 f_j)(0, 0) = 4i\chi_h \cdot \frac{i}{2} \mu_j \cdot \delta^j_h = -2\delta^j_h \mu_j \chi_j,
  \]

  \[
  (L_h^2 \phi_{hh})(0, 0) = 2\sqrt{\mu_h} + 4i\chi_h \epsilon_{h,hh},
  \]

  \[
  (L_h^2 \phi_{hj})(0, 0) = 4i\chi_h \epsilon_{h,j} \text{ for } 1 \leq j \leq n - 1, \ j \neq h,
  \]

  \[
  (L_h^2 \phi_{jk})(0, 0) = 0 \text{ for } j, k \neq h.
  \]
Setting $l = 1$ in (4.4), we obtain
\[\mu_h e_h, h_j = \frac{\mu_h}{\mu_1} \mu_{1j} e_{1,1j} = \mu_h A_j. \quad (5.6)\]

Thus we get
\[
\begin{align*}
\sum_{j=1}^{\kappa_0} (\mathcal{L}_h f_j)(0,0) \cdot \overline{f_j}(\chi) + \sum_{(\mu,\nu) \in S_0} (\mathcal{L}_h \phi_{\mu \nu})(0,0) \cdot \overline{\phi_{\mu \nu}(\chi)} \\
= (-2\mu_h \chi_h) \cdot \chi_h + \left(2\sqrt{\mu_h} + 4i\chi_h e_{h,hh}\right) \cdot \frac{\mu_h \chi_h^2}{1 + 2i\sum_{j=1}^{\kappa_0} A_j \chi_j} \\
+ \sum_{j \neq h, 1 \leq j \leq n-1} 4i\chi_h e_{h,hj} \cdot \frac{\mu_h \chi_h \chi_j}{1 + 2i\sum_{j=1}^{\kappa_0} A_j \chi_j} \\
= \frac{1}{1 + 2i\sum_{j=1}^{\kappa_0} A_j \chi_j} \left( -2\mu_h \chi_h^2 \cdot 2i \sum_{j=1}^{\kappa_0} A_j \chi_j + 4i\chi_h \sum_{j=1}^{\kappa_0} \mu_h e_{h,hj} \chi_j \right) = 0. \quad (5.7)
\end{align*}
\]

- Calculate $(\mathcal{L}_h \mathcal{L}_l H)(0,0)$ $(1 \leq h < l \leq \kappa_0)$ for $H = f_j, \phi_{jk}$. A direct computation shows that $\mathcal{L}_h \mathcal{L}_l = \frac{\partial^2}{\partial z_h \partial z_l} + 2i\chi_l \frac{\partial^2}{\partial z_h \partial w} + 2i\chi_h \frac{\partial^2}{\partial z_l \partial w} - 4\chi_h \chi_l \frac{\partial^2}{\partial w^2}$. At the point $(0,0)$, we have
\[
(\mathcal{L}_h \mathcal{L}_l f_j)(0,0) = 2i\chi_l \cdot \frac{i}{2} \mu_j \delta_l^j + 2i\chi_h \cdot \frac{i}{2} \mu_l \delta_j^l = -\mu_j \chi_l \delta_l^j - \mu_l \chi_h \delta_j^l,
\]
\[
(\mathcal{L}_h \mathcal{L}_l \phi_{hl})(0,0) = \mu_{hl} + 2i\chi_l \cdot e_{h,hl} + 2i\chi_h \cdot e_{l,hl},
\]
\[
(\mathcal{L}_h \mathcal{L}_l \phi_{jk})(0,0) = 2i\chi_l \cdot e_{h,jk} + 2i\chi_h \cdot e_{l,jk} \text{ for } (j,k) \neq (h,l) \text{ or } (l,h).
\]

Combining this with (5.6), we get
\[
\begin{align*}
\sum_{j=1}^{\kappa_0} (\mathcal{L}_h \mathcal{L}_l f_j)(0,0) \cdot \overline{f_j}(\chi) + \sum_{(\mu,\nu) \in S_0} (\mathcal{L}_h \mathcal{L}_l \phi_{\mu \nu})(0,0) \cdot \overline{\phi_{\mu \nu}(\chi)} \\
= -\mu_h \chi_l \cdot \chi_h - \mu_l \chi_h \cdot \chi_l + (\mu_{hl} + 2i\chi_l \cdot e_{h,hl} + 2i\chi_h \cdot e_{l,hl}) \cdot \frac{\mu_h \chi_h \chi_l}{1 + 2i\sum_{j=1}^{\kappa_0} A_j \chi_j} \\
+ \sum_{1 \leq j < k \leq \kappa_0, (j,k) \neq (h,l)} (2i\chi_l \cdot e_{h,jk} + 2i\chi_h \cdot e_{l,jk}) \cdot \frac{\mu_{jk} \chi_j \chi_k}{1 + 2i\sum_{j=1}^{\kappa_0} A_j \chi_j} \\
= \frac{1}{1 + 2i\sum_{j=1}^{\kappa_0} A_j \chi_j} \left( - (\mu_h + \mu_l) \chi_h \chi_l \cdot 2i \sum_{j=1}^{\kappa_0} A_j \chi_j + 2i\chi_l \sum_{1 \leq k \leq \kappa_0} e_{h,hl} \mu_{hk} \chi_h \chi_k \\
+ 2i\chi_h \sum_{1 \leq j \leq \kappa_0} e_{l,jh} \mu_{lj} \chi_j \chi_l \right) = 0. \quad (5.8)
\end{align*}
\]

- Calculate $(\mathcal{L}_h \mathcal{L}_l H)(0,0)$ $(1 \leq h \leq \kappa_0 < \alpha \leq n-1)$ for $H = f_i, \phi_{jl}$. A direct computation shows that $\mathcal{L}_h \mathcal{L}_\alpha = \frac{\partial^2}{\partial z_h \partial z_\alpha} + 2i\chi_\alpha \frac{\partial^2}{\partial z_h \partial w} + 2i\chi_h \frac{\partial^2}{\partial z_\alpha \partial w} + 2i\chi_h \cdot 2i\chi_\alpha \frac{\partial^2}{\partial w^2}$. At the point $(0,0)$, we have
\[
(\mathcal{L}_h \mathcal{L}_\alpha f_j)(0,0) = 2i\chi_\alpha \cdot \frac{i}{2} \mu_j \delta_\alpha^j = -\mu_h \chi_\alpha \delta_h^j,
\]
\[
(\mathcal{L}_h \mathcal{L}_\alpha \phi_{jk})(0,0) = 2i\chi_\alpha e_{h,jk},
\]
\[
(\mathcal{L}_h \mathcal{L}_\alpha \phi_{j\beta})(0,0) = \mu_{h\alpha} \delta_j^\beta \delta_\alpha^\beta.
\]
We get
\[ \sum_{j=1}^{\kappa_0} (\mathcal{L}_h \mathcal{L}_f f_j)(0, 0) \cdot \overline{f_j(x)} + \sum_{(\mu, \nu) \in S_0} (\mathcal{L}_h \mathcal{L}_o \phi_{\mu \nu})(0, 0) \cdot \overline{\phi_{\mu \nu}(x)} \]
\[ = -\mu_h \chi \cdot \chi_h + \sum_{1 \leq j, k \leq \kappa_0} 2i \chi_h \cdot e_{h, jk} \cdot \frac{\mu_{jk} \chi \chi_k}{1 + 2i \sum_{j=1}^{\kappa_0} A_j z_j} + \mu_{h0} \frac{\mu_{h0} \chi \chi_0}{1 + 2i \sum_{j=1}^{\kappa_0} A_j z_j} \]
\[ = \frac{1}{1 + 2i \sum_{j=1}^{\kappa_0} A_j z_j} (\mu_h \chi \chi_0 - 2i \sum_{j=1}^{\kappa_0} A_j z_j + 2i \chi_0 \chi h) = 0. \]
(5.9)

By all of the above, (5.4) is proved. This also finishes the proof of Lemma 5.1. □

Now we are in a position to complete the proof of Theorem 1.1.

**Proof of Theorem 1.1.** By Lemma 5.1, the degree of the map in Theorem 1.1 is at most 2. By Lebl’s Theorem [L11, Theorem 1.5], it must have the following form:
\[ (\sqrt{t_1} z_1, \sqrt{t_2} z_2, ..., \sqrt{t_n} z_n, \sqrt{1 - t_1 z_1^2}, \sqrt{1 - t_2 z_2^2}, ..., \sqrt{1 - t_n z_n^2}, \sqrt{2 - t_i - t_j z_i z_j})_{i \neq j} \]
(5.10)
where \( 0 \leq t_1 \leq ... \leq t_n \leq 1, (t_1, t_2, ..., t_n) \neq (1, 1, ..., 1). \)

Suppose that \( t_j = 0 \) for \( 1 \leq j \leq h, t_j = 1 \) for \( k + 1 \leq j \leq n \) and \( t_j \in (0, 1) \) for \( h \leq j \leq k. \)
Here \( 0 \leq h \leq k \leq n \) and \( h = 0 \) means that there is no \( t_j = 0. \) Then it has the following form:
\[ (\sqrt{t_{h+1}} z_{h+1}, ..., \sqrt{t_n} z_n, \sqrt{1 - t_1 z_1^2}, ..., \sqrt{1 - t_k z_k^2}, (\sqrt{2 - t_i - t_j z_i z_j})_{1 \leq i \leq k, 1 \leq j \leq n, i < j}) \]
(5.11)
Notice that this map is linear on \( z_j = c_j \) for \( 1 \leq j \leq k \) and can not be linear on any lower dimensional linear subspace. By [Hu03], the geometric rank of this map is \( k. \) Thus we have \( k = \kappa_0. \) By counting the dimension of the map, we have
\[ (n - h) + k + \frac{n(n - 1)}{2} - \frac{(n - k)(n - k - 1)}{2} = n + \frac{2n - \kappa_0 - 1}{2} \kappa_0. \]
Then we get \( h = k = \kappa_0. \) In this case, (5.11) is exactly the generalized Whitney map defined by (1.3)-(1.4). This completes the proof of Theorem 1.1. □

**REFERENCES**


[JX04] S. Ji and D. Xu, Maps between $\mathbb{B}^n$ and $\mathbb{B}^N$ with geometric rank $k_0 \leq n - 2$ and minimum $N$, Asian J. Math, 8 (2004), pp. 233–258.

