Complex Analysis and Complex Geometry

Shanyu Ji

December 5, 2008

\[1\] Lecture notes for 2008 fall.
## Contents

0.1 Introduction ................................................................. 4

1 Riemann-Roch Theorem .................................................... 5
  1.1 Complex Manifolds ..................................................... 5
  1.2 Holomorphic tangent bundles and cotangent bundles ............. 7
  1.3 Meromorphic functions on Riemann surfaces ...................... 13
  1.4 Complex projective space $\mathbb{C}P^n$ ............................ 15
  1.5 Complex Tori .......................................................... 16
  1.6 Divisors .............................................................. 23
  1.7 Statement of the Riemann-Roch theorem .......................... 29
  1.8 Bergmann metric on $M$ with $g(M) > 0$ ......................... 34
  1.9 More applications of the Riemann-Roch theorem ............... 35

2 Proof of the Riemann-Roch Theorem .................................. 41
  2.1 Holomorphic Line Bundles .......................................... 41
  2.2 Operators of line bundles ........................................... 46
  2.3 Sheaves ................................................................... 48
  2.4 Čech cohomology theory .............................................. 54
  2.5 De Rham theorem ....................................................... 60
  2.6 Dolbeault theorem ...................................................... 63
  2.7 Hermitian metric and connection .................................. 65
  2.8 Statement of Hodge Theorem ......................................... 68
  2.9 Serre Duality Theorem ............................................... 73
  2.10 Proof of the Riemann-Roch theorem ................................ 75

3 Proof of Hodge Theorem .................................................. 79
  3.1 Sobolev spaces .......................................................... 79
  3.2 Three theorems ......................................................... 83
  3.3 Garding Inequality ..................................................... 87
## Contents

3.4 Rellich Lemma and the proof of Theorem III ........................................ 91  
3.5 Sobolev Lemma ................................................................. 92  
3.6 Proof of Theorem II ........................................................... 95  

4 Positive Closed Currents Theory .................................................. 101  
4.1 Plurisubharmonic functions ..................................................... 101  
4.2 Positive closed currents ......................................................... 106  
4.3 Poincare-Lelong formula ....................................................... 113  
4.4 Wedge product of currents ..................................................... 117  
4.5 Lelong numbers ................................................................. 121  
4.6 Singular metric on line bundle ................................................ 126  

5 Appendix: Integral Theory .......................................................... 129  
5.1 Tensor product of vector spaces .............................................. 129  
5.2 (s,r)-tensors ................................................................. 132  
5.3 Symmetric and antisymmetric tensors ....................................... 136  
5.4 Exterior Algebra ............................................................... 138  
5.5 Tensor Fields ................................................................. 143  
5.6 Exterior differential forms .................................................... 145  
5.7 Integrals of exterior differential forms ...................................... 148  
5.8 Stokes Theorem ............................................................... 150  

6 Homework and solutions ............................................................ 153  
6.1 HW 1 ................................................................. 153  
6.2 HW 2 ................................................................. 154  
6.3 HW 3 ................................................................. 155  
6.4 HW 4 ................................................................. 157  
6.5 HW 5 ................................................................. 158  
6.6 HW 6 ................................................................. 159  

## 0.1 Introduction

We want to study the geometric, topological and cohomological properties of (compact) complex Kahler manifolds.  
To make easy to understand the material, we prefer to focus on Riemann surface. Much of the material could be extended to multivariable case.
Chapter 1

Riemann-Roch Theorem

1.1 Complex Manifolds

Definition 1.1 An n-dimensional complex manifold \( M \) in a Hausdorff paracompact topological space with local coordinates covering \( \{ U_i, \Phi_i \} \) such that

1. Each \( U_i \) is an open subset of \( M \) and \( \bigcup U_i = M \),
2. \( \Phi_i : U_i \to U_i^0 \) is a homeomorphism from \( U_i \) to an open subset \( U_i^0 \) of \( \mathbb{C}^n \),
3. If \( U_i \cap U_j \neq \emptyset \), then the transition function \( \Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \to \Phi_i(U_i \cap U_j) \) is holomorphic.

In the above definition, \( U_i \) is called a coordinate chart (or a coordinate system) and \( \Phi_i \) is called a holomorphic coordinate map.  

Definition 1.2 A one dimensional complex manifold is called a Riemann Surface and a two dimensional complex manifold is called a complex analytic surface.  

Definition 1.3 (1) \( f : M \to \mathbb{C} \) is called a holomorphic function if for each \( i \), \( f \circ \Phi_i^{-1} : \Phi_i(U_i) \to \mathbb{C} \) is a holomorphic function.

\(^1\)Here, we recall a continuous function \( h \) over an open subset of \( \mathbb{C}^n \) is said to be holomorphic if for any \( p \in \mathbb{C}^n \), \( h(z) \) near \( p \) has a uniformly convergent power series expansion in a small neighborhood of \( p \):

\[
h(z) = \sum_{|a| \geq 0} a_a (z - p)^a.
\]

\(^2\)In real geometry, by a surface we mean a 2 real dimensional manifold. In complex geometry, a complex surface means a 2 complex dimensional manifold while a Riemann surface means a 1 complex dimensional surface.
CHAPTER 1. RIEMANN-ROCH THEOREM

(2): A continuous map \( f : M^m \to N^n \), where \( M \) and \( N \) are complex manifolds with coordinate coverings \( \{ U_i, \Phi_i \} \) \( \{ V_j, \Psi_j \} \) respectively, is called holomorphic if

\[
\Psi_j \circ f \circ \Phi_i^{-1} : \Phi_i(U_i \cap f^{-1}(V_j)) \subset \mathbb{C}^n \to \Phi_j(V_j)
\]
is a holomorphic map.

For a map \( F = (f_1, \ldots, f_m) : \mathbb{C}^n \to \mathbb{C}^m \), its differential \( dF \) is a linear operator from \( T_a \mathbb{C}^n \) to \( T_{F(a)} \mathbb{C}^m \) given by

\[
\begin{bmatrix}
\frac{\partial}{\partial z_1} \\
\vdots \\
\frac{\partial}{\partial z_n}
\end{bmatrix} \mapsto
\begin{bmatrix}
\frac{\partial f_1}{\partial z_1} & \ldots & \frac{\partial f_1}{\partial z_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_m}{\partial z_1} & \ldots & \frac{\partial f_m}{\partial z_n}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial}{\partial z_1} \\
\vdots \\
\frac{\partial}{\partial z_n}
\end{bmatrix}
\]

The matrix \( F'(a) := \left( \frac{\partial f_j}{\partial z_i} \right) \) is called the Jacobian matrix. When \( n = m \), it is a square matrix.

If we regard \( \mathbb{C}^n = \mathbb{R}^{2n} \), \( \mathbb{C}^m = \mathbb{R}^{2m} \), \( z_j = x_j + iy_j \) and \( f_j = u_j + iv_j \), then the real Jacobian matrix is

\[
J_{\mathbb{R}}(F) := 
\begin{bmatrix}
\frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial y_1} & \ldots & \frac{\partial u_1}{\partial x_n} & \frac{\partial u_1}{\partial y_n} \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{\partial u_m}{\partial x_1} & \frac{\partial u_m}{\partial y_1} & \ldots & \frac{\partial u_m}{\partial x_n} & \frac{\partial u_m}{\partial y_n}
\end{bmatrix}
\]

Lemma 1.4 Any complex manifold is orientable. 3

By definition, a smooth manifold \( M \) with local coordinate covering \( \{ U_i, \Phi_i \} \) is called orientable if the Jacobian determinant

\[
\text{Jac}(\Phi_i \circ \Phi_j^{-1}) > 0 \quad \text{for all} \ i, j \quad \text{(the matrix is invertible.)}
\]

Let us prove the lemma for the case of Riemann surface. The proof for higher dimensional case is similar.

If \( M \) is a Riemann surface with local chart \( \{ U_i, \Phi_i \} \) then \( \Phi_i \circ \Phi_j^{-1} \) is a holomorphic map from

\[
\Phi_j(U_i \cap U_j) \to \Phi_i(U_i \cap U_j)
\]

\[
z \mapsto g(z)
\]

3In Differential Geometry, on an n dimensional manifold, integrals for differential forms can be defined if the manifold is orientable. For example, the Möbius strip.
Write \( z = x + iy, \ u = \text{Re} \ g, \ v = \text{Im} \ g \). Then by the Cauchy-Riemann equation

\[
\text{Jac}(\Phi_i \circ \Phi^{-1}_j) = \det \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = u_x^2 + u_y^2 = v_x^2 + v_y^2 = |g'|^2 = |(\Phi_i \circ \Phi^{-1}_j)'|^2 \neq 0,
\]

for \( \Phi_i \circ \Phi^{-1}_j \) is a biholomorphic map.

For higher dimensional case, we claim

\[
\det_{\mathbb{R}} F(z) = |\det F'(z)|^2 \geq 0.
\]

In fact, after a permutation of the rows and columns, one can write

\[
\det J_{\mathbb{R}} F = \det \begin{pmatrix} \frac{\partial f_k}{\partial x_j} & \frac{\partial f_k}{\partial y_j} \\ \frac{\partial f_k}{\partial x_j} & \frac{\partial f_k}{\partial y_j} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_k}{\partial x_j} + i \frac{\partial f_k}{\partial x_j} & \frac{\partial f_k}{\partial y_j} - i \frac{\partial f_k}{\partial y_j} \\ \frac{\partial f_k}{\partial y_j} - i \frac{\partial f_k}{\partial y_j} & \frac{\partial f_k}{\partial x_j} + i \frac{\partial f_k}{\partial x_j} \end{pmatrix}
\]

(Add \( i \) times the bottom blocks to the top
and use the Cauchy–Riemann equation is used)

\[
= \begin{pmatrix} \frac{\partial f_k}{\partial x_j} & 0 \\ * & \frac{\partial f_k}{\partial x_j} \end{pmatrix}
\]

(subtract \( i \) times the left blocks from the right side)

Here we used \( \frac{\partial f}{\partial z} = \frac{\partial f}{\partial x} \) for holomorphic function \( f \).

### 1.2 Holomorphic tangent bundles and cotangent bundles

Let \( D \subset \mathbb{C}^1 \) be a domain. For any point \( p \in D \), we have four types of the tangent spaces of \( D \) at \( p \).

(a): Real tangent space of \( D \) at \( p \rightarrow T_p D \).

\( T_p D : = \) the set of all 1st order linear differential operations without constant term acting on germs of smooth functions at \( p \) where all are assumed in the real category, i.e.,

\[
T_p D = \text{Span}_{\mathbb{R}} \left\{ \frac{\partial}{\partial x}|_p, \frac{\partial}{\partial y}|_p \right\} = \left\{ a \frac{\partial}{\partial x}|_p + b \frac{\partial}{\partial y}|_p, \ a, b \in \mathbb{R} \right\}.
\]

Here we use \( z = x + iy \) for the coordinates of \( \mathbb{C}^1 \).
(b): Complexified tangent space of $D$ at $p \rightarrow \mathbb{C}T_p D$.

\[ \mathbb{C}T_p D = \{ \text{all 1st order linear differential operators (without constant term) acting on germs of smooth functions at } p, \text{ where all are assumed in the } \mathbb{C}-\text{category} \} \]

\[ = \text{Span}_\mathbb{C} \left\{ \frac{\partial}{\partial x}|_p, \frac{\partial}{\partial y}|_p \right\}. \]

Write \( \frac{\partial}{\partial z} = \frac{1}{2} \left( \frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right), \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right), i = \sqrt{-1}. \) Then

\[ \mathbb{C}T_p D = \text{Span}_\mathbb{C} \left\{ \frac{\partial}{\partial z}|_p, \frac{\partial}{\partial \bar{z}}|_p \right\}. \]

(c): Holomorphic tangent space of $D$ at $p \rightarrow T_p^{(1,0)} D$.

\[ T_p^{(1,0)} D \subset \mathbb{C}T_p D \text{ and } T_p^{(1,0)} D = \text{Span}_\mathbb{C} \left\{ \frac{\partial}{\partial \bar{z}}|_p \right\}. \]

(d): Conjugate holomorphic tangent space of $D$ at $p$.

\[ T_p^{(0,1)} D \subset \mathbb{C}T_p D \text{ and } T_p^{(0,1)} D = \text{Span}_\mathbb{C} \left\{ \frac{\partial}{\partial z}|_p \right\}. \]

Apparently, we have

\[ \mathbb{C}T_p D = T_p^{(1,0)} D \oplus T_p^{(0,1)} D, \quad T_p^{(1,0)} D = \overline{T_p^{(0,1)} D}, \quad T_p^{(0,1)} D \cap T_p^{(0,1)} D = \{0\}. \]

Here for $\nu \in \mathbb{C}T_p D$, $\overline{\nu}$ is defined such that $\overline{\nu}(h) = \nu(\overline{h})$ for any $h \in C^\infty(D)$.

From the above four spaces, we have the following four types of vector bundles ($C^\infty$ vector bundle):

\[ TD = \Pi_{p \in D} T_p D \xrightarrow{\pi} D \rightarrow \text{real tangent bundle of } D, \]

\[ \mathbb{C}TD = \Pi_{p \in D} \mathbb{C}T_p D \xrightarrow{\pi} D \rightarrow \text{complexified tangent bundle of } D, \]

\[ T_p^{(1,0)} D = \Pi_{p \in D} T_p^{(1,0)} D \xrightarrow{\pi} D \rightarrow \text{holomorphic tangent bundle of } D, \]

\[ T_p^{(0,1)} D = \Pi_{p \in D} T_p^{(0,1)} D \xrightarrow{\pi} D \rightarrow \text{conjugate holomorphic tangent bundle of } D. \]

**Remark**
1. A (real or complex) vector bundle \( V \) over a manifold \( X \) is defined by

\[
\cup_{p \in X} V_p
\]

where \( V_p \) are vector spaces of the same (real or complex) dimension \( r \) that are smooth as \( p \) varies. The \( \dim V_p = r \) is called the rank of the bundle and \( V_p \) is called the fiber of \( V \) over \( p \).

2. For \( TD \), all fibers are isomorphic to \( \mathbb{R}^2 \)

\[
\begin{align*}
T_p D & \cong \mathbb{R}^2 \\
a \frac{\partial}{\partial x} |_p + b \frac{\partial}{\partial y} |_p & \mapsto (a, b)
\end{align*}
\]

so that the bundle

\[
TD \cong D \times \mathbb{R}^2, \quad a \frac{\partial}{\partial x} |_p + b \frac{\partial}{\partial y} |_p \mapsto (p, (a, b))
\]

is of real rank 2. A bundle like this is called a trivial bundle over \( D \).

3. For \( CTD \), all fibers are isomorphic to \( \mathbb{C}^2 \)

\[
\begin{align*}
CT_p D & \cong \mathbb{C}^2 \\
a \frac{\partial}{\partial z} |_p + b \frac{\partial}{\partial \bar{z}} |_p & \mapsto (a, b)
\end{align*}
\]

so that the bundle

\[
CTD \cong D \times \mathbb{C}^2, \quad a \frac{\partial}{\partial z} |_p + b \frac{\partial}{\partial \bar{z}} |_p \mapsto (p, (a, b))
\]

is of complex rank 2. \( CTD \) is also a trivial bundle.

4. For \( T^{(1,0)}D \), all fibers are isomorphic to \( \mathbb{C} \)

\[
\begin{align*}
T_p^{(1,0)} D & \cong \mathbb{C} \\
a \frac{\partial}{\partial z} |_p & \mapsto a
\end{align*}
\]

so that the bundle

\[
T^{(1,0)} D \cong D \times \mathbb{C}, \quad a \frac{\partial}{\partial z} |_p \mapsto (p, a)
\]

is of complex rank 1. \( T^{(1,0)} D \) is also a trivial bundle.

Similar consideration for \( T^{(0,1)} D \).
5. Since $T^{(1,0)}D$ is a vector bundle of rank 1, we call such bundle a line bundle.

Let $V = \Pi_{p \in D} V_p$ be one of the above bundles. A section (or a cross section) of $V$ is a map which assigns a point $p \in D$ to an element $s_p \in V_p$.

**Example 1.5** $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ are cross sections of $TD$ or $\mathbb{C}TD$:

$$\frac{\partial}{\partial x} : p \mapsto \frac{\partial}{\partial x} |_p, \quad \frac{\partial}{\partial y} : p \mapsto \frac{\partial}{\partial y} |_p.$$ 

**Definition 1.6** A cross section $\mathcal{L}$ of $TD$ is called a real vector field of $C^k$ smooth if $\mathcal{L}|_p = a(p, \bar{p}) \frac{\partial}{\partial x}|_p + b(p, \bar{p}) \frac{\partial}{\partial y}|_p$ with $a, b \in C^k(D)$.

When $k = \infty$, we simply say that $\mathcal{L}$ is a smooth vector field. Then $\mathcal{L}$ defines a map from $C^\infty(D)$ to $C^\infty(D)$, called a first order linear differential operator without constant terms over $D$. When $k = \omega$, we simply say that $\mathcal{L}$ is a real valued real analytic vector field. Similarly, we can define the notion of complex vector field of $C^k$ smooth.

A vector field $\mathcal{L}$ is called of type $(1,0)$ if $\mathcal{L} = a \frac{\partial}{\partial z}$. A vector field $\mathcal{L}$ is called of type $(0,1)$ if $\mathcal{L} = a \frac{\partial}{\partial \bar{z}}$. A vector field $\mathcal{L}$ is called a holomorphic vector field if $\mathcal{L} = a \frac{\partial}{\partial z}$ with $a \in \text{Hol}(D)$.

**Remarks**

1. Any continuous (resp. holomorphic function) over a complex manifold $X$ can be regarded as a continuous (resp. holomorphic) section of the trivial bundle $X \times \mathbb{C}$.

2. We can draw picture for a vector field over $\mathbb{R}^2$ or $\mathbb{R}^3$.

3. Let $f : M \to N$ be a differentiable map where $M, N$ are complex manifold. For any point $p \in M$, it induces a linear transformation

$$f_* : T_pM \to T_{f(p)}N, \quad a \frac{\partial}{\partial x}|_p + b \frac{\partial}{\partial y}|_p \mapsto f_*(a \frac{\partial}{\partial x}|_p + b \frac{\partial}{\partial y}|_p),$$

where for any germ of smooth function $g$ defined in a neighborhood of $f(p)$, we define

$$f_*(a \frac{\partial}{\partial x}|_p + b \frac{\partial}{\partial y}|_p)(g) = (a \frac{\partial}{\partial x}|_p + b \frac{\partial}{\partial y}|_p)(g \circ f).$$

4. Notice that in general we cannot define a pull-back map from $T_{f(p)}N$ to $T_pM$. For a vector field $L$, we cannot define the “pushforward” $f_*L$ or the “pull-back” $f^*L$.

However, for a differential form $\omega$, we can always define the “pull-back” $f^*\omega$. 
We next consider the dual space of the above tangent space.

\[ T^*D \longrightarrow \text{real cotangent bundle,} \]

\[ T^*D = \bigoplus_{p \in D} T^*_pD \quad \text{and} \quad T^*_pD = \text{Span}_{\mathbb{C}} \{ dx|_p, dy|_p \}, \]

where

\[ \langle dx|_p, \frac{\partial}{\partial x}|_p \rangle = 1, \quad \langle dx|_p, \frac{\partial}{\partial y}|_p \rangle = 0, \quad \langle dy|_p, \frac{\partial}{\partial x}|_p \rangle = 0, \quad \langle dy|_p, \frac{\partial}{\partial y}|_p \rangle = 1. \]

A cross section \( \alpha \) of \( T^*D \) is called a \textit{real 1-form} if \( \alpha = a(x,y)dx + b(x,y)dy \). When \( a, b \in C^k(D) \), we say \( \alpha \) has \( C^k \) smoothness. In particular, when \( k = \infty \), we say \( \alpha \) is a \textit{smooth real 1-form} over \( D \).

\[ C_T^*D = \bigoplus_{p \in D} C_T^*_pD \xrightarrow{\pi} D \longrightarrow \text{complexified cotangent bundle space,} \]

\[ C_T^*D = \text{Span}_{\mathbb{C}} \{ dz|_p, d\overline{z}|_p \} \]

\[ \langle dz_p, \frac{\partial}{\partial z}|_p \rangle = 1, \quad \langle dz_p, \frac{\partial}{\partial \overline{z}}|_p \rangle = 0, \quad \langle d\overline{z}_p, \frac{\partial}{\partial z}|_p \rangle = 0, \quad \langle d\overline{z}_p, \frac{\partial}{\partial \overline{z}}|_p \rangle = 1. \]

A cross section \( \alpha \) of \( C_T^*D \) is called a \textit{complex-valued 1-form} if \( \alpha = a(p,\overline{p})dz + b(p,\overline{p})d\overline{z} \) with \( a, b \in C^k(D) \). We usually call such an \( \alpha \) a \( C^k \)-smooth complex-valued 1-form.

\[ T^{*(1,0)}D = \bigoplus_{p \in D} T^{*(1,0)}_pD, \quad T^{*(1,0)}_pD = \text{Span}_{\mathbb{C}} \{ dz|_p \}. \]

A cross section \( \alpha \) of \( C_T^{*(1,0)}D \) is called a \textit{1-form of type} \((1,0)\). Write \( \alpha = a(p,\overline{p})dz + b(p,\overline{p})d\overline{z} \) with \( a \in C^k(D) \). We say \( \alpha \) is a \( C^k \) smooth 1-form of type \((1,0)\). When \( a \in \text{Hol}(D) \), we say that \( \alpha \) is a \textit{holomorphic 1-form} over \( D \).

\[ T^{*(0,1)}D = \bigoplus_{p \in D} T^{*(0,1)}_pD, \quad T^{*(0,1)}_pD = \text{Span}_{\mathbb{C}} \{ d\overline{z}|_p \}. \]

For a Riemann surface \( M \), we can similarly define

\[ T_pM, \quad TM = \bigoplus_{p \in M} T_pM, \]

\[ C_TpM \longrightarrow \text{the complexified tangent space at} \ p, \]

\[ C_TM = \bigoplus_{p \in M} C_TpM \longrightarrow \text{the complexified tangent bundle over} \ M. \]

We claim \( T_p^{*(1,0)}M \) is also well-defined. In fact, let \( (U_i \ni p, \Phi_i) \) be a holomorphic chart near \( p \), we define \( T_p^{*(1,0)}M = Spin \{ (\Phi_i^{-1})_*(\frac{\partial}{\partial z}|_{z_0}) \} \) with \( z_0 = \Phi_i(p) \), namely, for any smooth function \( \chi \) defined near \( p \) in \( M \).
\((\Phi^{-1})_*(\frac{\partial}{\partial z}|_{z_0})(\chi) = \frac{\partial}{\partial z}(\chi \circ \Phi^{-1})|_{z=z_0} = \frac{1}{2}(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y})(\chi \circ \Phi^{-1})|_{z=z_0}\). \(\partial\partial z \mid_{z_0} = 1\)

\((\chi \circ \Phi - 1)(\Phi^{-1})_*(\frac{\partial}{\partial w}|_{w_0}) = (\Phi^{-1})_*(\frac{\partial}{\partial z}|_{z_0})\)

To make sense that \(T_p^{(1,0)}M = \text{Span}\{((\Phi^{-1})_*(\frac{\partial}{\partial z}|_{z_0}))\}\) is well-defined, we need only to check if \(\Psi\) is another holomorphic coordinate map from a neighborhood \(\tilde{U}\) of \(p\) in \(M\) with \(\Psi(p) = w_0\), then

\((\Psi^{-1})_*(\frac{\partial}{\partial w}|_{w_0}) = \tilde{k}(\Phi^{-1})_*(\frac{\partial}{\partial z}|_{z_0})\)

for some \(\tilde{k}\). Indeed,

\((\Psi^{-1})_*(\frac{\partial}{\partial w}|_{w_0})(\chi) = \frac{\partial}{\partial w}(\chi \circ \Psi^{-1}) = \frac{\partial}{\partial w}(\chi \circ \Phi^{-1} \circ \Phi \circ \Psi^{-1})

= \frac{\partial}{\partial w}(\chi \circ \Phi^{-1} \circ h) \quad \text{(Here } h = \Phi \circ \Psi^{-1})

= \frac{\partial}{\partial z}(\chi \circ \Phi^{-1}|_{z_0}) \frac{\partial h}{\partial w}|_{w_0} + \frac{\partial}{\partial w}(\chi \circ \Phi^{-1})|_{z_0} \frac{\partial \bar{h}}{\partial w}|_{w_0}

= (\Phi^{-1})_*(\frac{\partial}{\partial z}|_{z_0}) \frac{\partial \bar{h}}{\partial w}|_{w_0}.

Similarly, \(T_p^{(0,1)}M\) is well-defined and

\(CT_pM = T_p^{(1,0)}M \oplus T_p^{(0,1)}M, \quad T_p^{(1,0)}M \cap T_p^{(0,1)}M = \{0\}, \quad \overline{T_p^{(1,0)}M} = T_p^{(0,1)}M.\)

As before, by studying the dual space of the tangent space, we get

\(T^*M, \quad CT^*M, \quad T^{(1,0)*}M, \quad T^{(0,1)*}M.\)

The last two spaces are called the holomorphic cotangent bundle and conjugate holomorphic contangent bundle, respectively.

Let \(U \subset M\). A cross section of \(T^*M, CT^*M, T^{(1,0)*}M, T^{(0,1)*}M\) over \(U\) is called a differential one form, complexified differential 1-form, a differential one-form of type \((1,0)\), etc.

Let \(S\) be a cross section of \(T^{(1,0)*}M\) over \(U\). We call \(S\) a smooth 1-form of type \((1,0)\) if for any \(p \in U\) and a coordinate map \(\Phi_p\) near \(p, z = \Phi_p: U_p \to U^0_p(\subset \mathbb{C}^1), S = a(p) \cdot \Phi_p^*(dz)\) with \(a(p) \in C^\infty(U_p)\).

We call \(S\) a holomorphic 1-form if for any \(p \in U\), a local chart \(U_p\) and the associate holomorphic map \(\Phi(p) = z, S = a(p)\Phi^*(dz)\) with \(a(p) \in \text{Hol}(U_p)\).

A holomorphic 1-form over \(M\) is called a global holomorphic 1-form over \(M\).

Remark
1.3. MEROMORPHIC FUNCTIONS ON RIEMANN SURFACES

1. Let $f : M \to N$ be a smooth map, as we discussed before, it induces $f_* T_p M \to T_{f(p)} N$, but in general it does not include a map from $T_p^{(1,0)} M$ to $T_{f(p)}^{(1,0)} N$. In fact, for a smooth map $f : M \to N$, $f$ is a holomorphic map if and only if

$$f_* (T_p^{(1,0)} M) \subseteq T_{f(p)}^{(1,0)} N, \quad \forall p \in M.$$

2. A smooth map $f$ can pull-back a smooth form $\omega$: $f^* \omega$.

1.3 Meromorphic functions on Riemann surfaces

By the Maximum Principle, if $M$ is a compact manifold, then

any holomorphic function $f : M \to \mathbb{C}$ must be a constant.

As a result, we have to study meromorphic functions on $M$ and holomorphic sections of a line bundle over $M$, instead of holomorphic functions on compact complex manifolds.

**Definition 1.7** Let $M$ be a Riemann surface. $f : M \to \mathbb{C}$ is called a meromorphic function if $\exists$ a discrete set $\{p_i\} \subset M$ such that $f : M \setminus \{p_i\}_{i=1}^{\infty} \to \mathbb{C}$ is holomorphic and for each $p_j \in M$, there is a neighborhood $U_j \ni p_j$ such that $f = \frac{g_j}{h_j}$ over $U_j \setminus \{p_j\}$ where $g_j, h_j$ are holomorphic over $U_j$. In the other word, let $\{U_j, \Phi_j\}$ be a holomorphic chart near $p_j$ with $\Phi_j(p_j) = 0$, $f \circ \Phi_j^{-1} = \sum_{m=0}^{\infty} a_j z^m$ for $m \in \mathbb{Z}$ and $z \neq 0$. If $a_m \neq 0$, then we call $f$ has a pole of order $m$ at $p_j$. When $p_j$ is a regular point of $f$ and $f \circ \Phi_j^{-1} = \sum_{m=0}^{\infty} a_j z^m \quad m \geq 0$, we say $f$ has a zero of order $m$ at $p_j$ if $m \geq 0$ and $a_m \neq 0$.

**Lemma 1.8** The pole or zero of order $m$ for $f$ is well-defined.

**Definition 1.9** Let $M$ be a Riemann surface. A 1-form of type $(1,0)$ $\omega$ is called a meromorphic 1-form if $\exists \{p_j\} \subset M$ discrete such that $\omega$ is a holomorphic 1-form over $M \setminus \bigcup_j \{p_j\}$ and let $\{U_j, \Phi_j\}$ be a holomorphic chart near $p_j$ with $\Phi_j(p_j) = 0$, $(\Phi_j^{-1})^* (\omega) = a(z) dz$ with $a(z)$ meromorphic at 0.

The study of holomorphic and meromorphic functions or forms is the basic problem for Riemann surfaces. Next, we give more definitions.

1. A function $u$ on $M$ is called harmonic if $u \circ \Phi_j^{-1} : U_j \to \mathbb{R}$ is harmonic for each $\Phi_j$. Apparently, the definition makes sense for if $u \circ \Phi_j^{-1}$ is harmonic, then $u \circ \Phi_j^{-1} = u \circ \Phi_i^{-1}(\Phi_j \circ \Phi_i^{-1}) = \text{harmonic} \circ \text{holomorphic}$ is still harmonic.
(2) A function $u$ on $M^1 \subset M$ is called subharmonic if $u \circ \Phi_j^{-1}$ is subharmonic.

(3) Let $u$ be a subharmonic function on $M$ and $M$ connected. Then if $u(p_0)$ contains the maximum value, then $u \equiv$ constant — openness+closedness argument.

(4) Let $M$ be a compact Riemann Surface. If $f$ is holomorphic on $M$, then $\text{Re} f, \text{Im} f$ are harmonic over $M$, Hence $f \equiv$ constant. Hence, there is no nontrivial holomorphic function over an compact Riemann Surface.

(5) Residue theorem

Let $f$ be a meromorphic function on compact $M$. Let $\{p_j\}$ be the zero set of $f$ and $\{q_j\}$ be the pole of $f$. Assume that $\text{ord}_{p_j} f = m_j, \text{ord}_{q_j} f = n_j$. Then

$$\sum m_j = \sum n_j$$

Proof: Let $\omega = \frac{df}{f}$. Then $\omega$ is a meromorphic $(1,0)$-form. Write $\{P_j\} = \{p_j, q_j\}$. Let $\Delta_j$ be a small disk centered at $P_j$ with smooth boundary $C_j$ of positive orientation w.r.t $\Delta_j$.

Assume $\overline{\Delta_j} \cap \{P_j\} = P_j$ and $\overline{\Delta_j} \subset U_j$ for a certain holomorphic coordinate chart $U_j$ with holomorphic coordinate function $\Phi_j$. By Stokes Theorem

$$\int_{M \setminus \overline{\Delta_j}} d\omega = -\sum \int_{C_j} \omega,$$

$$\int_{M \setminus \overline{\Delta_j}} d\omega = \int_{M \setminus \overline{\Delta_j}} \frac{df}{f} = \int_{M \setminus \overline{\Delta_j}} \frac{-f df \wedge df}{f^2} \equiv 0.$$

Hence, we conclude that

$$\sum_j \int_{C_j} \omega = 0$$

Now

$$\int_{C_j} \omega = \int_{\Phi_j \ast (C_j)} (\Phi_j^{-1})^* \omega = \int_{C_j} \frac{df \circ \Phi_j^{-1}}{f \circ \Phi_j^{-1}}$$

$$= 2\pi i \text{Res}_{\Phi_j(P_j)} \frac{f \circ \Phi_j^{-1}}{f \circ \Phi_j^{-1}} = \begin{cases} 2\pi i m_j, & P_j = p_j \\ -2\pi i n_j, & P_j = q_j. \end{cases}$$

Hence $\sum m_j - \sum n_j = 0$. We conclude that “the total number of poles” = “the total number of zeros”.
1.4 Complex projective space $\mathbb{CP}^n$

We know $\mathbb{C}$ is isomorphic to the Riemann sphere $S^3$ minus the north pole $\{N\}$. On the Riemann sphere $S^3$, every finite complex number and the “infinite” have equal position. $S^3$ can be regarded as the $\mathbb{CP}^1$, the one complex dimensional projective space.

We define an equivalent relation “$\sim$” in $\mathbb{C}^{n+1}\setminus\{0\}$,

$$(z_0,\ldots,z_n) \sim (w_0,\ldots,w_n) \iff (w_0,\ldots,w_n) = k(z_0,\ldots,z_n)$$

for some $k$. As a point set $\mathbb{C}P^n = (\mathbb{C}^{n+1} - \{0\})/\sim$.

We will define the holomorphic coordinates covering such that $\mathbb{C}P^n$ becomes a complex manifold.

Let $U_j = \{(z_0,\ldots,z_n) : z_j \neq 0\} \subset \mathbb{C}^{n+1}$ and let $U_j = \mathbb{C}P^n/\sim = \{(z_0,\ldots,z_n) : (z_0,\ldots,z_n) \in \mathbb{C}P^n\}.$

Let $\phi_j : U_j \to \mathbb{C}^n$ by $[z_0,\ldots,z_n] \mapsto (\overline{z_j}, \frac{z_{j+1}}{z_j}, \frac{z_{j+2}}{z_j}, \ldots, \frac{z_n}{z_j}).$ Then $\phi_j$ is well-defined. Moreover, $\phi_j$ is one to one and onto. Moreover,

$$\phi_j \circ \phi^{-1}_j : \phi_l(U_l \cap U_j) \to \phi_j(U_l \cap U_j)$$

$$\phi_j \circ \phi^{-1}_j : (\xi_1,\ldots,\xi_n) \mapsto [\xi_1,\ldots,\xi_{j-1},1,\xi_{j+1},\ldots,\xi_n] \mapsto (\frac{\xi_1}{\xi_l}, \ldots, \frac{\xi_{j-1}}{\xi_l}, \frac{1}{\xi_j}, \frac{\xi_{j+1}}{\xi_l}, \ldots, \frac{\xi_n}{\xi_l})$$

Since $\xi_l \neq 0$ on $\phi_l(U_l \cap U_j)$, $\phi_j \circ \phi_l^{-1}$ is holomorphic. Moreover, $\phi_l \circ \phi_j^{-1}$ is holomorphic. Thus $\phi_j \circ \phi_l^{-1}$ is biholomorphic. Now, we define the topology of $\mathbb{C}P^n$ as follows:

$U \subset \mathbb{C}P^n$ is open iff $\phi_j(U \cap U_j)$ is open in $\mathbb{C}^n$ for each $j$.

Then one can see that $\mathbb{C}P^n$ is a compact Hausdorff space. Also, the topology of $\mathbb{C}P^n$ is generated by a countable number of open subsets of $\mathbb{C}P^n$ called the basis.

By the definition, $\mathbb{C}P^n$ is a complex manifold of dim $n$. When $n = 1, \mathbb{C}P^1 = \{U_0, U_1\}$.

$$U_0 = \{[z_0, z_1] : z_0 \neq 0\} \xrightarrow{\phi_0} \frac{z_1}{z_0}, \text{ } \phi_0 \text{ is biholomorphic from } U_0 \to \mathbb{C}^1,$$

$$U_1 = \{[z_0, z_1] : z_1 \neq 0\} \xrightarrow{\phi_1} \frac{z_0}{z_1}, \text{ } \phi_1 \text{ is a biholomorphic map from } U_1 \to \mathbb{C}^1$$

$$\phi_0 \circ \phi_1^{-1} : \mathbb{C}\setminus\{0\} \to \mathbb{C}^1\{0\} : z \mapsto \frac{1}{z}.$$

Let $S^2 = \{(z, t) : z \in \mathbb{C}^1, |z|^2 + t^2 = 1\} \subset \mathbb{C} \times R$. We define $\mathbb{P} : S^2 \to \mathbb{C}P^1$ as follows: If $p(\neq (0,1)) \in S^2$, we define $\mathbb{P}(p)$ to be the intersection of $N\mathbb{P}$ with $R^2 = \mathbb{C}^1$, where $N = (0,1)$.

$$\mathbb{P}(p_1, p_2, p_3) = \frac{p_1 + ip_2}{1 - p_3} \mapsto \left[\frac{p_1 + ip_2}{1 - p_3}, 1\right],$$

$$\mathbb{P}(0,1) = [1,0] = \infty, \text{ } \mathbb{P} \text{ one-to-one and onto}.$$
Example 1.10 \( \mathbb{P} \) is a homeomorphism. Thus we can also regard \( \mathbb{C}P^1 \) as \( S^2 \). We normally write \( \mathbb{C}P^1 = \mathbb{C} \cup \{\infty\} = \hat{\mathbb{C}} \) the Gauss plane.

Example 1.11 Let \( M \) be a Riemann Surface. Let \( f \) be a meromorphic function. Then \( \hat{f} : M \to \mathbb{C}P^1 \)
\[
\begin{cases}
  f(z) & \text{, } f(z) \in \mathbb{C}, \\
  \infty & \text{, } f(z) \neq \text{ a number.}
\end{cases}
\]
is a holomorphic map.

Indeed, \( f \) is continuous. To see this, suppose \( p_0 \in M \) is a pole of \( f \), we need only to show that \( \hat{f} \) is continuous at \( p_0 \). Let \( p_j (\neq p_0) \to p_0 \). Then \( \hat{f}(p_j) = [f(p_j), 1] = [1, \frac{1}{f(p_j)}] \to [1, 0] = \infty = \hat{f}(p_0) \). Hence \( \hat{f} \) is continuous.

Next, we verify that \( f \) is holomorphic near \( p_0 \). Let \( \{U_{p_0}, \phi_{p_0}\} \) be a local holomorphic coordinate chart near \( p_0 \) with \( \phi_{p_0}(p_0) = 0 \). Then
\[
\psi_1 \circ \hat{f} \circ \phi_{p_0}^{-1} : \xi \mapsto \begin{cases}
  \frac{1}{f \circ \phi_{p_0}} = h(\xi), \\
  0.
\end{cases}
\]
h(\( \xi \)) hol for \( \xi \neq 0 \) and \( \lim_{\xi \to 0} h(\xi) = 0 \). Hence \( \phi_1 \circ \hat{f} \circ \phi_{p_0}^{-1} \) is holomorphic.

Conversely, one can verify that any holomorphic map \( f : M \to \mathbb{C}P^1 \) induce a meromorphic function on \( M \).

Corollary 1.12 Let \( f : M \to \mathbb{C}P^1 \) be a holomorphic map where \( M \) is a compact Riemann surface. Then for each \( p \in \mathbb{C}P^1 \), the number of the preimage set \( \sharp\{f^{-1}(p)\} \) counting multiplies is independent of the choice of \( p \). In fact, by the Residue theorem, \( \sharp\{f^{-1}(p)\} = \sharp \) of poles of \( f = \sharp\{f^{-1}(\infty)\} \).

1.5 Complex Tori

We’ll study “genus” \( g \) of a compact Riemann surface \( M \), the number of “holes” of \( M \). When \( g = 0 \), \( M \) is biholomorphic to \( \mathbb{C}P^1 \). When \( g = 2 \), it is torus. Let us study a tours.

Let \( M \) be a complex manifold of dimension \( n \). Write
\[
\text{Aut}(M) = \{ f : M \to M, f \text{ biholomorphic} \}. \]
1.5. **COMPLEX TORI**

Then Aut(M) is a group under the composition law, called the *automorphism group* of M. Let \( \Gamma \subset Aut(M) \) be a subgroup.

(i) \( \Gamma \) is called *discrete* if \( \forall p_0 \in M, \Gamma(p_0) = \{ r(p_0) : r \in \Gamma \} \) is a discrete subset.

(ii) \( \Gamma \) is said to be *fixed point free* if for any \( g \in \Gamma, g \neq id, g \) has no fixed point.

(iii) \( \Gamma \) is called *properly discontinuous* if for any \( K_1, K_2 \subset M, \{ r \in \Gamma : r(K_1) \cap K_2 \neq \emptyset \} \) is a finite set of \( \Gamma \).

**Remarks**

1. The conditions (ii)+(iii) \( \Rightarrow \) (i).

2. In general, \( M/\Gamma \) may not be a complex manifold. Some conditions are required. For example, let \( g : \mathbb{C}^2 \to \mathbb{C}^2, (z_1, z_2) \mapsto (-z_1, -z_2) \) be an element in \( Aut(\mathbb{C}^2) \). Then \( \Gamma = \{ g, Id \} \) defines a subgroup. \( \mathbb{C}^2/\Gamma \) is not a smooth manifold. Here \( \Gamma \) is not fixed point free because \( g(0,0) = (0,0) \) so that \( g \) has a fixed point \( (0,0) \). In fact, consider a \( \Gamma \)-invariant map

\[
L : \mathbb{C}^2 \to \mathbb{C}^3, (z_1, z_2) \mapsto (z_1^2, z_2^2, z_1z_2).
\]

Notice \( L(z_1, z_2) = L(z_1, z_2) \) if and only if either \( (z_1, z_2) = (\tilde{z}_1, \tilde{z}_2) \) or \( (z_1, z_2) = (-\tilde{z}_1, -\tilde{z}_2) \). It induces a quotient map

\[
L : \mathbb{C}^2/\Gamma \to A = \{ (z_1, z_2, z_3) \in \mathbb{C}^3, z_1z_2 = z_3^2 \}.
\]

Here \( \mathbb{C}^2/\Gamma \) can be identified with \( A \) which is a variety on \( \mathbb{C}^2 \) with singularity 0.

**Example** Let

\[
M = \mathbb{C} \text{ and } \Gamma = \{ g(z) = z + m_1\omega_1 + m_2\omega_2, m_1, m_2 \in \mathbb{Z} \}
\]

where \( \omega_1, \omega_2 \in \mathbb{C} \) that are \( \mathbb{R} \)-linearly independent. \( \Gamma \) is discrete, fixed point free and properly discontinuous.

Assume that \( \Gamma \) is properly and discontinuously acting on \( M \) and \( \Gamma \) is fixed point free. We want to show that \( M/\Gamma \) has a canonical complex structure of a complex manifold induced from that of \( M \).

We claim that for each \( p \in M, \exists U_p \) a neighborhood of \( p \) in \( M \), such that \( g(U_p) \cap U_p = \emptyset \) for \( g \neq Id \). In fact, let \( U \) be a neighborhood of \( p \) such that \( U \subset M \). By (iii), \( \exists \) only \( \{ g_1, \ldots, g_l \} \subset \Gamma \) such that \( g_j(U) \cap U \neq \emptyset \). Assume \( g_j \neq Id \). Let \( U_p \) be a sufficiently small neighborhood of \( p \) in \( M \) such that \( g_j(U_p) \cap U_p = \emptyset \) for each \( j \). Then \( U_p \) is the desired one.
For any $p, q \in M$, we define an equivalence relation:
\[ p \cong q \text{ if and only if } q = g(p) \text{ for some } g \in \Gamma. \]

We denote by $[p]$ the equivalence class. Let $M/\Gamma = \{ [p] \mid p \in M \}$ be the quotient space.

**Example**  Let $M = \mathbb{C}$ and $\Gamma = \{ g(z) = z + m_1 \omega_1 + m_2 \omega_2, \ m_1, m_2 \in \mathbb{Z} \}$. Then $M/\Gamma$ is a torus.

In Topology, we can define a quotient topology on $M/\Gamma$. Namely, $\hat{U} \subset M/\Gamma$ is open if and only if $\pi^{-1}(\hat{U})$ is open in $M$, where $\pi : M \to M/\Gamma$. $\pi$ is now a covering map. Indeed, let
\[ \nu = \{ [U] = U/\sim : U \text{ is open in } M \text{ such that } g(U) \cap U = \emptyset \text{ for } g \neq \text{Id}, \ g \in \Gamma \}. \]

Then $\nu$ forms a basis of the topology of $M/\Gamma$. Now, for any $p \in M/\Gamma$, let $[U_p] \subset \nu$. Then
\[ \pi^{-1}([U_p]) = \bigcup_{g \in \Gamma} g(U_p) \text{ and,} \]
\[ g(U_p) \cap g'(U_p) \neq \emptyset \leftrightarrow g = g'. \]

Moreover, $\pi|_{g(U_p)} : g(U_p) \to [U_p]$ is a homeomorphism.

**Proposition 1.13** For each $[p] \in M/\Gamma$ with $p \in M$, let $U_p$ be the coordinate neighborhood of $p$ in $M$ with holomorphic coordinate map $\phi_p$ such that $U_p \cap g(U_p) = \emptyset$ for $g \in \Gamma$ with $g \neq \text{Id}$. Define
\[ [\phi_p] : [U_p] \subset M/\Gamma \to \mathbb{C}^n [q] \mapsto \phi_p(q) \]
where $q \in M$, $q \in U_p$. Then $M/\Gamma$ with $\{(U_p), [\phi_p]\}$ form a complex manifold of dim = dim$M$. Moreover, the covering map $\pi : M \to M/\Gamma$ is holomorphic, called the holomorphic covering map.

**Proof:** $[\phi_p][\phi_q]^{-1} = \phi_p \circ r_{pq} \circ \phi_q^{-1}$ for a certain $r_{pq} \in \Gamma$. Hence it is holomorphic whenever it is well-defined. □

**Example**  Let $M = \mathbb{C}^n$. Take $2n$ vectors $\{\omega_1, \ldots, \omega_{2n}\}$, $\omega_k \in \mathbb{C}^n$ such that $\{\omega_j\}$ are $\mathbb{R}$-linearly independent. Let
\[ \Gamma = \{ g \mid g(z) = z + \sum_{k=1}^{2n} m_k \omega_k, m_k \in \mathbb{Z} \}. \]
Then, (i) $\Gamma$ is fixed-point free, and (ii) $\Gamma$ acts properly discontinuous on $\mathbb{C}^n$.

To see (ii), let $B(z_0, r)$ be the ball centered at $z_0$ with radius $r \ll 1$.

Suppose $g \neq \text{Id}$ and $\sharp \{ g : g(B(z_0, r)) \cap B(z_0, r) \neq \emptyset \} = \infty$. Then $\exists z' \in B(z_0, r)$ such that $|z' + \sum m_k' \omega_k - z_0| < r$ for a certain $m_k'$, depending on $r$. Let $m_{k_0}' = \max_{1 \leq k \leq 2n} |m_k'|$.

It follows that $|\sum m_k' \omega_k + \frac{z' - z_0}{m_{k_0}'}| \leq \frac{r}{|m_{k_0}'|} \rightarrow 0$ as $r \rightarrow 0$. Passing to a limit, it is easy to see that $\{ \omega_k \}$ is $\mathbb{R}$-linearly dependent.

Remarks

1. When $n = 1$, we get $\mathbb{C}/\Gamma$. Topologically speaking, it is a real torus of dimension two or a real (oriented) surface with genus $g = 1$.

2. (cf. p. 46, Kodaira, Complex manifolds and deformation of complex structure, 1985) When $n = 1$, $\omega_1 = \omega$ and $\omega_2 = 1$. Define

$$p = \frac{1}{z^2} + \sum_{(m,n) \neq (0,0)} \left( \frac{1}{(z - \omega_{mn})^2} - \frac{1}{\omega_{mn}^2} \right)$$

where $\omega_{mn} = m\omega + n$, which is called the Weierstrass $p$-function. $p$ is a meromorphic function defined on $\mathbb{C}/\Gamma$. It gives a biholomorphic map

$$\mathbb{C}/\Gamma \rightarrow \mathbb{CP}^2,$$

$$z \mapsto (1 : p'(z) : p(z)).$$

$p$ satisfies

$$p'(z)^2 - 4p(z)^3 + g_2p(z) + g_3 = 0$$

where $g_2$ and $g_3$ are constant numbers defined by $p(z) = \frac{1}{z^2} + \frac{1}{20}g_2z^2 + \frac{1}{28}g_3z^4 + O(z^6)$. We can identify $\mathbb{C}/\Gamma$ with an algebraic variety, a cubic curve in $\mathbb{CP}^2$.

**Lemma 1.14** $\mathbb{C}/\Gamma_1 \xrightarrow{f} \mathbb{C}/\Gamma_2$ is a biholomorphic map $f$ iff $\exists F = az + b$ with $a \neq 0$ such that $F$ map the equivalent classes w.r.t $\Gamma_1$ to equivalent classes w.r.t $\Gamma_2$.

Proof: Suppose $\mathbb{C}/\Gamma_1 \xrightarrow{f} \mathbb{C}/\Gamma_2$ is a holomorphic equivalent map. $\mathbb{C}$ is the universal covering space of $\mathbb{C}/\Gamma_j$, $j = 1, 2$. Consider the diagram

$$\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\pi_1} & \mathbb{C} \\
\downarrow F \quad & & \downarrow \pi_2 \\
\mathbb{C}/\Gamma_1 & \xrightarrow{f} & \mathbb{C}/\Gamma_2
\end{array}$$
by the lifting lemma, for any \( z_0 \in \mathbb{C}^1 \), and \( w_0 \in \mathbb{C} \) s.t \( f(\pi_1(z_0)) = \pi_2(w_0) \), there is a unique continuous map \( F: \mathbb{C}^1 \rightarrow \mathbb{C}^1 \) with \( F(z_0) = w_0 \). Since locally, \( F = \pi_2 \circ f \circ \pi_1^{-1} \), we see that \( F \) is holomorphic. Moreover, \( \pi_1 \circ f = F \circ \pi_2 \). Thus, \( F \) maps any equivalent class w.r.t \( \Gamma_1 \) to an equivalent class w.r.t \( \Gamma_2 \).

Let \( g = f^{-1} \). Then we can similarly show that \( \exists \) a unique \( G \) with \( G(w_0) = z_0 \) and \( G \) is holomorphic. Moreover, \( F \circ G = G \circ F = Id \).

This proves \( F \in \text{Aut}(\mathbb{C}^1) \) and hence \( F = az + b \) for some \( a \neq 0 \). In fact, we write \( F = \sum_k a_k z^k \) as a power series on \( \mathbb{C} \). Since \( F \) is one-to-one, \( \{\infty\} \) cannot be an essential singularity of \( F \) in the Riemann sphere by Weierstrass’ theorem. So \( \{\infty\} \) is a pole of \( F \) so that \( F \) is rational and hence is polynomial. Again by the injectivity, \( F \) is affine linear.

Conversely, let \( F = az + b \) with \( a \neq 0 \). If \( F \) maps equivalent classes of \( \Gamma_1 \) to equivalent classes w.r.t \( \Gamma_2 \), then \( f: [F([z])] \) defines a holomorphic equivalent map of \( \mathbb{C}/\Gamma_1 \) to \( \mathbb{C}/\Gamma_2 \).

Now, suppose that \( F = az + b \) preserves the equivalent classes of \( \Gamma \). Then so is \( F = az \). Hence, we get

**Corollary 1.15** \( \mathbb{C}/\Gamma_1 \) is biholomorphic to \( \mathbb{C}/\Gamma_2 \) iff \( \exists a \neq 0 \) such that \( F = az \) sends an equivalent class with respect to \( \Gamma_1 \) to an equivalent class with respect to \( \Gamma_2 \).

Let \( \Gamma = \{ g = z + \sum^2_{j=1} m_j \omega_j, m_j \in \mathbb{Z} \} \), \( \Gamma' = \{ g = z + \sum^2_{j=1} m'_j \omega'_j, m'_j \in \mathbb{Z} \} \). Suppose that \( F \) is as in corollary 1.15. Then

\[
F(\omega_1) = a \omega_1 = a_{11} \omega'_1 + a_{12} \omega'_2, \quad F(\omega_2) = a \omega_2 = a_{21} \omega'_1 + a_{22} \omega'_2, \quad \text{with } a_{ij} \in \mathbb{Z} \quad (1.1)
\]

or

\[
F \left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right) = \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \left( \begin{array}{c} \omega'_1 \\ \omega'_2 \end{array} \right) = A \left( \begin{array}{c} \omega'_1 \\ \omega'_2 \end{array} \right),
\]

where \( A = \left( \begin{array}{cc} a_{11} & a_{12} \\ a_{21} & a_{22} \end{array} \right) \in \text{GL}(2, \mathbb{Z}) \), i.e., the group of \( 2 \times 2 \) non-singular matrices with integer entries.

Consider \( F^{-1} \) we obtain

\[
F^{-1} \left( \begin{array}{c} \omega'_1 \\ \omega'_2 \end{array} \right) = \left( \begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right) \left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right) = B \left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right),
\]

where \( B = \left( \begin{array}{cc} b_{11} & b_{12} \\ b_{21} & b_{22} \end{array} \right) \in \text{GL}(2, \mathbb{Z}) \).

Then

\[
\left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right) = F^{-1} \circ F \left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right) = F^{-1} \circ A \left( \begin{array}{c} \omega'_1 \\ \omega'_2 \end{array} \right) = F^{-1} \left( \begin{array}{c} a_{11} \omega'_1 + a_{12} \omega'_2 \\ a_{21} \omega'_1 + a_{22} \omega'_2 \end{array} \right)
\]
\[ \left( \begin{array}{c} a_{11}F^{-1}(\omega_1') + a_{12}F^{-1}(\omega_2') \\ a_{21}F^{-1}(\omega_1') + a_{22}F^{-1}(\omega_2') \end{array} \right) = AF^{-1} \left( \begin{array}{c} \omega_1' \\ \omega_2' \end{array} \right) = AB \left( \begin{array}{c} \omega_1 \\ \omega_2 \end{array} \right). \]

Hence \( AB = Id \) so that \( \det(B) \cdot \det(A) = 1 \). Since \( A \) and \( B \in GL(2, \mathbb{Z}) \), it concludes \( \det(A) = \pm 1 \) so that \( A \in SL(2, \mathbb{Z}) \).

Let \( \tau = \frac{\omega_1}{\omega_2} \) and \( \tau' = \frac{\omega_1'}{\omega_2'} \). Then

\[ \tau = \frac{a^{-1}(a_{11}\omega_1' + a_{12}\omega_2')}{a^{-1}(a_{21}\omega_1' + a_{22}\omega_2')} = \frac{a_{11}\omega_1' + a_{12}\omega_2'}{a_{21}\omega_1' + a_{22}\omega_2'} = a_{11}a_{22} - a_{12}a_{21} = \pm 1. \]

and hence

\[ \tau = \frac{a_{11}\tau' + a_{12}}{a_{21}\tau' + a_{22}}, \quad a_{11}a_{22} - a_{12}a_{21} = \pm 1. \quad (1.2) \]

If \( \mathbb{C}/\Gamma \cong \mathbb{C}/\Gamma' \), then (1.2) holds. Conversely, if (1.2) holds, then from (1.1) we have

\[ a\omega_1 = a_{11}\omega_1' + a_{12}\omega_2', \text{ and } a\omega_2 = a_{12}\omega_1' + a_{22}\omega_2'. \]

In order to define \( F(z) = az \), we need

\[ \frac{a_{11}\omega_1' + a_{12}\omega_2'}{\omega_1} = \frac{a_{21}\omega_1' + a_{22}\omega_2'}{\omega_2}, \]

which is true because of (1.2). Therefore we have proved:

\[ \mathbb{C}/\Gamma \cong \mathbb{C}/\Gamma' \iff (1.2) \text{ holds.} \]

In particular, we have proved:

\[ \mathbb{C}/\Gamma \cong \mathbb{C}/\Gamma_0 \quad (1.3) \]

where \( \Gamma \) is generated by \( (\omega_1, \omega_2) \) and \( \Gamma_0 \) is generated by \( (\omega, 1) \) where \( \omega = \frac{\omega_1}{\omega_2} \). Notice the torus generated by \( (\frac{\omega_1}{\omega_2}, 1) \) is isomorphic to the one generated by \( (1, \frac{\omega_1}{\omega_2}) \). For the non zero complex numbers \( \frac{\omega_1}{\omega_2} \) and \( \frac{\omega_2}{\omega_1} \), one of them must have positive imaginary part, we can further assume that

\[ \Gamma_0 = (\omega, 1) \text{ with } \text{Im}(\omega) > 0. \]
Since \( \omega = \frac{a_{11}\omega' + a_{12}}{a_{21}\omega' + a_{22}} \), we have

\[
I m \; \omega = \frac{\omega - \overline{\omega}}{2i} = \frac{1}{2i} \left( a_{11}\omega' + a_{12} - \frac{a_{11}\overline{\omega'} + a_{12}}{a_{21}\omega' + a_{22}} \right)
\]

\[
= \frac{1}{2i|a_{21}\omega' + a_{12}|^2} \left( (a_{11}a_{21} |\omega'|^2 + a_{12}a_{21} \overline{\omega'} + a_{11}a_{22}\omega' + a_{12}a_{22}) - (a_{11}a_{21} |\omega'|^2 + a_{12}a_{21} \overline{\omega'}) \right)
\]

\[
+ a_{12}a_{21}\omega' + a_{11}a_{22}\overline{\omega'} + a_{12}a_{22})|a_{21}\omega' + a_{12}|^2)
\]

\[
= \frac{1}{2i} \cdot \frac{(a_{12}a_{21} - a_{11}a_{22})\overline{\omega'} + (a_{11}a_{22} - a_{12}a_{21})\omega}{|a_{21}\omega' + a_{12}|^2} \cdot \frac{a_{11}a_{22} - a_{12}a_{21}}{|a_{21}\omega' + a_{12}|^2} \cdot I m \omega'.
\]

As a result, from \( \omega > 0 \), it implies that \( \text{det}(A) = a_{11}a_{22} - a_{12}a_{21} = 1 \). Hence, we get the following

**Theorem 1.16** Let \( \Gamma = \text{Span}_\mathbb{Z}\{1, \omega\} \), \( \Gamma' = \text{Span}_\mathbb{Z}\{1, \omega'\} \) with \( I m \omega, I m \omega' > 0 \), Then \( \mathbb{C}^1/\Gamma \) is biholomorphic to \( \mathbb{C}^1/\Gamma' \iff \omega' = \frac{a_{11}\omega + a_{12}}{a_{21}\omega + a_{22}} \) (1.4)

where \( a_{11}, a_{12}, a_{21}, a_{22} \in \mathbb{Z} \) and \( a_{11}a_{22} - a_{12}a_{21} = 1 \).

**Remarks**

1. Fix any torus \( \mathbb{C}/\Gamma \), there are only a countable number of discrete group \( \Gamma' = \text{Span}_\mathbb{Z}(\omega', 1) \) such that \( \mathbb{C}/\Gamma \cong \mathbb{C}/\Gamma' \).

2. We introduce an equivalent relation \( \sim \) into the set \( \mathcal{A}_1 \) of all tori \( \{\mathbb{C}/\Gamma\} \), up to isomorphisms:

\[ \mathbb{C}/\Gamma \sim \mathbb{C}/\Gamma' \iff \mathbb{C}/\Gamma \text{ and } \mathbb{C}/\Gamma' \text{ are biholomorphic.} \]

Then \( \mathcal{A}_1 = \{[\omega, 1]\} \) where \([\omega, 1] = [(\omega', 1)]\) if and only if (1.4) holds. From (1) above, \( \mathcal{A}_1 \) is an infinite set.

3. We could describe \( \mathcal{A}_1 \) more clearly. Let \( \mathbb{H} := \{z \in \mathbb{C} \mid I m(z) > 0\} \) be the upper half plane. Then all the maps of the form of (1.4) maps \( \mathbb{H} \) onto itself. Such maps form a group \( \text{sl}(2, \mathbb{Z}) \), called the modular group.

We define its fundamental region to be a subset \( D \subset \mathbb{H}^+ \)
1.6. DIVISORS

(i) if $\tau \in \mathbb{H}^+, \exists \tau' \in D$ s.t $\tau' \sim \tau$ in $SL(2, \mathbb{Z})$,

(ii) For any $\tau, \tau' \in D, \tau \not\sim \tau'$ in $SL(2, \mathbb{Z})$.

One of the fundamental region is shown as above. $A_1$ and $D$ are bijective. Such $D$ is called the moduli space of tori.

4. We will prove that any compact Riemann Surface of $g = 1$ is biholomorphic to $\mathbb{C}^1 / \Gamma$. Hence $D$ gives the modular space of all compact Riemann Surface of $g = 1$. $D$ has a compactification $\hat{D}$ with a singularity.

5. Recall that any complex torus $\mathbb{C}/\Gamma$ is always algebraic curve. But in higher dimensional case, $\mathbb{C}^n / \Gamma$ is not necessary an algebraic manifold.

Let $\Gamma = Span_{\mathbb{Z}}(\omega_1, ..., \omega_{2n})$. Denote $\omega_j = (\omega_j^1, ..., \omega_j^n) \in \mathbb{C}^n$, which are called the periods of the torus. Denote

$$\Omega = \begin{pmatrix}
\omega_1^1 & ... & \omega_1^n \\
\omega_2^1 & ... & \omega_2^n \\
... & ... & ...
\end{pmatrix}
$$

which is called the period matrix.

It is known by Siegel (1946) that $\mathbb{C}^n / \Gamma$ is algebraic if and only if $\Omega$ satisfies the following: there exists a $2n \times 2n$ integral alternating matrix $J$ such that

(i) $^t\Omega J^{-1}\Omega = 0$.

(ii) $^t\Omega J^{-1}\Omega > 0$.

If $\mathbb{C}^n / \Gamma$ is algebraic, it is called an abelian variety. Majority of complex tori are not abelian varieties.

1.6 Divisors

A divisor $D$ on $M$ is a locally finite sum of the form: $\sum_{p \in M} n(p) p$ where $n(p) \in \mathbb{Z}$ and $n(p) \neq 0$ only for a discrete subset of $p$’s.

**[Example]** Let $f$ be a meromorphic functions. Let $\{p_j\}$ and $\{q_j\}$ be its zero and pole set with multiplies $n_j$ and $m_j$, respectively. Then we define the divisor $(f)$ associated with $f$.

$$ (f) = \sum n_j p_j - \sum m_j q_j = \sum_{p \in M} v_p(f)p $$
Hence

\[ v_p(f) = \begin{cases} 
0 & \text{if } f(p) \neq 0, \text{ finite} \\
\ord_p f & \text{if } p \text{ is a zero} \\
-\ord_p f & \text{if } f(p) = \infty.
\end{cases} \]

[Example] Let \( M = S^2 = \mathbb{C}\mathbb{P}^1 \). Then

\[ f = \begin{cases} 
z, & \text{at } [1 : z] \\
\infty, & \text{at } [0 : 1]
\end{cases} \]

Then \( (f) = 0 - \infty \).

For two divisors \( D_1 = \sum n_1(p)p, D_2 = \sum n_2(p)p \), we define \( D_1 \pm D_2 = \sum (n_1(p) \pm n_2(p))p \).

The collection of all divisors \( \mathcal{D} \) forms an abelian group, called the group of divisors.

Write

\[ \mathcal{P} = \{(f) : f \text{ meromorphic over } M \}. \]

\( \mathcal{P} \) is called the group of principle divisors. For \( D = \sum n(p)p \), we define

\[ \deg(D) := \sum n(p), \]

called the degree of \( D \). By the Residue theorem, for \( D \in \mathcal{P}, \deg(D) = 0 \). The quotient group \( \mathcal{D}/\mathcal{P} := \mathcal{D} \) will be important for our later discussions.

Definition For \( D, D' \in \mathcal{D} \), if \( D - D' \in \mathcal{P} \), we say \( D \) and \( D' \) are linearly equivalent.

Let \( M \) be a Riemann surface. Let \( (U, z_U) := (U, \Phi) \) be a holomorphic coordinate system. Let \( f_U \) be a meromorphic function on \( U \). Then \( f_Udz_U := f\Phi^*dz \) is called a meromorphic differential (or meromorphic form) over \( U \). For any point \( p \in U \), the order \( n_p \) of this differential is defined to be

\[ n_p := \begin{cases} 
\text{the order of zeros of } f_U \text{ at } p, & \text{if } f_U(p) = 0, \\
-\text{the order of poles of } f_U \text{ at } p, & \text{if } f_U(p) = \infty, \\
0, & \text{if } f_U(p) \neq 0, \infty.
\end{cases} \]

When \( f_U \) is holomorphic, \( f_Udz_U \) is called a holomorphic differential (or holomorphic 1-form).

\( \omega \) is called a meromorphic differential (or meromorphic 1-form) if on any coordinate system \( (U, z_U), \omega|_U = f_Udz_U \) where \( f_U \) is a meromorphic function on \( U \), and if for another coordinate system \( (W, dz_W) \), we have on \( U \cap W \):

\[ f_U \frac{dz_U}{dz_W} = f_W, \]
where $\frac{dz_U}{dz_W}$ is the determinant of the Jacobian matrix. Then $f_Udz_U = f_Wdz_W$ holds on $U \cap W$. We can define the order of $\omega$ at $p$ to be the one for $f_Udz_U$ where $p \in U$. This definition is independent of choice of $U$. A meromorphic differential $\omega$ is called a holomorphic differential (or holomorphic form) if $f_U$ is holomorphic on any coordinates system $(U, dz_U)$.

Let $\omega$ be a meromorphic differential. We define $(\omega) = \sum_{p \in M} n_p(\omega)p$ where $n_p(\omega)$ is the order of $\omega$ at $p$.

**Remarks**

1. For complex manifold of complex dimension $n > 1$, a divisor $D$ is defined to be

$$\sum_j a_j V_j$$

where $\sum$ is locally finite sum, $a_j \in \mathbb{Z}$ and $V_j$ are irreducible analytic hypersurfaces. All divisors form a group $\mathcal{D}$. $D = \sum_j a_j V_j$ is called effective if $a_j \geq 0$ for all $j$.

2. Let $p \in M$ and $g$ a holomorphic function defined near $p$. Let $V$ be any irreducible analytic hypersurface in $M$. Let $U$ be an open subset in $M$ containing $p$. Then we can write $V \cap U = \{z \in U \mid f(z) = 0\}$ where $f \in \text{Hol}(U)$. We can define the order of $g$ along $V$ at $p$ by

$$\text{ord}_{V,p}(g) = a$$

where $a$ is the largest integer such that

$$g = f^ah$$

in the local ring $\mathcal{O}_{M,p}$. $\text{ord}_{V,p}(g)$ is independent of choice of $p$ so that we can define the order of $g$ along $V$

$$\text{ord}_V(g) := \text{ord}_{V,p}(g).$$

If $g = \frac{g_1}{g_2}$ is meromorphic function near $p$ where $g_1, g_2$, which are relatively prime, are holomorphic functions near $p$, and $V$ is irreducible analytic hypersurface in $M$, we define the order of $g$ along $V$ by

$$\text{ord}_V(g) = \text{ord}_V(g_1) - \text{ord}_V(g_2).$$

We say that $g$ has a zero of order $a$ along $V$ if $\text{ord}_V(f) = a > 0$, and that $f$ has a pole of order $a$ along $V$ if $\text{ord}_V(g) = -a < 0$. 


3. For any meromorphic function \( g \) on \( M \), we define a divisor
\[
(g) := \sum_V \text{ord}_V(g)V
\]
where \( V \) runs through all irreducible analytic hypersurfaces.
Write \( g = \frac{g_1}{g_2} \) locally where \( g_j \) are holomorphic, we can write
\[
(g) = (g)_0 - (g)_{\infty}
\]
where \( (g)_0 = (g_1) \) and \( g_{\infty} = (g_2) \).

4. We cannot define \( \deg(D) \) when \( n > 1 \) as did on Riemann surfaces. We can define the group \( \mathcal{P} \) and \( \mathcal{D} = \mathcal{D}/\mathcal{P} \) in a different point of view.

For the meromorphic forms \( \omega \) and \( \omega' \), \( \frac{\omega'}{\omega} = f \) is a meromorphic function if \( \omega' \neq 0 \). Hence \( (\omega) \) and \( (\omega') \) are linearly equivalent. Therefore we can define the following.

**Definition** Write \( K = [(\omega)] \) for the equivalent class of divisor of meromorphic differential in \( \mathcal{D} \).

By the definitions, \( \deg(K) := \deg(\omega) \) which is independent of choice of \( \omega \). If \( r \) is a holomorphic 1-form, then \( K = [(r)] \), and \( \deg(r) = \)the sum of zeros of \( r = d(K) \).

We’ll show from Riemann-Roch theorem that \( \deg(K) = -\chi(M) = 2g - 2 \), where \( \chi(M) \) is the Euler characteristic number of \( M \) and \( g \) is the genus of \( M \). Notice that \( \chi(M) \) and \( g \) are topological invariants while \( \deg(K) \) is a holomorphic invariant.

**Definition**

1. For any \( D \in \mathcal{D} \), we call \( D \geq 0 \) or \( D \) effective if \( D = \sum n(p)p \) with \( n(p) \geq 0 \),
2. \( \ell(D) = \{ f \in \mathcal{M}(M), \text{ the collection of meromorphic functions over } M, (f) + D \geq 0 \} \).
   Since \( v_p(f + g) \geq \min\{v_p(f), v_p(g)\} \), we see that \( \ell(D) \) is a vector space over \( \mathbb{C} \).
3. \( i(D) = \{ \omega : \text{meromorphic } 1 \text{-form over } M, (\omega) + D \geq 0 \} \). \( i(D) \) is also a vector space over \( \mathbb{C} \).

[Example]

1. If \( D < 0 \), then \( \ell(D) = \emptyset \).
   In fact, from \( (f) + D = \sum_p \text{ord}_p(f)p + \sum_p n_pp \geq 0 \) where \( n_p = 0 \) or \( n_p < 0 \), it implies \( f \) is holomorphic function so that it is a constant. Also if \( n_p < 0 \) for some \( p \in M \), \( f \) must order of zero \( \text{ord}_p(f) \geq |n_p| \), which is impossible for a constant function. So such \( f \) does not exist.
2. If $D \geq 0$, then $\dim_{\mathbb{C}} \ell(D) \geq 1$.

In fact, from $(f) + D = \sum_p \text{ord}_p(f)p + \sum_p n_p p \geq 0$ where $n_p \geq 0$, it implies that $f$ is a meromorphic function. Then $\ell(D)$ contains at least all constant functions which is of dimension 1.

3. If $p \in M$, then $\{p\} \in \mathcal{D}$ and $\dim_{\mathbb{C}} \ell(\{p\}) \leq 2$.

In fact, from $(f) + \{p\} \geq 0$, it implies that $f$ is a meromorphic function with one and only one pole at $p$ with order 1. All constant functions form 1 dimensional vector space. If there exists a meromorphic function $f$ with a simple pole at $p$, it generates a 1-dim vector space. In that case, $\dim_{\mathbb{C}} \ell(D) = 2$. Otherwise, $\dim_{\mathbb{C}} \ell(D) = 1$.

4. If $D = 5p - q$, then $\dim_{\mathbb{C}} \ell(D) = \{\text{all meromorphic functions } f \text{ such that it has exactly one pole } p \text{ with } |\text{ord}_p(f)| \leq 5 \text{ and exactly one zero } q \text{ with } \text{ord}_q(f) \geq 1\}$.

Lemma 1.17 $\dim_{\mathbb{C}} \ell(D)$, $\dim_{\mathbb{C}} i(D) < \infty$.

Proof: Write $D = D^+ - D^-$ with $D^+ \geq 0, D^- \geq 0$. Then

$$\ell(D) \subseteq \ell(D^+) = \{f \in \mathcal{M}(M), (f) \geq -D^+ = -\sum_{j=1}^m n_j p_j, n_j \geq 0\}$$

and hence $\dim_{\mathbb{C}} \ell(D) \leq \dim_{\mathbb{C}} \ell(D^+)$.

It suffices to prove the claim:

$$\dim_{\mathbb{C}} \ell(D^+) \leq \deg(D^+) + 1,$$  \hspace{1cm} (1.5)

which will be proved by using induction on $\deg D^+$ as follows.

(a) When $\deg D^+ = 0$, it is obvious, for $\ell(D^+) = \{\text{constant functions}\}$.

(b) Suppose it is true when $\deg D^+ \leq N$.

Now, consider $\deg D^+ = N + 1$. Write $D = \sum_p n_p p_j$. Without loss of generality, we assume $n_{p_1} > 0$.

Consider two cases.
The first case: for any \( f \in \ell(D), f(p_1) \neq \infty \) (i.e., \( f \) is holomorphic at \( p_1 \)). Then

\[
\dim_{\mathbb{C}} \ell(D^+) = \dim_{\mathbb{C}} \ell((D')^+) \quad (\text{here } D' = (n_{p_1} - 1)p_1 + \sum_{j=2}^{m} n_{p_j}p_j)
\]

\[
\leq \deg((D')^+) + 1 \quad (\text{by the induction assumption})
\]

\[
= [(N + 1) - 1] + 1 \quad (\text{deg}((D')^+) = \det(D^+) - 1 = (N + 1) - 1)
\]

\[
= N + 1.
\]

The second case: there exists \( f_1 \in \ell(D') \) such that \( f_1 \) has pole at \( p_1 \). Then for any \( f \in \ell(D) \), there is a constant \( C_f \) such that \( f = C_ff_1 + \mathfrak{g} \) with \( \text{ord}_{p_1}\mathfrak{g} < n_{p_1} \). Thus \( \ell(D) = \text{Span}_{\mathbb{C}}\{f_1\} + \ell(D') \) with \( D' = (n_{p_1} - 1)p_1 + \sum_{j=2}^{m} n_{p_j}p_j \). Then

\[
\dim_{\mathbb{C}} \ell(D^+) = 1 + \dim \ell((D')^+)
\]

\[
\leq 1 + N. \quad (\text{by the induction assumption to } D')
\]

We have proved that \( \dim_{\mathbb{C}} \ell(D) \leq N + 1 \).

\( \dim_{\mathbb{C}} i(D) < \infty \) follows from a more general lemma below. \( \Box \)

**Lemma 1.18** \( i(D) \cong \ell(K - D) \).

**Proof:** Let \( \omega \) be a meromorphic 1-form, \( K = [(\omega)]. \eta \in i(D) \iff (\eta) - (D) \geq 0 \iff (\eta) - (\omega) + (\omega) - D \geq 0 \iff (\frac{\eta}{\omega}) + ((\omega) - D) \geq 0 \). Here \( \frac{\eta}{\omega} \) is a meromorphic function. Hence, the map \( \eta \mapsto \frac{\eta}{\omega} \) gives the isomorphism. \( \Box \)

**Lemma 1.19** If \( D_1 \sim D_2 \), then \( \ell(D_1) \cong \ell(D_2) \) and \( i(D_1) \cong i(D_2) \).

**Proof:** \( \ell(D_1) = \{f : (f) \geq -D_1\} \) and \( \ell(D_2) = \{f : (f) \geq -D_2\} \). If \( D_1 = D_2 + (h) \) where \( h \) is a meromorphic function over \( M \), then

\[
\ell(D_1) = \{f : (f) \geq -D_2 - (h)\} = \{f : (f) + (h) \geq -D_2\} = \{f : (fh) \geq -D_2\}.
\]

Define \( \phi : \ell(D_1) \mapsto \ell(D_2) \),

\[
f \mapsto fh,
\]

which is an isomorphism. Then \( \ell(D_1) \sim \ell(D_2) \) holds.

The second equality follows from above lemma. \( \Box \)


1.7 Statement of the Riemann-Roch theorem

Let $M$ be a compact Riemann surface. Then topologically, it is an oriented smooth manifold of dim 2. By the classification theorem of real surfaces, if $g = \frac{1}{2} \dim H^1(M, \mathbb{Z})$, then $M$ is diffeomorphic to a sphere with $g$-bundles attached to it.

The famous Riemann-Roch theorem is to connect the meromorphic objects with topology.

**Theorem 1.20** Let $M$ be a compact Riemann Surface. Let $D$ be a divisor on $M$, then

$$\dim \ell(D) - \dim i(D) = \deg(D) + (1 - g)$$

or

$$\dim \ell(D) - \dim l(K - D) = \deg(D) + (1 - g).$$

We will prove the theorem later. In this section, we’ll give some applications of the R.-R. theorem.

**Lemma 1.21** Let $M$ be a compact Riemann surface. There is a point $p \in M$ such that

$$\dim \ell((p)) \geq 2 \iff M \text{ is biholomorphic to } S^2 = \mathbb{CP}^1.$$

**Proof:**

$$\ell((p)) = \{f \in \mathcal{M}(M) \mid (f) + (p) \geq 0\} = \{f \in \mathcal{M}(M) \mid f \text{ only has at most a simple pole at } p\}.$$

Then

$$\dim \ell(p) \geq 2 \iff \exists f \in \ell(p) \text{ with } f \neq \text{constant}.$$

Hence

$$\hat{f} : M \rightarrow \mathbb{CP}^1$$

$$z \mapsto f(z) := \begin{cases} f(z), & f(z) \in \mathbb{C}, \\ \infty, & f(z) = \infty. \end{cases}$$

Claim: $\hat{f}$ is a biholomorphic map.

$\hat{f}$ is one-to-one and onto: Since

$$0 = \int_C \frac{df}{f} = \#(\text{zeros of } f) - \#(\text{poles of } f) = \#(\text{zeros of } f) - 1,$$

the equation $f(z) = 0$ has exactly one point. Similarly consider $f - a$ instead of $f$ for any $a \in \mathbb{C}$. We know that the equation $f(z) - a = 0$ has exactly one solution. This proves one-to-one. Also, it shows that $\hat{f}$ is surjective.
Theorem 1.22 Let $M$ be a compact Riemann surface with $g(M) = 0$. Then $M$ is biholomorphic to $S^2 = \mathbb{CP}^1$.

Proof: Fixing any $p \in M$, we apply the Riemann-Roch theorem with $D = (p)$

$$\dim \mathcal{O}(D) - \dim \mathcal{I}(D) = \deg(D) + (1 - g)$$

$$- \dim \{\omega \mid (\omega) \geq (p)\} \parallel 1$$

$$\parallel 0$$

to get $\dim \mathcal{O}((p)) = 2$. Here we use the fact that $\{\omega \mid (\omega) \geq (p)\} \subset \mathcal{H}^{1,0}(M)$ and $\dim \mathcal{H}^{1,0}(M) = g = 0$ from the result below. Next we apply Theorem above to conclude that $M$ is biholomorphic to $\mathbb{CP}^1$. \(\square\)

Theorem 1.23 Let $M$ be a compact Riemann surface. Denote $\mathcal{H}^{1,0}(M) := \{\text{holomorphic 1-forms on } M\}$. Then $\dim \mathcal{H}^{1,0}(M) = g$.

Proof: Applying the Riemann-Roch theorem with $D = 0$, we have

$$\dim \mathcal{O}(D) - \dim \mathcal{I}(D) = \deg(D) + (1 - g)$$

$$\dim \{f \in \mathcal{M}(M) \mid (f) \geq 0\} - \dim \{\omega \mid (\omega) \geq 0\} \parallel 0$$

$$\parallel \dim \mathcal{O} \mathcal{H}(M)$$

$$\parallel 1 - \dim \mathcal{H}^{1,0}(M)$$

\(\square\)

Corollary 1.24 (1) $\deg(K) = -\chi(M) = 2g - 2$ where $\chi(M)$ is the Euler characteristic of $M$.

(2) Let $\omega$ be a holomorphic 1-form on $M$. Then the zeros of $\omega$, counting multiplicity, is equal to $-\chi(M) = 2g - 2$.

\(\square\)

\(5\)The Euler characteristic $\chi$ was classically defined for polyhedra, according to the formula: $\chi = V - E + F$ where $V, E$, and $F$ are respectively the numbers of vertices's (corners), edges and faces in the given polyhedron. $\chi$ is a topological invariant.
1.7. STATEMENT OF THE RIEMANN-ROCH THEOREM

Proof: (1) Applying the Riemann-Roch theorem with $D = 0$:

$$\dim_\mathbb{C} \ell(D) - \dim_\mathbb{C} \ell(K - D) = \deg(D) + (1 - g)$$

$$\begin{array}{c|c|c}
1 & - & \dim_\mathbb{C} \ell(K) \\
\hline
\end{array}$$

Hence $\dim_\mathbb{C} \ell(K) = g$. Also, we apply the Riemann-Roch theorem with $D = K$:

$$\dim_\mathbb{C} \ell(K) - \dim_\mathbb{C} \ell(K - K) = \deg(K) + (1 - g)$$

$$\begin{array}{c|c|c}
g & - & \dim_\mathbb{C} \ell(0) \\
\hline
1 & & \\
\end{array}$$

Here we used $\dim_\mathbb{C} \ell(K) = g$. Then $\deg(K) = 2g - 2$.

(2) By $K = [(\omega)]$, $\deg(K) = \deg(\omega) = \#(\text{zeroes of } \omega)$ and (1) above. □

Remark

1. If $g(M) > 0$, $\forall p \in M$, $\dim_\mathbb{C} \ell((p)) = 1$ because of Lemma 1.21.

2. Classical result: If $g = 1$, $\dim_\mathbb{C} \ell(2p) \geq 2$ (existence of elliptic functions).

3. Long before the Riemann-Roch theorem, people knew that for majority Riemann surfaces with $g = 3$, $\dim_\mathbb{C} \ell((2p)) = 1$ but there are a very few ones with $\dim \ell((2p)) = 2$, which are called the hypo-elliptic Riemann surfaces.

Remarks

1. When $n = 1$, a divisor $D = \sum p_n p$. Denote by $\mathcal{D}$ the group of divisors.

When $n > 1$, a divisor $D = \sum V_a V$ where $V$ are irreducible analytic hypersurfaces. Denote $\text{Div}(M) = H^0(M, \mathcal{M}^*/\mathcal{O}^*)$, also called the group of divisors. In fact, if locally $D$ is given by $f_\alpha$ where $f_\alpha \in \mathcal{M}(U_\alpha)$, then $f := \frac{f_\alpha}{f_\beta} \in H^0(M, \mathcal{M}^*/\mathcal{O}^*)$ is a global meromorphic section of the sheaf $\mathcal{M}^*/\mathcal{O}^*$.

2. When $n = 1$, $\forall f \in \mathcal{M}(M), (f) = \sum p \text{ord}_p(f) p \in \mathcal{D}$. Denote $\mathcal{P} = \{(f) \mid f \in \mathcal{M}(M)\}$. When $n > 1$, $\forall f \in \mathcal{M}(M), (f) = \sum V \text{ord}_V(f) V$ where $V$ are irreducible analytic hypersurfaces. we could denote $\mathcal{P} = \{(f) \mid f \in \mathcal{M}(M)\}$. $\mathcal{P} \cong H^0(M, \mathcal{M}^*)$. 
3. When $n = 1$, we denote $D = \overline{D}/\mathcal{P}$. $D \sim D'$ if and only if $D - D' \in \mathcal{P}$ if and only if $D' = D + (f)$ for some $f \in \mathcal{M}(M)$. We say that $D$ and $D'$ are linearly equivalent.

When $n > 1$, we could consider $\text{Div}(M)/\mathcal{P}$ which is embedded into the Picard group (see below). $D \sim D'$ if and only if $D - D' \in \mathcal{P}$ if and only if $D' = D + (f)$ for some $f \in \mathcal{M}(M)$. We say that $D$ and $D'$ are linearly equivalent.

4. When $n = 1$, the same as below.

When $n > 1$, any divisor $D$ is associated a line bundle $[D]$. In fact, if $D$ is given by $f_\alpha \in \mathcal{M}(U_\alpha)$, then $g_{\alpha\beta} = \frac{f_\beta}{f_\alpha}$ gives transition functions which define a line bundle $[D]$.

5. When $n = 1$, the same as below. We have a homomorphism $[\ ] : \overline{\mathcal{D}} \to \text{Pic}(M)$, $D \mapsto [D]$. In the case $n = 1$, it is indeed an isomorphism.

When $n > 1$, Denote by $\text{Pic}(M)$ the Picard group, the set of all line bundles. We have $\text{Pic}(M) \cong H^1(M, \mathcal{O}^*)$. In fact, a line bundle is given by transition functions $\{g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)\}$ which is an element of $H^1(M, \mathcal{O}^*)$. We define a homomorphism: $[\ ] : \text{Div}(M) \to \text{Pic}(M)$, $D \mapsto [D]$. The kernel is $\mathcal{P}$. In the case $n > 1$, this may not be surjective (see an explanation below).

In fact, a line bundle $L = [D]$ for some divisor $D$ if and only if there is a meromorphic section of $L$. In fact if $D$ is given by $f_\alpha \in \mathcal{M}(U_\alpha)$, $\{f_\alpha\}$ gives a nontrivial meromorphic section $s$ of $[D]$ such that $(s) = D$. Similarly, a line bundle $L = [D]$ for some effective divisor $D$ if and only if there is a nontrivial holomorphic section of $L$.

6. When $n = 1$, the same as below.

When $n > 1$, from the exact sequence $0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^* \to 0$, one has the exact sequence

$$
\begin{array}{cccc}
H^0(M, \mathcal{M}^*) & \xrightarrow{\alpha} & H^0(M, (O)^*/\mathcal{O}^*) & \xrightarrow{\beta} & H^1(M, \mathcal{O}^*) & \xrightarrow{\gamma} & H^1(M, \mathcal{M}^*) \\
\mathcal{P} & \xrightarrow{\text{Div}(M)} & \text{Pic}(M) & & & & \\
\end{array}
$$

$\beta$ is surjective if $\ker \gamma = 0$. In case $\dim H^0(M, (O)^*/\mathcal{O}^*) \neq 0$, $\text{Div}(M)/\mathcal{P}$ may not be isomorphic to $\text{Pic}(M)$.

7. When $n = 1$, we define a vector space $\ell(D) = \{f \in \mathcal{M}(M) \mid (f) + D \geq 0\}$. Let $D = \sum p_{\alpha}n_{\alpha}$. Then $f \in \ell(D)$ if and only if $\text{ord}_{\alpha}(f) \geq -n_{\alpha}$, $\forall \alpha$.

Theorem (cf. p. 161, Griffiths and Harris, Principle of Algebraic Geometry) Let $M \subset \mathbb{C}P^N$ be a submanifold. Then every line bundle on $M$ is of the form $L = [D]$ for some divisor $D$. 


When \( n > 1 \), we define \( \mathcal{L}(D) := \{ f \in \mathfrak{M}(M) \mid (f) + D \geq 0 \} \). Let \( D = \sum a_V V \). Then \( f \in \mathcal{L}(D) \) if and only if \( \text{ord}_V(f) \geq -a_V, \forall V \). Also we have an isomorphism \( \otimes s_0 : \mathcal{L}(D) \to H^0(M, \mathcal{O}[D]) \), \( f \mapsto s := fs_0 \) where \( s_0 \) is a global meromorphic section of \( [D] \); conversely, \( s \mapsto f := \frac{s}{s_0} \).

8. When \( n = 1 \), the same as below.

When \( n > 1 \), any two holomorphic sections of a line bundle \( [D] \) differ by a constant, so that we can consider \( \mathbb{P}(\mathcal{L}(D)) \). More precisely, we denote by \( [D] \subset \text{Div}(M) \) the set of all effective divisors linearly equivalent to \( D \). In fact, for every effective \( D' \in [D] \), there exists \( f \in \mathcal{L}(D) \) such that \( D' = D + (f) \), where \( f = \frac{s}{s_0} \) where \( s_0 \) is meromorphic section of \( [D] \) and \( s \) is holomorphic section with \( (s) = D' \). Any two such functions \( f, f' \) differ by a nonzero constant. Thus we have \( [D] \cong \mathbb{P}(\mathcal{L}(D)) \cong \mathbb{P}(H^0(M, \mathcal{O}([D]))) \).

As a result, if \( s \) is any holomorphic section (i.e., an effective divisor) of the line bundle \( [D] \), then \([\text{zero}(s)] = [D]\) in \( \text{Pic}(M) \).

9. When \( n = 1 \), for any divisor \( D = \sum_p n_p p \), we define \( \deg(D) = \sum_p n_p \) which is an integer. We have formula: \( \deg(D) = \langle c_1([D]), [M] \rangle = \frac{i}{2\pi} \int_M \Theta \) where \( [D] \) is the associated line bundle, \( c_1([D]) \) is the first Chern class of the line bundle \( [D] \) and \( \Theta \) is the curvature form.

When \( n > 1 \), for any line bundle \( L \) with the curvature form \( \Theta \), \( c_1(L) = \frac{i}{2\pi} \Theta \) \( \in H^2(M, \mathbb{Z}) \). This comes from the short exact sequence: \( 0 \to \mathbb{Z} \to \mathcal{O} \to \mathcal{O}^* \to 0 \) and hence \( H^1(M, \mathcal{O}) \to H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z}) \).

10. When \( n = 1 \), we know \( \deg(f) = 0 \) for any \( f \in \mathfrak{M}(M) \) by the Residue theorem.

When \( n > 1 \), we have \( c_1((f)) = \int_M \Theta = 0 \) because \( (f) \in \mathcal{P} \) means \([f] = 0\) in \( \text{Div}(M)/\mathcal{P} \) and \( \delta : H^1(M, \mathcal{O}^*) \to H^2(M, \mathbb{Z}) \) is a group homomorphism.

11. When \( n = 1 \), \( K = [\omega] \) where \( \omega \) is a meromorphic or holomorphic 1-form, i.e. \( \omega \) is a meromorphic or holomorphic section of \( T^*M \).

When \( n > 1 \), the canonical line bundle \( K := \wedge^n T^*M \). Notice that \( T^*M \) is a vector bundle of rank \( n = \dim M \) but \( \wedge^n T^*M \) is a line bundle.
CHAPTER 1. RIEMANN-ROCH THEOREM

1.8 Bergmann metric on $M$ with $g(M) > 0$

We use the Riemann-Roch theorem to construct the Bergman metric on any compact Riemann Surface $M$ with $g(M) > 0$. We first prove the following:

**Lemma 1.25** Let $M$ be a compact Riemann surface with $g = g(M) > 0$. For any $p \in M$, $\exists \omega \in \mathcal{H}^{1,0}(M)$ such that $\omega(p) \neq 0$.

**Proof:** Suppose that there is some point $p \in M$ such that $\forall \text{ holomorphic 1-form } \omega \text{ over } M \text{ with } \omega(p) = 0$. (1.6)

Let $D = (p)$. Then

$$i(D) := \{ \text{ meromorphic 1-form } \omega : (\omega) \geq (p) \}$$
$$= \{ \text{ holomorphic 1-form } \omega : (\omega) \geq (p) \}$$
$$= \mathcal{H}^{1,0}(M) \quad (By \text{ (1.6), there is no restriction}).$$

By the Riemann-Roch theorem with $D = (p)$,

$$\dim \mathcal{H}(D) - \dim \mathcal{H}(i(D)) = deg(D) + (1 - g)$$
$$\|1| - \dim \mathcal{H}(\mathcal{H}^{1,0}(M))|1 = -g$$

Then we get $1 - g = 1 + 1 - g$ or $1 = 0$ that is a contradiction. □

Now, we introduce a Hermitian metric $\langle , \rangle$ on $\mathcal{H}^{1,0}(M)$ as follows: For $\xi, \eta \in \mathcal{H}^{1,0}$, we define

$$\langle \xi, \eta \rangle = \int_M \xi \wedge \overline{\eta}$$

If $\xi \neq 0$, then apparently, $\langle \xi, \xi \rangle > 0$. By Lemma above, such $\xi$ exists.

Recall $\dim \mathcal{H}^{1,0}(M) = g$. We can take $\omega_1, \ldots, \omega_g$ as an orthonormal basis of $\mathcal{H}^{1,0}, \langle , \rangle \}$.

We define $G = \sum_{i=1}^{g} \omega_i \otimes \overline{\omega_i}$, namely,

$$G : T_p^{1,0}(M) \times T_p^{1,0}(M) \rightarrow \mathbb{C}$$
$$(T_1, T_2) \mapsto \sum_{j=1}^{g} \omega_j(T_1)\overline{\omega_j(T_2)}.$$

Hence, by Lemma 1.25, $G$ is indeed a Hermitian metric.

Let $F : M \rightarrow M'$ be a biholomorphic map. Then if $\{\omega'_i\}$ is an orthonormal basis of $\mathcal{H}^{1,0}(M')$, then it is easy to see that $\{F^*(\omega'_i)\}$ is also an orthonormal basis for $T^{1,0}(M)$. Hence $F^*G' = G$. Namely, $F$ is an isometry w.r.t the Bergmann metric.
1.9 More applications of the Riemann-Roch theorem

We present more application of the Riemann-Roch theorem on the structure of meromorphic functions. As above, we always consider compact Riemann surfaces.

**Proposition 1.26** Let $M$ be a compact Riemann surface and $D$ a divisor on $M$.

(a) Let $D$ be a divisor such that $\deg(D) \geq 2g - 1$. Then $\dim_{\mathbb{C}} \ell(D) = \deg(D) + (1 - g)$.

(b) If $n > 0$ and $\deg(D) = g + n$, then $\dim_{\mathbb{C}} \ell(D) \geq 1 + n$.

**Proof:** (a) Apply the Riemann-Roch theorem with $D$,

$$\dim_{\mathbb{C}} \ell(D) = \dim_{\mathbb{C}} \ell(K - D) + \deg(D) + (1 - g) = 0 + \deg(D) + (1 - g).$$

Here we use the fact that $\dim_{\mathbb{C}} \ell(K - D) = 0$ because of the following argument. First,

$$\deg(K - D) = \deg(K) - \deg(D)$$

$$= (2g - 2) - \deg(D) \quad (because \ \deg(K) = 2g - 2)$$

$$\leq (2g - 2) - (2g - 1) < 0.$$

Second, for any divisor $D_1 < 0$, we must have $\ell(D_1) = \{f \in \mathcal{M}(M) \mid (f) \geq -D_1\} = \{f \in \text{Hol}(M) \mid (f) \geq -D_1\} = \emptyset$ so that $\deg(D_1) = 0$. As a result, $\dim_{\mathbb{C}} \ell(K - D) = 0$.

(b) By the Riemann-Roch theorem,

$$\dim_{\mathbb{C}} \ell(D) = \dim_{\mathbb{C}} \ i(D) + \deg(D) + (1 - g)$$

$$\geq 0 + \deg(D) + (1 - g)$$

$$\geq (g + n) + (1 - g) \quad (because \ \deg(D) = g + n)$$

$$= n + 1.$$

**Corollary 1.27** For any point $p \in M$, $\exists f \in \mathcal{M}(M)$ such that $f$ only has pole at $p$ with $v_p(f) \geq -2g$.

**Proof:** Let $D = 2gp$. Then $\deg(D) = 2g \geq 2g - 1$. By Proposition 1.26, $\dim_{\mathbb{C}} \ell(D) = 2g + (1 - g) = g + 1 \geq 2$; otherwise $g = 0 \Leftrightarrow M \cong \mathbb{CP}^1$ but Corollary is true in this case.

This implies that there is $f \in \mathcal{M}(M)$ such that $(f) + (2gp) \geq 0$, i.e., $f$ has only pole at $p$ with $v_p(f) \geq -2g$. □

**Corollary 1.28** $\forall p, q \in M, p \neq q, \exists f \in \mathcal{M}(M)$ such that $f(p) \neq f(q)$. 
CHAPTER 1. RIEMANN-ROCH THEOREM

Proof: \( \forall p, p' \in M, p \neq p', q \neq p' \). Let

\[ D_1 := (2g + 2)p' - p \implies \deg(D_1) = (2g + 2) - 1 = 2g + 1, \]

\[ D_2 := (2g + 2)p' - p - q \iff \deg(D_2) = (2g + 2) - 1 - 1 = 2g. \]

By Proposition 1.26, we obtain

\[ \dim_{\mathbb{C}} \ell(D_1) = (2g + 1) + (1 - g) = g + 2, \]

\[ \dim_{\mathbb{C}} \ell(D_2) = 2g + (1 - g) = g + 1. \]

Then \( \exists f \in \ell(D_1) \) but \( f \not\in \ell(D_2) \), i.e.,

\[ (f) + (2g + 1)p' - p \geq 0 \text{ and } “(f) + (2g + 2)p' - p - q \geq 0” \text{ does not hold.} \]

Then

\[ f \text{ has zero at } p \text{ and } f \text{ cannot have zero at } q, \]

i.e., \( f(p) = 0 \) but \( f(q) \neq 0 \). \( \square \)

**Corollary 1.29** If \( g(M) = 1 \), \( \exists f \in \mathfrak{M}(M) \) such that \( f \) only has a double pole at \( p \in M \).

**Proof:** Let \( D = 2p \). Then \( \deg(D) = 2 \geq 2 \cdot 1 - 1 \). We apply Proposition 1.26 to have

\[ \dim_{\mathbb{C}} \ell(D) = 2 + (1 - 1) = 2. \]

Then \( \exists f \in \ell(D) \). \( \square \)

**Remark** Let \( \Gamma = \text{Span}_\mathbb{Z}\{1, \omega\} \) as before \( Im \omega > 0 \). Then \( M = \mathbb{C}^1/\Gamma \) is a torus with \( g(M) = 1 \). Let \( p \in D \) with \( D \) a fundamental region of \( \Gamma \). Let \( f \in \mathfrak{M}(M) \) only having a double pole at \( p \). Then \( \tilde{f} = f \circ \pi : \mathbb{C}^1 \to \mathbb{C}^1 \) is a meromorphic function on \( \mathbb{C}^1 \) with double poles at \( \{\Gamma(0)\} \). Apparently, \( \tilde{f}(z + 1) = \tilde{f}(z) \), and \( \tilde{f}(z + \omega) = \tilde{f} \). \( \tilde{f} \) is what people called an elliptic function. Hence, we derive the elliptic functions by making use of the Riemann-Roch theorem. Historically, elliptic functions were constructed more directly.

**Proposition 1.30** For any \( p \in M \), \( \exists f \in \mathfrak{M}(M) \), such that \( f(p) = 0 \), \( df(p) \neq 0 \).
1.9. MORE APPLICATIONS OF THE RIEMANN-ROCH THEOREM

Proof: Let $p \neq q$. By Proposition 1.26 we obtain
\[
\dim \mathcal{L}((2g+1)q - p) = (2g+1) - 1 + (1-g) = g + 1
\]
and
\[
\dim \mathcal{L}((2g+1)q - 2p) = (2g+1) - 2 + (1-g) = g.
\]
Then
\[
\dim \mathcal{L}((2g+1)q - p) - \dim \mathcal{L}((2g+1)q - 2p) = (g + 1) - g = 1
\]
so that there exists some $f \in \mathcal{L}((2g+1)q - p) - \mathcal{L}((2g+1)q - 2p)$ and such function $f$ has a simple zero at $p$. □

To conclude this subsection, we prove the following embedding theorem.

**Theorem 1.31** Let $M$ be a compact Riemann surface. Then there is a holomorphic map $F : M \to \mathbb{CP}^{g+1}$ such that $F$ is one-to-one and $dF_p \neq 0$ for each $p \in M$. Namely, $F$ is a holomorphic embedding.

Proof: Let $D = (2g+1)p$ where $p$ is fixed. By Proposition 1.26
\[
\dim \mathcal{L}(D) = \deg(D) + 1 - g = 2g + 2 - g = g + 2.
\]
Let $s_0 = 1$ and \( \{s_j\}_{j=1}^{g+1} \) be a basis of $\mathcal{L}(D)$. Define
\[
F : \begin{array}{cc}
M & \to \mathbb{CP}^{g+1} \\
q & \mapsto [1 : s_1(q) : \ldots : s_{g+1}(q)]
\end{array}
\]
\( \forall q \neq p \). Here $s_j$ are holomorphic at $q$, but has pole at $p$.

\[ \implies F \text{ is a holomorphic map on } M - \{p\} \].

When $q \in M$ near $p$, we let $z = \Phi$ be the holomorphic coordinate function with $\Phi(p) = 0$. Then
\[
s_j(z) = \frac{\tilde{s}_j(z)}{z^n}
\]
where \( \tilde{n} = \max|v_p(s_j)| \), $\tilde{s}_j(z)$ are holomorphic at $p$, $\forall j$, and $s_{j_0}(p) \neq 0$ for at least for a certain $j_0$.

\[ \implies \text{When } q \in M \text{ near } p, \]
\[
F(q) = [z^{\tilde{n}} : \tilde{s}_1(q) : \ldots : \tilde{s}_{g+1}(q)]
\]
so that $F$ is holomorphic at $p$.

Next we’ll prove that $F$ is one-to-one and is locally diffeomorphic (i.e., $dF_p \neq 0, \forall p \in M$).

**Injectivity:** \( \forall q, q' \in M, q \neq q' \). Let \( D_1 = D - q \) and \( D_2 = D - q - q' \). Since \( \deg(D_1) = 2g + 1 - 1 = 2g \), by Proposition 1.26, \( \dim_{\mathbb{C}} \ell(D_1) = 2g + (1 - g) = g + 1 \). Similarly, \( \dim_{\mathbb{C}} \ell(D_2) = g \). Then \( \ell(D_2) \subsetneq \ell(D_1) \subsetneq \ell(D) \).

and hence \( \exists f \in \ell(D_1) - \ell(D_2) \)

because \( \dim_{\mathbb{C}} \ell(D_1) - \dim_{\mathbb{C}} \ell(D_2) = 1 \). Then \( f(q) = 0, \text{ and } f(q') \neq 0 \).

Recall \( f = \sum_{j=0}^{g+1} a_j s_j \) because \( \{s_j\} \) form a basis. The above implies \( \sum_{j=0}^{g+1} a_j s_j(q) = 0, \text{ but } \sum_{j=0}^{g+1} a_j s_j(q') \neq 0, \)

i.e., \( [s_0 : s_1 : \ldots : s_{g+1}](q) \neq [s_0 : s_1 : \ldots : s_{g+1}](q') \).

In fact, otherwise \( s_j(q) = \lambda s_j(q'), 0 \leq j \leq g + 1, \) which gives a contradiction. We have proved that $F$ is a one-to-one map.

**Local diffeomorphism:** To show: \( \forall q \in M, dF(q) \neq 0 \). We separate two cases.

Case 1: \( q \neq p \). Let \( D_1 := D - q = (2g + 1)p - q \) and \( D_2 := D - 2q = (2g + 1)p - 2q \). As above, there exists \( f \in \ell(D_1) - \ell(D_2) \), i.e., \( f(q) = 0, \text{ and } df(q) \neq 0 \) (simple zero)

Now, to show that \( dF(q) \neq 0 \), by considering non-homogeneous coordinates, we need only to show \( (ds_1(q), \ldots, ds_{g+1}(q)) \neq 0 \). Suppose that \( (ds_1(q), \ldots, ds_{g+1}(q)) = 0 \). Write \( f = \sum_{j=0}^{g+1} a_j s_j(q) \) and thus \( df(q) = \sum_{j=1}^{g+1} a_j ds_j(q) = 0 \). That is a contradiction.

Case 2: \( q = p \). Since \( \dim_{\mathbb{C}} \ell(D) = \dim_{\mathbb{C}} \ell(D - p) + 1 = \dim_{\mathbb{C}} \ell(D - 2p) + 2 \), we can choose, without loss of generality, that \( v_p(s_{g+1}) = 2g + 1, \ v_p(s_g) = 2g \ v_p(s_j) < 2g, \text{ for } j < g. \)
Then

\[ F = \left[ \frac{s_0}{s_{g+1}}, \ldots, \frac{s_{g-1}}{s_{g+1}}, \frac{s_g}{s_{g+1}} : 1 \right]. \]

Now, \(\frac{s_g}{s_{g+1}}\) has a simple zero at \(p\). Hence \(dF(p) \neq 0\).
Chapter 2

Proof of the Riemann-Roch Theorem

2.1 Holomorphic Line Bundles

A holomorphic line bundle on a complex manifold $M$ is roughly an assignment of a complex space of dimension 1 to each point in $M$. Also the complex space depends holomorphically on the base point. More precisely, we have

**Definition** $\pi : E = \cup_{x \in M} E_x \to M$ is called a *holomorphic vector bundle* over a complex manifold $M$, where $E_x$ is a complex vector space of dimension $k$ for any $x \in M$, if

1. $E$ is a complex manifold and $\pi : E \to M$ is a holomorphic map with $\pi^{-1}(x) = E_x$ for any point $x \in M$.
2. There is a covering $\{W_\alpha\}$ of $M$ and biholomorphic map $\{\psi_\alpha\}$,

$$\psi_\alpha : \pi^{-1}(W_\alpha) \to W_\alpha \times \mathbb{C}^k$$

such that

$$\psi_\alpha|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \{x\} \times \mathbb{C}^k$$

is an isomorphism between the vector spaces $E_x = \pi^{-1}(x)$ and $\{x\} \times \mathbb{C}^k$.

The vector space $\pi^{-1}(x)$ is called the *fiber* of $E$ at $x$, the dimension of the fiber $k = \dim_{\mathbb{C}} E_x$ is called the *rank* of $E$, and the function $\psi_\alpha$ is called a *trivialization function*.

When $k = 1$, $E$ is called a *holomorphic line bundle* or simply a *line bundle* over $M$.

Let $E$ be a holomorphic vector bundle over $M$. We define

$$g_{\alpha \beta} : W_\alpha \cap W_\beta \to GL(k, \mathbb{C})$$

as

$$x \mapsto g_{\alpha \beta}(x) := \psi_\alpha(x) \psi^{-1}_\beta(x) \mid_{\{x\} \times \mathbb{C}^k}$$
for any $W_\alpha \cap W_\beta \neq \emptyset$. Such a function $g_{\alpha\beta}$ are called the transition function of $E$ related to the trivializations $\psi_\alpha$ and $\psi_\beta$. It satisfies
\[ g_{\alpha\alpha} = Id, \quad \forall x \in W_\alpha, \quad (2.1) \]
\[ g_{\alpha\beta}(x) \cdot g_{\beta\alpha}(x) = Id, \quad \forall x \in W_\alpha \cap W_\beta, \quad (2.2) \]
\[ g_{\alpha\beta}(x) \cdot g_{\beta\gamma}(x) \cdot g_{\gamma\alpha}(x) = Id, \quad \forall x \in W_\alpha \cap W_\beta \cap W_\gamma. \quad (2.3) \]

Conversely, for a complex manifold $M$ with an open covering $\{W_\alpha\}$, if there are holomorphic functions $\{g_{\alpha\beta}\}$ such that (2.1), (2.2) and (2.3) are satisfied, then there is a unique complex vector bundle $E \to M$ with transition functions $\{g_{\alpha\beta}\}$. In fact, define
\[ E := \cup_\alpha (W_\alpha \times \mathbb{C}^k)/\sim \]
where $\sim$ is an equivalent relation defined by
\[ (x, \lambda_\alpha) \sim (x, \lambda_\beta) \iff \lambda_\beta = g_{\alpha\beta} \lambda_\alpha, \quad \forall x \in W_\alpha \cap W_\beta. \]
We denote by $[x, \lambda_\alpha]$ the equivalent class of $(x, \lambda_\alpha)$. The coordinates systems $\{U_\alpha, \Psi_\alpha\}$ of $E$ are defined by
\[ U_\alpha := \{[x, \lambda_\alpha] \mid (x, \lambda_\alpha) \in W_\alpha \times \mathbb{C}^k\} \]
and
\[ \Psi_\alpha : \quad U_\alpha \to W_\alpha \times \mathbb{C}^k \]
\[ [x, \lambda_\alpha] \mapsto (\psi_\alpha(x), \lambda_\alpha) \]
and
\[ \pi : \quad E \to M \]
\[ [x, \lambda_\alpha] \mapsto x \]

Then $E$ is a holomorphic vector bundle over $M$ with the trivializations
\[ \psi_\alpha : \quad \pi^{-1}(W_\alpha) \to W_\alpha \times \mathbb{C}^k \]
\[ [x, \lambda_\alpha] \mapsto (x, \lambda_\alpha). \]
This gives the transitions $\{g_{\alpha\beta}\}$.

In particular, a line bundle $L$ over a Riemann surface $M$ is equivalent to an open covering $\{W_\alpha\}$ of $M$ and a family of holomorphic functions $\{f_{\alpha\beta} : W_\alpha \cap W_\beta \to \mathbb{C}^* = \mathbb{C} - \{0\}\}$ such that
\[ f_{\alpha\alpha} = 1, \text{ on } W_\alpha; \quad f_{\alpha\beta} f_{\beta\alpha} = 1, \text{ on } W_\alpha \cap W_\beta; \quad f_{\alpha\beta} \cdot f_{\beta\gamma} \cdot f_{\gamma\alpha} = 1, \text{ on } W_\alpha \cap W_\beta \cap W_\gamma. \]
2.1. HOLOMORPHIC LINE BUNDLES

So we can denote

\[ L \iff \{ W_\alpha, f_{\alpha\beta} \}. \]

[Example] (Holomorphic tangent bundle \(\pi : T^{(1,0)} M \to M\)) Let \(\{ W_\alpha \} \) be a local coordinate covering with coordinate functions \(\{ z_\alpha : W_\alpha \to W^0_\alpha \subset \C \}. \) Here we use \(z\) to denote coordinate function. Then for any \(p \in W_\alpha, \pi^{-1}(z) = \{ a \frac{d}{dz_\alpha}\} : a \in \C \}. \) We define

\[
\psi_\alpha : \pi^{-1}(W_\alpha) \to W_\alpha \times \C \to W^0_\alpha \times \C,
\]

\[ a \frac{d}{dz_\alpha} \big|_p \mapsto (p, a) \mapsto (z_\alpha(p), a). \]

\(T^{(1,0)} M\) becomes a complex manifold of dimension 2 with coordinate covering \(\{ \pi^{-1}(W_\alpha) \}\) and coordinate map \(\{ \psi_\alpha \}\). In fact,

\[
\psi_\beta \circ \psi_\alpha^{-1} : (z_\alpha(p), a) \mapsto (p, a) \mapsto a \frac{d}{dz_\alpha} \big|_p \mapsto a \frac{d}{dz_\alpha} \frac{d}{dz_\beta} \big|_p \mapsto (z_\beta z_\alpha^{-1}(p), a \frac{dz_\beta}{dz_\alpha} \big|_p). \]

Here \(\psi_\beta \circ \psi_\alpha^{-1}\) is holomorphic, and \(\pi\) is obviously holomorphic.

For any \(\alpha, \beta, \) with \(W_\alpha \cap W_\beta \neq \phi, \)

\[
\psi_\alpha^{-1}(x, y_\alpha) = \psi^{-1}(x, y_\beta) \iff y_\beta = y_\alpha \frac{dz_\beta}{dz_\alpha}. \]

Let \(f_{\alpha\beta} = \frac{dz_\alpha}{dz_\beta}\). Then \(T^{(1,0)} M\) is also a holomorphic line bundle with the transition functions \(\{ f_{\alpha\beta} = \frac{dz_\alpha}{dz_\beta} \}, \)

\[
T^{(1,0)} M \leftrightarrow \{ W_\alpha, f_{\alpha\beta} = \frac{dz_\alpha}{dz_\beta} \}. \]

[Example] (Holomorphic cotangent bundle \(\pi : T^{* (1,0)} M \to M\)) Let \(\{ W_\alpha, z_\alpha \} \) be a coordinate covering. Then

\[
\pi^{-1}(p) = \{ a \frac{dz_\alpha}\} : a \in \C \}. \]

\[
\psi_\alpha : \pi^{-1}(W_\alpha) \to W_\alpha \times \C \to W^0_\alpha \times \C.
\]

\[ a \frac{dz_\alpha}\big|_p \mapsto (p, a) \mapsto (z_\alpha(p), a). \]

\[
\psi_\beta : \pi^{-1}(W_\beta) \to W^0_\beta \times \C.
\]

\[ bdz_\beta\big|_p \mapsto (z_\beta(p), b). \]

We can also see that \(\{ \pi^{-1}(W_\alpha), \psi_\alpha \}\) gives a complex structure to \(T^{* (1,0)} M\) to make it into a complex line bundle. Now,
\[ \psi_{-1}^{-1}(z_{\alpha}, y_{\alpha}) = \psi^{-1}(z_{\beta}, y_{\beta}) \iff y_{\alpha} dz_{\alpha} = y_{\beta} dz_{\beta} = y_{\beta} \frac{dz_{\beta}}{dz_{\alpha}} dz_{\alpha}. \]

Here we get \( f_{\alpha \beta} = \frac{dz_{\beta}}{dz_{\alpha}} \). We see that

\[ T^{\ast}(1,0) M \leftrightarrow \{ W_{\alpha}, f_{\alpha \beta} = \frac{dz_{\beta}}{dz_{\alpha}} \}. \]

**Definition**  Let \( \pi_{1} : L_{1} \rightarrow M \) and \( \pi_{2} : L_{2} \rightarrow M \) be two holomorphic line bundles. We call a biholomorphic map \( h : L_{1} \rightarrow L_{2} \) a bundle isomorphism if the diagram

\[ \begin{array}{ccc}
\mathcal{L}_{1} & \xrightarrow{h} & \mathcal{L}_{2} \\
\pi_{1} \downarrow & & \downarrow \pi_{2} \\
\mathcal{M} & = & \mathcal{M}
\end{array} \]

commutes and

1. \( h \) preserves the fibers.
2. \( h|_{\pi_{1}^{-1}(z)} \) is a vector space isomorphism.

**Definition**  Let \( \{ V_{i} \} \) be an open covering of \( M \). We call another open covering \( \{ W_{\alpha} \} \) a refinement of \( \{ V_{i} \} \) if for any \( \alpha \), \( \exists i(\alpha) \) such that \( W_{\alpha} \subset V_{i(\alpha)} \). Let \( \pi : L \rightarrow M \) be a holomorphic line bundle given by \( \{ V_{j}, f_{jk} \} \). Let \( \{ W_{\alpha} \} \) be a refinement of \( \{ V_{j} \} \), then \( L \) is given by \( \{ W_{\alpha}, f_{j(\alpha)j(\beta)}|_{W_{\alpha} \cap W_{\beta}} \} \).

**Lemma 2.1**  \( L \) and \( L' \) are isomorphic holomorphic line bundle over \( M \) if and only if there exists a common open covering refinement \( \{ W_{\alpha} \} \) of \( M \) such that \( L \) and \( L' \) are given by \( \{ W_{\alpha}, f_{\alpha \beta} \} \) and \( \{ W_{\alpha}, f'_{\alpha \beta} \} \) respectively, and holomorphic functions \( f_{\alpha} : W_{\alpha} \rightarrow \mathbb{C}^{\ast} \) such that

\[ f_{\alpha \beta} = f_{\alpha}^{-1} f'_{\alpha \beta} f_{\beta}, \text{ on } W_{\alpha} \cap W_{\beta} \]

where \( f_{\alpha}^{-1} = \frac{1}{f_{\alpha}} \).

**Proof:**  \( (\iff) \) Let \( L \) be given by \( \{ W_{\alpha}, f_{\alpha \beta} \} \) and \( L' \) be given by \( \{ W_{\alpha}, f'_{\alpha \beta} \} \).

For any \( \alpha \), we have trivializations

\[ \psi_{\alpha} : \pi^{-1}(W_{\alpha}) \rightarrow W_{\alpha} \times \mathbb{C}, \quad \psi'_{\alpha} : \pi'^{-1}(W_{\alpha}) \rightarrow W_{\alpha} \times \mathbb{C}. \]
We define \( h_\alpha : \psi^{-1}_\alpha(z, y_\alpha) \mapsto \psi^{-1}_\alpha(z, f_\alpha y_\alpha) \). Then over \( W_\alpha \cap W_\beta \neq \emptyset \), if \( \psi^{-1}_\alpha(z, y_\alpha) = \psi^{-1}_\beta(z, y_\beta) \), then

\[
\begin{align*}
\psi^{-1}_\alpha(z, f_\alpha y_\alpha) &= \psi^{-1}_\alpha(z, f_\alpha f_\alpha \beta y_\beta) \quad \text{(because } y_\alpha = f_\alpha \beta y_\beta) \\
&= \psi^{-1}_\alpha(z, f_\alpha \beta f_\beta y_\beta) \quad \text{(because } f_\alpha \beta = f^{-1}_\alpha f'_\alpha \beta f_\beta) \\
&= \psi^{-1}_\beta(z, f_\beta y_\beta).
\end{align*}
\]

Hence \( h \) is well-defined. Apparently, \( h \) is an isomorphism.

\((\implies)\) From the commutative diagram

\[
\begin{array}{ccc}
\pi^{-1}(W_\alpha) & \xrightarrow{\psi_\alpha} & W_\alpha \times \mathbb{C}^1 \\
\downarrow h & & \downarrow f_\alpha \\
\pi'^{-1}(W_\alpha) & \xrightarrow{\psi'_\alpha} & W_\alpha \times \mathbb{C}^1,
\end{array}
\]

we define \( f_\alpha(z) := \psi'_\alpha \circ h \circ \psi^{-1}_\alpha(z, 1) \). Then

\[
\begin{align*}
f_\beta(z) &= \psi'_\beta \circ \psi^{-1}_\beta(z, 1) \\
&= \psi'_\beta \circ \psi^{-1}_\alpha \circ \psi'_\alpha \circ \psi^{-1}_\alpha \circ \psi_\alpha \circ \psi^{-1}_\beta(z, 1) \\
&= f'_\beta \circ f_\alpha \circ f_\alpha \beta(z) \\
&= f'_\alpha \circ f_\alpha \circ f_\alpha \beta(z).
\end{align*}
\]

Then \( f_\alpha \beta = f_\beta f'_\alpha \beta f^{-1}_\alpha \) is proved. \( \square \)

**Definition** We consider a line bundle \( \pi : M \times \mathbb{C} \to \mathbb{C} \), \( (x, \lambda) \mapsto x \) with transition functions \( f_\alpha \beta = 1 \). Any line bundle \( L \) over \( M \) that is bundle isomorphic to \( M \times \mathbb{C} \) is called a trivial line bundle.

**Corollary 2.2** \( \pi : L \to M \) is trivial \( \iff \exists \) open covering and transition functions \( \{ W_\alpha, f_\alpha \beta \} \) and holomorphic functions \( f_\alpha : W_\alpha \to \mathbb{C}^* \) such that

\[
f_\alpha \beta = \frac{f_\beta}{f_\alpha}, \quad \text{on } W_\alpha \cap W_\beta.
\]

**Example** Let \( D = \sum_{p \in M} n_p p \) be a divisor on a Riemann surface \( M \). Since the summation is locally finite, we can take local coordinates \( \{ W_\alpha \} \) of \( M \) such that \( W_\alpha \) contains at most one \( p_\alpha \) of these \( p \) with \( n_p \neq 0 \). Take a holomorphic function \( g_\alpha \in Hol(W_\alpha - \{ p_\alpha \}) \) such that
if \( n_{p_a} > 0 \), then \( g_\alpha \) has zero at \( p_\alpha \) with order \( n_{p_a} \); if \( n_{p_a} < 0 \), then \( g_\alpha \) has pole at \( p_\alpha \) with order \( -n_{p_a} \). Then we let

\[
f_{\alpha \beta} := \frac{g_\alpha}{g_\beta}, \quad W_\alpha \cap W_\beta.
\]

\( \{W_\alpha, f_{\alpha \beta}\} \) defines a holomorphic line bundle \([D]\) over \( M \).

**Example**  Let \( f \in \mathcal{M}(M) \). \((f)\) is the induced divisor. Then the induced line bundle \([((f))]\) is given by the transition functions

\[
f_{\alpha \beta} = \frac{f_{\mid W_\alpha}}{f_{\mid W_\beta}} = 1, \quad \text{on } W_\alpha \cap W_\beta
\]

so that \([((f))]\) is a trivial line bundle over \( M \).

### 2.2 Operators of line bundles

Let

\[
L \leftrightarrow \{W_\alpha, f_{\alpha \beta}\} \text{ and } L' \leftrightarrow \{W_\alpha, f'_{\alpha \beta}\}.
\]

We define \( L + L' \) or \( L \otimes L' \) to be the holomorphic line bundle given by

\[
\{W_\alpha, f_{\alpha \beta} f'_{\alpha \beta}\}
\]

and define the dual line bundle \(-L\) to be given by

\[
\{W_\alpha, f_{\beta \alpha} := \frac{1}{f_{\alpha \beta}}\}.
\]

**Lemma 2.3**  If \( L_1 \sim L'_1, L_2 \sim L'_2 \), then \( L_1 + L_2 \sim L'_1 + L'_2 \) and \( -L_1 \sim -L'_1, -L_2 \sim -L'_2 \).

**Proof:** Let \( \exists \) common open covering \( \{W_\alpha\} \) of \( M \) with \( L_j \leftrightarrow \{W_\alpha, f_{j \alpha \beta}\} \), and \( L'_j \leftrightarrow \{W_\alpha, f'_{j \alpha \beta}\}, \ j = 1, 2 \).

Then \( L_1 \sim L'_1, L_2 \sim L'_2 \iff \exists \ f_{j \alpha}, f'_{j \alpha} \in Hol(W_\alpha, \mathbb{C}^*), j = 1, 2 \), such that

\[
f_{1 \alpha \beta} = f^{-1}_{1 \alpha} f'_{1 \alpha \beta} f_{1 \beta}, \quad f_{2 \alpha \beta} = f^{-1}_{2 \alpha} f'_{2 \alpha \beta} f_{2 \beta}.
\]

Hence

\[
f_{1 \alpha \beta} f_{2 \alpha \beta} = (f_{1 \alpha} f_{2 \beta})^{-1} (f'_{1 \alpha \beta} f'_{2 \beta}) (f_{1 \beta} f_{2 \beta}).
\]
Then $L_1 + L_2 \sim L'_1 + L'_2$. □

**Definition**  Let $U \subset M$ be an open subset. $s : U \subset M \to L$ is called a *holomorphic section* (or *holomorphic cross section*) over $U$ if $s$ is holomorphic and $\pi \circ s = id|_U$.

We write $\Gamma(U, L)$ for the set of all holomorphic sections $s$ over $U$. Then $\Gamma(U, L)$ forms a vector space over $\mathbb{C}$. Let $\{W_\alpha\}$ be an open covering of $M$. Let $L$ be given by $\{W_\alpha, f_{\alpha\beta}\}$, $s \in \Gamma(U, L)$ and $\psi_\alpha(s) = (z, s_\alpha)$, if $W_\alpha \cap U \neq \emptyset$, where $\psi_\alpha$ is a trivialization, then $s_\alpha \in \text{Hol}(W_\alpha \cap U)$ and $s_\alpha = \psi_\alpha^{-1} \circ \psi_\beta \circ \psi_\beta^{-1} = f_{\beta\alpha} \circ s_\beta$, i.e.,

$$s_\alpha = f_{\beta\alpha} s_\beta, \text{ if } W_\alpha \cap W_\beta \cap U \neq \emptyset. \quad (2.4)$$

Conversely, if we have $\{s_\alpha \in \text{Hol}(W_\alpha \cap U)\}$ such that $s_\alpha = f_{\beta\alpha} s_\beta$ on $W_\alpha \cap W_\beta \cap U \neq \emptyset$, then we have a well-defined holomorphic section $s := \psi_\alpha^{-1}(p, s_\alpha)$. Hence

$$\Gamma(U, L) = \{s_\alpha : s_\alpha = f_{\beta\alpha} s_\beta\}. \quad (2.5)$$

**Remarks**

1. Let $D$ be a divisor of $M$ given by $\{W_\alpha, g_\alpha \in \mathfrak{M}(W_\alpha)\}$ and $L(D)$ the holomorphic line bundle given by $\{W_\alpha, f_{\alpha\beta}\}$. $L(D)$ is called the *holomorphic line bundle associate with the divisor* $D$. $L(D)$ is independent of the choice of $\{g_\alpha\}$ that defines the divisor under the equivalent relation.

2. In particular, if $D = (f)$ for $f \in \mathfrak{M}(M)$, then $g_\alpha = f|_{W_\alpha}$ so that the transition function $f_{\alpha\beta} = 1$. Thus $L(D)$ is trivial.

3. $\Gamma(U, L)$ gives a sheaf of germs of holomorphic sections of $L$, denoted by $\mathcal{O}(L)$. Then $\Gamma(U, L) = \Gamma(U, \mathcal{O}(L))$.

4. If there are meromorphic functions $s_\alpha \in \mathfrak{M}(W_\alpha \cap U)$ such that $2.4$ holds for any $W_\alpha \cap W_\beta \cap U \neq \emptyset$. Then we say that it defines a *meromorphic section*, still denoted by $s$.

5. A line bundle $L$ is determined by a divisor $D$ if and only if there is a meromorphic section $s$ of $L$.

A line bundle $L$ is determined by an effective divisor $D$ if and only if there is a holomorphic section $s$ of $L$. 
We also have

\[ L(D_1) + L(D_2) \sim L(D_1 + D_2), \quad L(-D) \sim -L(D). \]

Hence, we get a map \([\cdot, \cdot] : \mathcal{D} \to \text{Pic}(M)\) and \([\cdot, \cdot]\) is an abelian group homomorphism, where \(\mathcal{D}\) is \(\text{Div}(M)/\sim\), and \(\text{Pic}(M)\) is the set of all holomorphic line bundles / \(\sim\).

Next, if \(L(D) = [D]\) is given by \(\{W_\alpha, \phi_\alpha \in \mathfrak{M}(W_\alpha)\}\), we define

\[ i : \Gamma(L(D)) \to \ell(D) \quad s \mapsto \frac{s_\alpha}{\varphi_\alpha}, \]

where \(s\) is given by \(\{W_\alpha, s_\alpha\}\). Then

\[ \frac{s_\alpha}{\varphi_\alpha} = \frac{f_\beta s_\beta}{f_\alpha \varphi_\beta} = \frac{s_\beta}{\varphi_\beta}. \]

Thus \(i(s)\) is well defined meromorphic function over \(M\). Apparently, \((\frac{s_\alpha}{\varphi_\alpha}) + D \geq 0\). Conversely, we define

\[ i^{-1} : \ell(D) \to \Gamma(L(D)) \quad f \mapsto (f \varphi_\alpha : W_\alpha \to \mathbb{C}). \]

Hence we have proved the following

**Lemma 2.4** \(i : \Gamma(M, L(D)) \to \ell(D)\) is an isomorphism.

**[Examples]**

\[ \Gamma(T^{*1,0} M) = \{\text{holomorphic 1-forms over } M\}, \]

\[ \Gamma(T^{1,0} M) = \{\text{holomorphic vector fields over } M\}. \]

### 2.3 Sheaves

**Model of sheaves** Let \(M\) be a Riemann surface, \(\mathcal{O}(W)\) be the ring of holomorphic functions over an open subset \(W \subset M\). Let

\[ \mathcal{O} = \{\mathcal{O}(W) : W \text{ are open in } M\}, \]

Then \(\mathcal{O}\) has the following properties:
(1) For any open subset $V \subset W \subset M$, let 
$$ \rho_{WV} : \mathcal{O}(W) \to \mathcal{O}(V) $$
be the restriction map $\rho(f) = f|_V$. Then for open subsets $U \subset V \subset W$, we have,
$$ \rho_{WU} = \rho_{VU} \circ \rho_{WV}. $$

(2) Let $W = \bigcup_j W_j$ where $W_j$ are open subsets, if $\exists s_j \in \mathcal{O}(W_j)$, such that on $W_j \cap W_k$, 
$$ \rho_{W_j,W_j \cap W_k}(s_j) = \rho_{W_k,W_j \cap W_k}(s_k), $$
then $\exists s \in \mathcal{O}(W)$ such that 
$$ \rho_{WW_j}(s) = s_j. $$

(3) If $W = \bigcup_j W_j$ where $W_j$ are open subsets and $s \in \partial(W)$, $\rho_{W,W_j}(s) = 0$, then $s = 0$. With the above properties in (1)-(3), we call $\mathcal{O}$ the structure sheaf of $M$.

For $x \in M$, let $\mathcal{O}_x = \varprojlim_{x \in U} \mathcal{O}(U)$. We call $\mathcal{O}_x$ germ of holomorphic functions over $x$. Here $\varprojlim$ is the direct limit, namely
$$ \mathcal{O}_x = \bigcup \{ \mathcal{O}(U) : x \in U \} / \sim. $$

Here for $s_W \in \mathcal{O}(W)$, $s_V \in \mathcal{O}(V)$, we say 
$$ s_W \sim s_V \iff \rho_{W,W \cap V}(s_W) = \rho_{V,W \cap V}(s_V) $$
for some $x \in U \subset W \cap V$.

Let $\tilde{\mathcal{O}} = \{ \mathcal{O}_x : x \in M \}$. We can introduce a topology over $\tilde{\mathcal{O}}$: For any $f \in \mathcal{O}(W)$,
$$ \bigcup_{x \in W} [f]_x, $$
where we denote by $[f]_x$ the equivalent class of $f$ at $x$, is an open subset of $\tilde{\mathcal{O}}$ (the open subset of $\tilde{\mathcal{O}}$ is generated by all the above sets).

Let $\pi : \tilde{\mathcal{O}} \to M$, $\pi|_{\mathcal{O}_x} = x$. $\pi$ is then a local homomorphism.

Let $W \subset M$ be an open subset. A section (or cross section) over $W$ is a map $s : W \to \tilde{\mathcal{O}}$ such that $\pi \circ s = \text{id}|_W$. Let $\Gamma(\tilde{\mathcal{O}},W)$ be the space of all cross sections over $W$. Then $\Gamma(\mathcal{O},W) = \mathcal{O}(W)$. We call $\{ \tilde{\mathcal{O}}, \pi \}$ the associate space of $\mathcal{O}$. We can identify $\mathcal{O}$ with $\tilde{\mathcal{O}}$. 
More generally, we have the following

**Definition of a sheaf** Let $M$ be a topology space. A sheaf $\mathcal{F}$ over $M$ is a collection of abelian groups:

$$\{\mathcal{F}(U) \mid U \text{ is open in } M\},$$

where $\mathcal{F}(U)$ is an abelian group such that for open subsets $U \subset W \subset M$, there is a group homomorphism $\rho_{WV} : \mathcal{F}(W) \to \mathcal{F}(V)$ such that

1. For open subsets $U \subset V \subset W \subset M$, we have,

$$\rho_{WU} = \rho_{VU} \circ \rho_{WV}.$$

2. Let $W = \bigcup_j W_j$ where $W_j$ are open subsets, if $\exists s_j \in \mathcal{F}(W_j)$, such that on $W_j \cap W_k$,

$$\rho_{W_j,W_j \cap W_k}(s_j) = \rho_{W_k,W_j \cap W_k}(s_k),$$

then $\exists s \in \mathcal{F}(W)$ such that

$$\rho_{WW_j}(s) = s_j.$$

3. If $W = \bigcup_j W_j$ where $W_j$ are open subsets and $s \in \vartheta(W)$, $\rho_{W,W_j}(s) = 0$, then $s = 0$.

For each $x \in M$

$$\mathcal{F}_x = \lim_{x \in U} \mathcal{F}(U) \text{ and } \pi : \tilde{\mathcal{F}} = \bigcup_{x \in M} \mathcal{F}_x \xrightarrow{\pi} M.$$  

We can similarly define the topology over $M$. $\pi^{-1}(x) = \mathcal{F}_x$ is called the stalk at $x \in M$. $\mathcal{F}_x$ is also an abelian group.

**Remarks** Notice that $M$ may not have to be a manifold so that we can define sheaves over a variety which has singularities.

**[Examples]**

1. $\mathcal{O}$ as before is called the structure sheaf of $M$, or $\tilde{\mathcal{O}}$ — sheaf of germ of holomorphic functions over $M$ (In the future, we don’t distinguish $\mathcal{O}$ and $\tilde{\mathcal{O}}$).

2. $\mathcal{A}^p$ is the sheaf of germ of smooth $p$-forms over $M$. $\mathcal{A}^p(U)$ is the smooth $p$-forms over $U$. In particular $\mathcal{A}^0$ is the sheaf of germ of smooth functions over $M$.

3. $\mathcal{A}^{(p,q)}$ is the sheaf of germs of smooth $(p,q)$-forms over $M$, defined by $\mathcal{A}^{(p,q)}(U)$ for any open subset $U$ of $M$. 
4. Let $\pi : L \to M$ be a line bundle. $\mathcal{O}(L)$ is the sheaf of germs of holomorphic sections. 

$$ \mathcal{O}(L)(U) = \{ \text{holomorphic sections of } L \text{ over } U \}. $$

5. $\mathcal{A}^0(L) = \{ \mathcal{A}^0(L)(U) \} \text{ smooth sections of } L \text{ over } U \}$.

$$ \mathcal{A}^p(L)(W) = \mathcal{A}^p(W) \otimes \mathcal{A}^0(L)(W) $$

$$ = \{ \sum \omega_i s_i : \omega_i \in \mathcal{A}^p(W), s_i \in \Gamma(W, L) \}. $$

Notice that $\mathcal{A}^p(L)(W)$ is a modular over $\mathcal{A}^0(W)$:

$$ \omega(s + s') = \omega s + \omega s', $$

$$ (\omega + \omega') s = \omega s + \omega' s, $$

$$ f(\omega s) = f\omega s = \omega(fs). $$

6. $\mathcal{A}^{p,q}(L)$—sheaf of germs of $L$-valued $(p,q)$-forms

$$ \mathcal{A}^{p,q}(L)(W) = \mathcal{A}^p(W) \otimes \mathcal{A}^q(L)(W) $$

$$ = \{ \sum \omega_j s_j : \omega_j \in \mathcal{A}^{p,q}(W), s_j \in \Gamma(W, L) \}. $$

7. Ideal sheaf $\mathcal{I}_p : p \in M, \mathcal{I}_p(U) = \{ f \in \mathcal{O}(U), f(p) = 0 \}$, then

$$ \mathcal{I}_p|_x = \begin{cases} \partial_x & \text{for } x \neq p, \\ \mathfrak{m}_x & \text{for } x = p. \end{cases} $$

8. Constant sheaf $\mathbb{Z}, \mathbb{R}, \mathbb{C}$

$$ \mathbb{Z} = \{ \text{constant functions evaluate in } \mathbb{Z} \}, $$

$$ \mathbb{R} = \{ \text{constant functions evaluate in } \mathbb{R} \}, $$

$$ \mathbb{C} = \{ \text{constant functions evaluate in } \mathbb{C} \}. $$

**Definition of sheaf maps** Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves over $M$. Suppose $\exists \{ \varphi_U \}, \varphi_U : \mathcal{F}(U) \to \mathcal{G}(U)$ homomorphism such that the following diagram commutes:

$$ \begin{array}{ccc} \mathcal{F}(W) & \xrightarrow{\varphi_W} & \mathcal{G}(W) \\
\downarrow \rho_{W,V} & & \downarrow \rho_{W,V} \\
\mathcal{F}(V) & \xrightarrow{\varphi_V} & \mathcal{G}(V). \end{array} $$
CHAPTER 2. PROOF OF THE RIEMANN-ROCH THEOREM

for any open subsets $V \subset W \subset M$, then we can well-define $\varphi_x : \mathcal{F}_x \to \mathcal{G}_x$, $\forall x$, and $\varphi : \mathcal{F} \to \mathcal{G}$ is continuous, $\varphi_x$ is a group homomorphism, we call $\varphi$ is a sheaf map from $\mathcal{F}$ to $\mathcal{G}$. $\varphi([f]_x) = [\varphi(f)_x]$.

**Kernel sheaf and image sheaf**

Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a sheaf map:

$\ker \varphi(U) = \ker \{ \varphi_U : \mathcal{F}(U) \to \mathcal{G}(U) \} \subset \mathcal{F}(U)$,

$\text{Im} \varphi(U) = \text{Im} \{ \varphi_U : \mathcal{F}(U) \to \mathcal{G}(U) \} \subset \mathcal{G}(U)$.

One can verify that $\ker \varphi, \text{Im} \varphi$ also give the well-defined sheaves.

**Quotient sheaf**

Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a sheaf map. If we define a quotient sheaf (or cokernel sheaf) by

$\text{Coker}(\varphi)(U) := \mathcal{G}(U)/\varphi_U(\mathcal{F})(U)$,

then the condition (3) in the definition of sheaf may not be satisfied. For example, let $M = \mathbb{C} - \{0\}$ with an open covering $\{U\}$ where each $U$ is contractible to a point, we consider a sheaf map:

$exp : \mathcal{O} \to \mathcal{O}^*$

over the manifold $M$, by sending $f \in \mathcal{O}(U)$ to $e^{2\pi i f} \in \mathcal{O}^*(U)$, for any open subset $U \subset M$. The section $z \in \mathcal{O}^*(\mathbb{C} - \{0\})$ is not equal to the image of $\mathcal{O}(\mathbb{C} - \{0\})$ under the map $exp$, but its restriction to any $U_0 \in \{U\}$ is the image of $\mathcal{O}(U)$.

In order to make $\text{Coker}(\varphi)$ a sheaf, we define the quotient sheaf $\mathcal{G}/\text{Im} \varphi$ as follow. A section $s \in \mathcal{G}(U)/\varphi_U(\mathcal{F})(U)$ if and only if there is an open covering of $U$: $U = \cup \alpha U_\alpha$ and $s_\alpha \in \mathcal{G}(U_\alpha)$ such that for all $U_\alpha \cap U_\beta \neq \emptyset$, we have

$s_\alpha|_{U_\alpha \cap U_\beta} - s_\beta|_{U_\alpha \cap U_\beta} \in \varphi_U(\mathcal{F}(U_\alpha \cap U_\beta))$.

We identify two such collections $s = \{(U_\alpha, s_\alpha)\}$ and $s' = \{(U'_\alpha, s'_\alpha)\}$ in $\mathcal{G}/\text{Im} \varphi(U)$ if $\forall x \in U$ and $x \in U_\alpha \cap U'_\beta$, $\exists U_x$ with $x \in U_x \subset U_\alpha \cap U'_\beta$ such that

$s_\alpha|_{U_x} - s'_\beta|_{U_x} \in \varphi(\mathcal{F}(U_x))$.

In summary, for a sheaf map $\varphi : \mathcal{F} \to \mathcal{G}$, we have defined three sheaves:

$\ker \varphi, \text{Im} \varphi,$ and $\mathcal{G}/\text{Im} \varphi$.

$0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G} \xrightarrow{\psi} \mathcal{H} \to 0$ is exact

$\iff \varphi_x$ is injective, $\psi_x$ is surjective and $\ker \psi_x = \text{Im} \varphi_x, \forall x \in M$. 


Short exact sequence  We say that a sequence of sheaf maps
\[ 0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0 \] (2.6)
is exact if \( \text{Im}(\alpha) = \text{Ker}(\beta) \) and \( \mathcal{G} = \mathcal{F}/\text{Im}(\alpha) \). In this case, the above is called a short exact sequence, \( \mathcal{E} \) is called a subsheaf of \( \mathcal{F} \) and \( \mathcal{G} \) the quotient sheaf of \( \mathcal{F} \) by \( \mathcal{E} \), written \( \mathcal{F}/\mathcal{E} \).

[Example]
1. Ideal sheaf \( \mathcal{I}_p : p \in M, \mathcal{I}_p(U) = \{ f \in \mathcal{O}(U), f(p) = 0 \} \), then
\[
\mathcal{I}_p|_x = \begin{cases} \vartheta_x, & \text{for } x \neq p, \\ m_x, & \text{for } x = p. \end{cases}
\]
and we define the quotient sheaf:
\[
\vartheta/\mathcal{I}_p = \begin{cases} \mathbb{C}, & \text{at } p, \\ 0, & \text{otherwise.} \end{cases}
\]
2. We have a short exact sequence
\[ 0 \to \mathbb{Z} \xrightarrow{\text{inclusion}} \mathcal{O} \xrightarrow{\exp} \mathcal{O}^* \to 0. \]
which is called the exponential sheaf sequence
3. Let \( M \) be a complex manifold, and \( V \subset M \) a complex manifold. Let \( \mathcal{O}_V \) be considered a sheaf on \( M \) by extension by zero. Then we obtain a short exact sequence
\[ 0 \to \mathcal{I}_V \xrightarrow{i} \mathcal{O}_M \xrightarrow{r} \mathcal{O}_V \to 0 \]
where \( i \) is the inclusion, \( \mathcal{I}_V \) is the sheaf defined by holomorphic functions that vanish on \( V \), and \( r \) is the restriction.

Exact sequence  More generally, we consider a sequence of sheaf maps
\[ \mathcal{F}_1 \xrightarrow{\varphi_1} \mathcal{F}_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} \mathcal{F}_n \xrightarrow{\varphi_n} \mathcal{F}_{n+1} \xrightarrow{\varphi_{n+1}} \cdots \]
We say the sequence is an exact, if
\[ \forall n, \ \ker\varphi_n(x) = \text{Im}\varphi_{n-1}(x). \] (2.7)

Remarks
CHAPTER 2. PROOF OF THE RIEMANN-ROCH THEOREM

1. We have $\phi_j \circ \phi_{j-1} = 0$, $\forall j$ by (2.7).

2. For a short exact sequence of sheaf maps (2.6), by the definition of the quotient sheaf, it does not imply

$$0 \to \mathcal{E}(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} \mathcal{G}(U) \to 0.$$ 

It only implies the following: for any section $\sigma \in \mathcal{G}(U)$ and any point $p \in U$, $\exists$ a neighborhood $V_p$ of $p$ in $U$ such that $\sigma|_{V_p}$ is the image of $\beta_{V_p}$.

In may applications, we need to have existence of section in $\mathcal{F}(U)$. To do that, we need to introduce cohomology and vanishing theorems.

[Examples]

1. By the Poincare lemma, we have the long exact sequence

$$0 \to \mathbb{R} \to C^\infty \xrightarrow{d} \mathcal{A}^1 \xrightarrow{d} \mathcal{A}^2 \to ...$$

2. By the $\overline{\partial}$-Poincare lemma, we have the long exact sequence

$$0 \to \Omega^p \to \mathcal{A}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,1} \xrightarrow{\overline{\partial}} \mathcal{A}^{p,2} \to ...$$

where $\Omega^p$ is the sheaf of germs of holomorphic $p$-forms over $M$.

2.4 Čech cohomology theory

We mentioned that sheaf and Čech cohomology theory are introduced to study the existence of global sections by gluing local sections. Before we proceed, we first introduce

Origin: the Millag-Leffter Problem  Let $M$ be a Riemann surface, not necessary compact. Let $\mathcal{P} = \{p_i\}$ be a discrete subset of $M$, and $D = \sum n_j p_j$ with $n_j > 0$. We will try to find a meromorphic function $f$ on $M$ such that $f$ has poles only at $\{p_j\}$, and the order of pole at $p_j$ is $n_j$. More precisely, let $\{U_j, z_j\}$ be a coordinate chart at $\mathcal{P} = p_j$, and $h_j = \sum_{j=1}^{n_j} \frac{a_{nj}}{z_j}$, with $a_{nj} \neq 0$. Can we find $f \in \mathcal{M}(M)$, such that $f - h_j$ is holomorphic near $p_j$? This problem is called the Mittag-Leffler problem.

The question is locally trivial. The problem is one of passage from local data to global data.
2.4. 

CZECH COHOMOLOGY THEORY

Cohomology approach Let \( \{W_\alpha\} \) be a coordinate covering for \( M \) such that \( W_\alpha \cap \mathcal{P} = \emptyset \) or one point \( W_\alpha \cap \mathcal{P} = \{p_\alpha\} \in \mathcal{P} \).

Let \( g_\alpha \in \mathfrak{M}(W_\alpha) \) be such that

\[
 g_\alpha = \begin{cases} 
 1, & \text{if } W_\alpha \cap \mathcal{P} = \phi, \\
 h_j, & \text{if } p_j \in W_\alpha.
\end{cases}
\]

Now we want to know if we can “glue” \( \{g_\alpha\} \) to a globally defined meromorphic function.

For this purpose, let \( g_{\alpha\beta} = g_\beta - g_\alpha \in \mathcal{O}(W_\alpha \cap W_\beta) \).

Then

\[
 g_{\alpha\beta} = -g_{\beta\alpha}, \quad (\text{it'll be called a } 1-\text{cochain})
\]

\[
 g_{\beta\gamma} - g_{\alpha\gamma} + g_{\alpha\beta} = 0. \quad (\text{it'll be called closed})
\]

(2.8)

(Solving the problem globally is equivalent to finding \( \varphi_\alpha \in \mathcal{O}(W_\alpha) \) such that

\[
 g_{\alpha\beta} = \varphi_\beta - \varphi_\alpha, \quad \text{on } W_\alpha \cap W_\beta.
\]

(2.9)

(In this case \( \{g_{\alpha\beta}\} \) is called an exact 1-cochain).

Then

\[
 f := g_\beta - \varphi_\beta = g_\alpha - \varphi_\alpha
\]

is a globally defined meromorphic function. Clearly \( f \in \mathfrak{M}(M) \) has exactly the same pole as \( g_j \) near \( p_j \) so that it is the desired function.

Formally, we write

\[
 Z^1(\{W_\alpha\}, \mathcal{O}) = \{g_{\alpha\beta} \in \mathcal{O}(W_\alpha \cap W_\beta), \ g_{\alpha\beta} = -g_{\beta\alpha}, \ g_{\alpha\beta} + g_{\beta\gamma} = g_{\alpha\gamma}\}.
\]

\[
 B^1(\{W_\alpha\}, \mathcal{O}) = \{g_{\alpha\beta} \in \mathcal{O}(W_\alpha \cap W_\beta), \ g_{\alpha\beta} = \varphi_\beta - \varphi_\alpha, \varphi_\alpha \in \mathfrak{M}(W_\alpha)\}.
\]

We write

\[
 H^1(\{W_\alpha\}, \mathcal{O}) = Z^1(\{W_\alpha\}, \mathcal{O})/B^1(\{W_\alpha\}, \mathcal{O})
\]

the 1st-cohomology group of sheaf of germs of holomorphic functions with respect to \( \{W_\alpha\} \).

The solvability of the Mittag-Leffter problem is equivalent to the vanishing of \( H^1(\{W_\alpha\}, \mathcal{O}) \) for a sufficient fine covering \( \{W_\alpha\} \).

So the Millag-Leffler Problem on \( M \) can be solved for any \( D \) if and only if the group

\[
 H^1(\{W_\alpha\}, \mathcal{O}) = 0.
\]
CHAPTER 2. PROOF OF THE RIEMANN-ROCH THEOREM

PDE approach Another way to look at the Mittag-Leffler problem is to use \( \bar{\partial} - \)equation. Let \( \chi_j \) be a cut-off function near \( p_j \), namely,

\[
\chi_j = \begin{cases} 
1, & \text{near } p_j, \\
0, & \text{elsewhere,}
\end{cases}
\]

Let \( \{W_j\} \) be a locally finite covering of \( M \). Let \( g = \sum \chi_j h_j \in C^\infty(M) \setminus \mathcal{P} \)

\[
\bar{\partial}g = \sum \bar{\partial} \chi_j h_j + \sum \chi_j \bar{\partial} h_j = \sum \bar{\partial} \chi_j h_j = \omega \in A^{(0,1)}(M).
\]

Then \( \bar{\partial} \omega = 0 \) because \( \bar{\partial}^2 = 0 \). If \( \exists u \in C^\infty(M) \) such that \( \bar{\partial} u = \omega \), then \( \bar{\partial}(g - u) = 0 \), and hence \( g - u \in \text{Hol}(M - \mathcal{P}) \), \( g - u \) is also the function we are looking for.

Solvability of the partial differential equation \( \bar{\partial}u = \omega \) if and only if the cohomology group \( H^1(\{W_\alpha\}, \mathcal{O}) = 0 \). The Millag-Leffler problem can also be studied by the PDE method. This gives a connection between topology and PDE.

There are three kinds of sheaves:

1. Holomorphic sheaves, e.g., \( \mathcal{O}, J_L, \mathcal{O}(L) \), etc. Analytic information.
2. \( C^\infty \) sheaves, e.g. \( \mathcal{A}^p, \mathcal{A}^{p,q} \), etc. Differential information.
3. Constant sheaves, e.g., \( \mathbb{Z}, \mathbb{R}, \mathbb{C} \), etc. Topological information about the underlying manifold.

Čech cohomology theory Let \( M \) be a topology space, \( \mathcal{U} = \{U_\alpha\} \) an open covering of \( M \), and \( \mathcal{F} \) a sheaf over \( M \). We define

\[
\begin{align*}
C^0(\mathcal{U}, \mathcal{F}) &= \prod_\alpha \mathcal{F}(U_\alpha), \\
C^1(\mathcal{U}, \mathcal{F}) &= \prod_{\alpha \neq \beta} \mathcal{F}(U_\alpha \cap U_\beta), \\
& \vdots \\
C^p(\mathcal{U}, \mathcal{F}) &= \prod_{\alpha_0 \neq \alpha_1 \neq \ldots \neq \alpha_p} \mathcal{F}(U_{\alpha_0} \cap \ldots \cap U_{\alpha_p}).
\end{align*}
\]

An element \( \sigma \) of \( C^q(\mathcal{U}, \mathcal{F}) \) is called a \( q \)-cochain if

\[
\sigma = \begin{cases} 
\sigma_{(i_0, i_1, \ldots, i_q)} \in \mathcal{F}(U_0 \cap U_1 \cap \cdots \cap U_q), & \forall U_0, U_1, \ldots, U_q \in \mathcal{U} \text{ with } \bigcap_{j=0}^q U_j \neq \emptyset, \\
0, & \forall U_0, U_1, \ldots, U_q \in \mathcal{U} \text{ with } \bigcap_{j=0}^q U_j = \emptyset.
\end{cases}
\]
Also, we request that \( \sigma \) is skew symmetric in \((U_0, U_1, \cdots, U_q)\). Now
\[
C^q(\mathcal{U}, \mathcal{F}) = \{ \text{collection of all } q \text{-cochains over } M \},
\]
and \( C^q(\mathcal{U}, \mathcal{F}) \) forms an Abelian group.

We define the boundary operator \( \delta_q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F}) \) by
\[
(\delta_q f)(U_0, U_1, \cdots, U_{q+1}) = \sum_{i=0}^{q+1} (-1)^i f(U_0, \cdots, \overset{\wedge}{U_i}, \cdots, U_{q+1}).
\]
Here we use \( \wedge \) to denote “omit”. We obtain a sequence of group homomorphism
\[
C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_1} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_2} \cdots
\]
It is easy to verify that \( \delta_q \circ \delta_{q-1} = 0 \).

Let \( Z^q = \ker(\delta_q) \), and \( B^q = \text{Im}(\delta_{q-1}) \) be the kernel and image sheaves. Then
\[
H^q(\mathcal{U}, \mathcal{F}) := Z^q(\mathcal{U}, \mathcal{F}) / B^q(\mathcal{U}, \mathcal{F}),
\]
which is called the \( q \)-th cohomology group of the sheaf \( \mathcal{F} \) with respect to \( \mathcal{U} \).

Hence
\[
H^q(\mathcal{U}, \mathcal{F}) = 0 \iff Z^q(\mathcal{U}, \mathcal{F}) = B^q(\mathcal{U}, \mathcal{F}).
\]

[Example] (a) \( g \in C^0(\mathcal{U}, \mathcal{F}) \) \( \iff \) \( \forall U \in \mathcal{U}, \exists g_U \in \mathcal{F}(U) \).
\( (\delta_0 g) \in C^1(\mathcal{U}, \mathcal{F}) \) is defined by
\[
(\delta_0 g)_{U_0, U_1} = g_{U_1} - g_{U_0}, \quad \forall U_0, U_1 \in \mathcal{U}.
\]

(b) \( f \in C^1(\mathcal{U}, \mathcal{F}) \) \( \iff \) \( \forall U_0, U_1 \in \mathcal{U}, \exists f_{U_0, U_1} \in \mathcal{F}(U_0 \cap U_1) \) with \( f_{U_0, U_1} = -f_{U_1, U_0} \).
\( \delta_1 f \in C^2(\mathcal{U}, \mathcal{F}) \) is defined by
\[
(\delta_1 f)_{U_0, U_1, U_2} = f_{U_1, U_2} - f_{U_0, U_2} + f_{U_0, U_1}, \quad \forall U_0, U_1, U_2 \in \mathcal{U}.
\]

(c) Now \( H^1 = 0 \) iff \( Z^1 = B^1 \) iff \( \forall f \in C^1(\mathcal{U}, \mathcal{F}) \) with \( \delta_1 f = 0 \), \( \exists g \in C^0(\mathcal{U}, \mathcal{F}) \) such that \( \delta_0(g) = f \). Coming back to the the Millag-Leffler Problem, this is what happened in (2.8) and (2.9).

For the coverings \( \mathcal{W} \) and \( \mathcal{B} \), \( H^q(\mathcal{W}, \mathcal{F}) \) may not be isomorphic to \( H^q(\mathcal{B}, \mathcal{F}) \). Now we assume that \( M \) is Hausdoff and paracompact and assume that \( \mathcal{W} \) and \( \mathcal{B} \) locally finite.
covering of $M$. Then by a result from the general topology, we can find a common locally finite covering $\mathcal{U}$ which is a refinement of both $\mathcal{W}$ and $\mathcal{B}$.

Now there is a homomorphism $\tau(\mathcal{B}, \mathcal{U}) : H^q(\mathcal{B}, \mathcal{F}) \to H^q(\mathcal{U}, \mathcal{F})$ defined as follows: let $\mathcal{B} = \{B_\alpha\}_{\alpha}$, and $\mathcal{U} = \{U_\beta\}_{\beta}$, and

$$\lambda : \wedge \beta \to \Theta_\alpha$$

be a map such that $U_\beta \subset B_{\lambda(\beta)}$. For $\sigma_B \in H^q(\mathcal{B}, \mathcal{F})$, we define

$$\tau(\mathcal{B}, \mathcal{U})(\sigma_B)(U_0, U_1, \cdots, U_q) = \sigma_B(U_{\lambda(0)}, U_{\lambda(1)}, \cdots, U_{\lambda(q)}).$$

Then one can verify that $\tau(\mathcal{B}, \mathcal{U})$ is independent of the choice of $\lambda$.

Similarly, we can define a homomorphism $\tau(\mathcal{W}, \mathcal{U}) : H^q(\mathcal{W}, \mathcal{F}) \to H^q(\mathcal{U}, \mathcal{F})$. For $\alpha \in H^q(\mathcal{B}, \mathcal{F})$ and $\beta \in H^q(\mathcal{W}, \mathcal{F})$, we say $\alpha \sim \beta$ if and only if there is a common refinement $\mathcal{U}$ of $\mathcal{B}$ and $\mathcal{W}$, such that $\tau(\mathcal{B}, \mathcal{U})(\alpha) = \tau(\mathcal{W}, \mathcal{U})(\beta)$. Next, we define

$$H^q(M, \mathcal{F}) = \bigsqcup H^q(\mathcal{W}, \mathcal{F}) / \sim$$

where $H^q(\mathcal{W}, \mathcal{F})$s are locally finite covering of $M$, and " $\sim$ " is defined as above. $H^q(M, \mathcal{F})$ is called the $q$-th cohomology group of $M$ with coefficients in $\mathcal{F}$. We write

$$H^q(M, \mathcal{F}) = \lim_{\leftarrow} H^q(\mathcal{W}, \mathcal{F}).$$

There is a natural map

$$i : H^q(\mathcal{W}, \mathcal{F}) \to H^q(M, \mathcal{F}).$$

**Remark** $H^q(M, \mathcal{F})$ is an invariant of $M$ with respect to biholomorphic maps. In practice, it is more or less impossible to work with $H^q(M, \mathcal{F})$. What is needed is a simple sufficient condition on a cover $\mathcal{U}$ for

$$H^*(\mathcal{U}, \mathcal{F}) = H^*(M, \mathcal{F}).$$

**Lemma 2.5** Let $M$ be a Hausdorff and paracompact space, $\mathcal{W}$ be a covering. Then,

$$i : H^1(\mathcal{W}, \mathcal{F}) \to H^1(M, \mathcal{F}).$$

is injective. Moreover, $H^0(\mathcal{W}, \mathcal{F}) = \mathcal{F}(M) = H^0(M, \mathcal{F})$. 
Proof: We skip the proof for the first statement because we shall not use it here (for reference, see R.C. Gunning and H. Rossi, Analytic Functions of Several Complex Variables, 1965). Let us prove the second statement.

Consider $B^0(W, \mathcal{F}) = 0$, $H^0(W, \mathcal{F}) = Z^0(W) = 0$.

For $f \in Z^0(W, \mathcal{F})$, $\delta f \equiv 0$ if and only if for any $W_\alpha$ and $W_\beta \in W$, $\delta f|_{W_\alpha \cap W_\beta} = f(W_\beta) - f(W_\alpha)|_{W_\alpha \cap W_\beta} = 0$ or $f(W_\beta) = f(W_\alpha)$ on $W_\alpha \cap W_\beta$. By the gluing property of a sheaf, we see that $f$ defines a section on $M$. Hence

$$H^0(M, \mathcal{F}) = \lim_{\leftarrow} W H^0(W, \mathcal{F}) = \mathcal{F}(M).$$

Lemma 2.6 Assume the short exact sequence

$$0 \rightarrow E \xrightarrow{i} \mathcal{F} \xrightarrow{j} G \rightarrow 0.$$

Then there is a long exact sequence in the cohomology groups:

$$0 \rightarrow E(M) \xrightarrow{i^*} \mathcal{F}(M) \xrightarrow{j^*} G(M) \xrightarrow{\delta^*} H^1(M, E) \xrightarrow{i^*} H^1(M, \mathcal{F}) \xrightarrow{j^*} H^1(M, G) \xrightarrow{\delta^*} H^2(M, E) \xrightarrow{i^*} H^2(M, \mathcal{F}) \xrightarrow{j^*} H^2(M, G) \xrightarrow{\delta^*} \cdots.$$

Proof: Given a short exact sequence

$$0 \rightarrow E \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} G \rightarrow 0.$$

It induces group homomorphisms

$$\begin{align*}
C^n(U, E) &\xrightarrow{\alpha} C^n(U, \mathcal{F}) \xrightarrow{\delta} C^n(U, G) \\
C^{n+1}(U, E) &\xrightarrow{\alpha} C^{n+1}(U, \mathcal{F}) \xrightarrow{\delta} C^{n+1}(U, G)
\end{align*}$$

be a short exact sequence of sheaf maps for any open subset $U$. Then

$$0 \rightarrow E(U) \xrightarrow{\alpha_U} \mathcal{F}(U) \xrightarrow{\beta_U} G(U) \rightarrow 0.$$

may not be exact. It only implies that this sequence is exact at the first two stages for all open sets $U$; and that for any section $\sigma \in G(U)$ and any point $p \in U$, there exists a neighborhood $V$ of $p$ in $U$ such that $\sigma|_V$ is in the image of $\beta_V$. 

\[1\text{Let} \quad 0 \rightarrow E \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} G \rightarrow 0. \]
CHAPTER 2. PROOF OF THE RIEMANN-ROCH THEOREM

Then we have well-defined homomorphisms

\[ H^p(M, \mathcal{E}) \xrightarrow{\alpha^*} H^p(M, \mathcal{F}), \text{ and } H^p(M, \mathcal{F}) \xrightarrow{\alpha^*} H^p(M, \mathcal{G}). \]

It remains to define a homomorphism

\[ \delta^*: H^p(M, \mathcal{G}) \to H^{p+1}(M, \mathcal{E}). \]

Consider

\[
\begin{array}{c}
H^p(M, \mathcal{E}) \xrightarrow{\alpha^*} H^p(M, \mathcal{F}) \ni \tau \\
\downarrow \delta \\
H^{p+1}(M, \mathcal{E}) \ni \mu \xrightarrow{\alpha^*} H^{p+1}(M, \mathcal{F}) \ni \delta^\tau \\
\downarrow \delta \\
H^{p+1}(M, \mathcal{G}) \ni \sigma
\end{array}
\]

\[ \forall \sigma \in H^p(M, \mathcal{G}), \text{ i.e., } \sigma \in C^p(U, \mathcal{G}) \text{ with } \delta \sigma = 0. \] By the Remark 2 in page 54, there exists a refinement \( U' \) of \( U \) and an element \( \tau \in C^p(U', \mathcal{F}) \) such that

\[ \beta(\tau) = \rho \sigma \]

where \( \rho: H^p(U, \mathcal{F}) \to H^p(U', \mathcal{F}) \) is the homomorphism induced by restriction. Then

\[ \beta \delta \tau = \delta \beta \tau = \delta (\rho \sigma) = 0 \]

because \( \delta \alpha = 0 \). Then there is a further refinement \( U'' \) of \( U \) and an element \( \mu \in C^{p+1}(U'', \mathcal{E}) \) such that

\[ \alpha \mu = \delta \tau. \]

Here we used the fact in Remark 2, p. 54 again. Then we have defined a homomorphism

\[ \delta^*: H^p(M, \mathcal{G}) \to H^{p+1}(M, \mathcal{E}) \]

\[ \sigma \mapsto \delta^* \sigma = \mu. \]

2.5 De Rham theorem

Definition A sheaf \( \mathcal{F} \) is called a fine sheaf on \( M \), if for any locally finite covering \( \mathcal{W} = \{W_\alpha\} \), there is a set of sheaf maps \( \{\eta_\alpha\} \) such that \( \eta_\alpha: \mathcal{F} \to \mathcal{F} \)

1. for any \( W_\alpha \), \( \exists K_\alpha \subset \subset W_\alpha \), \( \eta_\alpha|_x: \mathcal{F}(x) \to \mathcal{F}(x) \) is 0 for \( x \notin K_\alpha \).
2. \( \sum \eta_\alpha = \text{id.} \)

[Example] Let \( \mathcal{F} = \mathcal{A}^p, \mathcal{A}^{p,q}, \mathcal{A}^p(L), \) or \( \mathcal{A}^{p,q}(L) \). Then \( \mathcal{F} \) is a fine sheaf.

Proof Let \( \mathcal{W} = \{W_\alpha\} \) be a locally finite covering of \( M \). Then there exists a partition of unity \( \{\varphi_\alpha\} \) such that \( \text{supp } \varphi_\alpha \subset \subset W_\alpha \), and \( \sum \varphi_\alpha \equiv 1 \). Define \( \eta_\alpha: s \mapsto \varphi_\alpha s \) for \( s \in \Gamma(U, \mathcal{F}) \). Then \( \{\eta_\alpha\} \) satisfies the above condition.
2.5. DE RHAM THEOREM

Theorem 2.7 Let \( \mathcal{W} = \{W_\alpha\} \) be a locally finite covering of \( M \), and \( \mathcal{F} \) a fine sheaf over \( M \). Then \( H^q(\mathcal{W}, \mathcal{F}) = 0 \), \( q \geq 1 \). Hence \( H^q(M, \mathcal{F}) = 0, \forall q \geq 1 \).

Proof: We only give the proof when \( q = 1 \), the other cases can be done similarly.

Let \( \mathcal{W} = W_\alpha \) be a locally finite covering of \( M \), and \( f \in Z^1(\mathcal{W}, \mathcal{F}) \) be any element. Then
\[
f(W_\alpha, W_\beta) + f(W_\beta, W_\gamma) = f(W_\alpha, W_\gamma).
\]
Now, we want to find \( g \in C^0(\mathcal{W}, \mathcal{F}) \) such that
\[
f(W_\alpha, W_\beta) = g(W_\alpha) - g(W_\beta).
\]
Let \( \{\eta_r\} \) be the partition of unity related to \( \{\mathcal{W}, \mathcal{F}\} \). We define \( g(W_\alpha) = \sum_j \eta_j(f(W_j, W_\alpha)) \). For a fixed \( r \), suppose \( \text{supp} \eta_r \subset K_r \subset W_r \), and we claim
\[
\eta_r(f(W_r, W_\alpha)) \in \mathcal{F}(W_\alpha).
\]
In fact, \( f(W_r, W_\alpha) \in \mathcal{F}(W_r \cap W_\alpha) \) and \( W_\alpha \) has an open covering: \( W_\alpha = (W_\alpha - K_r) \cup (W_\alpha \cap W_r) \). We have \( \eta_r f(W_r, W_\alpha) = 0 \) on \( W_\alpha - K_r \) and \( \eta_r f(W_r, W_\alpha)|_{W_r \cap W_\alpha} \in \mathcal{F}(W_r \cap W_\alpha) \).

By the condition (ii) of the definition of sheaf (i.e., if \( W = \bigcup_j W_j \), \( \exists s_j \in \mathcal{F}(W_j) \) such that \( \rho_{W_j, W_j \cap W_k}(s_j) = \rho_{W_j, W_j \cap W_k}(s_k), \forall W_j \cap W_k \neq \emptyset \), then there is \( s \in \mathcal{F}(W) \) such that \( \rho_{W, W_j}(s) = s_j \)), there is an element, denoted by \( \eta_r f(W_r, W_\alpha) \), in \( \Gamma(W_\alpha, \mathcal{F}) \). Our Claim is proved.

Now
\[
g(W_\alpha) - g(W_\beta) = \sum_r \{\eta_r f(W_r, W_\alpha) - \eta_r f(W_r, W_\beta)\}
\]
\[
= \sum_r \eta_r \{f(W_r, W_\alpha) - f(W_r, W_\beta)\}
\]
\[
= \sum_r \eta_r f(W_r, W_\alpha) = f(W_\beta, W_\alpha).
\]
Hence, \( H^1(\mathcal{W}, \mathcal{F}) = 0 \) and
\[
H^1(M, \mathcal{F}) = \lim_{\mathcal{W}} H^1(\mathcal{W}, \mathcal{F}) = 0.
\]

Definition Let \( \mathcal{F}, \{\mathcal{F}_j\}_{j=0}^\infty \) be sheaf over \( M \) and \( \mathcal{F}_j \) be fine sheaf. We call \( 0 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_0 \xrightarrow{d_0} \mathcal{F}_1 \xrightarrow{d_1} \cdots \) a fine decomposition of \( \mathcal{F} \) if the sequence is exact.

[Example] Let \( M \) be a Riemann surface
\[
0 \rightarrow \mathbb{C} \rightarrow \mathcal{A}^0 \xrightarrow{d_0} \mathcal{A}^1 \xrightarrow{d_1} \mathcal{A}^2 \rightarrow 0
\]
is a fine decomposition of the constant \( \mathbb{C} \)-sheaf. The exactness of the above sequence follows from the Poincaré lemma.
Theorem 2.8 (De Rham theorem) Let $M$ be a Hausdorff, paracompact topological space. Suppose

$$0 \to \mathcal{F} \to \mathcal{F}_0 \xrightarrow{d_0} \mathcal{F}_1 \xrightarrow{d_1} \mathcal{F}_2 \xrightarrow{d_2} \cdots$$

is a fine sheaf decomposition of $\mathcal{F}$. Let

$$0 \to \mathcal{F}(M) \to \mathcal{F}_0(M) \xrightarrow{d_0^*} \mathcal{F}_1(M) \xrightarrow{d_1^*} \mathcal{F}_2(M) \xrightarrow{d_2^*} \cdots$$

be the induced sequence. Then

$$H^p(M, \mathcal{F}) = \ker d^*_p / d_{p-1}^*(\mathcal{F}_{p-1}(M)) \text{ for } p \geq 1.$$  

Proof: Let $E_p = \ker d_p$. Then $E_p$ also defines a sheaf over $M$, and $E_p \subset \mathcal{F}_p$. Hence, we have the following exact sequence:

$$0 \to \mathcal{F} \to \mathcal{F}_0 \xrightarrow{d_0} E_1 \to 0,$$

$$0 \to E_1 \to \mathcal{F}_1 \xrightarrow{d_1} E_2 \to 0,$$

$$\cdots.$$  

From the first short exact sequence, we get the long exact sequence:

$$0 \to \mathcal{F}(M) \to \mathcal{F}_0(M) \xrightarrow{d_0^*} E_1(M) \to H^1(M, \mathcal{F}) \to 0$$

$$\to H^1(M, E_1) \to H^2(M, \mathcal{F}) \to 0 \to H^2(M, E_1) \to H^3(M, \mathcal{F}) \to \cdots.$$  

Here we used the fact that $H^p(M, \mathcal{F}_0) = 0, \forall p \geq 1$. Hence

$$H^1(M, \mathcal{F}) \cong E_1(M) / d_0^*(\mathcal{F}_0(M)) = \ker d_1^* / d_0^*(\mathcal{F}_0(M)). \quad (2.10)$$

because $E_1 = \ker d_1$ and

$$H^1(M, E_1) \cong H^2(M, \mathcal{F}). \quad (2.11)$$

From the second short exact sequence, we get the long exact sequence:

$$0 \to E_1(M) \to \mathcal{F}_1(M) \xrightarrow{d_1^*} E_2(M) \to H^1(M, E_1) \to 0 \to H^1(M, E_2) \to$$

$$\to H^2(M, E_1) \to 0 \to H^2(M, E_2) \to H^3(M, E_1) \to H^3(M, \mathcal{F}_1) \to \cdots.$$  

Here we used the fact that $H^p(M, \mathcal{F}_1) = 0, \forall p \geq 1$. Then

$$H^1(M, E_1) \cong E_2(M) / d_1^*(\mathcal{F}_1(M)) = \ker d_2^* / d_1^*(\mathcal{F}_1(M))$$
because $E_2 = \ker d_2$. We use (2.11) to get

$$H^2(M, \mathcal{F}) \cong \ker d^*_2/d^*_1(\mathcal{F}_1(M)).$$  \hspace{1cm} (2.12)

Similarly, we can prove the theorem for any $p \geq 3$.  \hspace{1cm} \square

**Corollary 2.9**

$$0 \rightarrow \mathbb{C} \rightarrow \mathcal{A}^0(M) \xrightarrow{d_0} \mathcal{A}^1(M) \xrightarrow{d_1} \mathcal{A}^2(M) \rightarrow \cdots$$

we get

$$H^p(M, \mathbb{C}) \cong \ker d^*_p/\text{Im } d^*_{p-1} = \frac{\{\text{global closed smooth } p\text{-forms over } M\}}{\{\text{exact global smooth } (p-1)\text{-forms over } M\}}.$$  

**Remark** On the left hand side, it is topological invariant and on the right hand side, it is differential invariant.

### 2.6 Dolbeault theorem

We now prove the complex version of the de Rham theorem, which is called the Dolbeault Theorem.

**Lemma 2.10** (Dolbeault lemma) Let $\omega$ be a $(p, q)$-form in a neighborhood of $0 \in \mathbb{C}$. Assume that $\omega$ is $\bar{\partial}$-closed, namely, $\bar{\partial}\omega = 0$. Then there is a small neighborhood of $0$ in $\mathbb{C}^1$, and a $(p, q - 1)$-form $\eta$, such that $\bar{\partial}\eta = \omega$.

**Proof:** Write $\omega = h \, d\bar{z}$, or $\omega = h dz \wedge d\bar{z}$. Then by the Cauchy-formula in one complex variable, we know that there is a smooth function $f$ such that $\frac{\partial f}{\partial \bar{z}} = h$ in a neighborhood of $0$. Now let $\eta = f$ or $\eta = f \, dz$. Then $\bar{\partial}\eta = \omega$ neighborhood of $0$.  \hspace{1cm} \square

**Remark:** Dolbeault lemma also holds on higher dimensional complex manifolds.

Let $M$ be a Riemann surface. We consider the sequence

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{A}^0 \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,1)} \xrightarrow{\bar{\partial}} 0,$$

$$0 \rightarrow \Omega^1 \rightarrow \mathcal{A}^{1,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{(1,1)} \xrightarrow{\bar{\partial}} 0.$$

By the Dolbeault lemma, the above gives the fine sheaf decomposition for $\mathcal{O}$ and $\Omega^1$ respectively. Hence, by the de Rham theorem, we get
CHAPTER 2. PROOF OF THE RIEMANN-ROCH THEOREM

Theorem 2.11 (Dolbeault theorem)

\[
H^1(M, \mathcal{O}) \cong \frac{\text{closed } (0,1)\text{-forms over } M}{\bar{\partial}\text{-exact } (0,1)\text{-forms over } M}.
\]

\[
H^1(M, \Omega^1) \cong \frac{\text{closed } (1,1)\text{-forms over } M}{\bar{\partial}\text{-exact } (1,1)\text{-forms over } M}.
\]

Remark:

1. Dolbeault theorem also holds on higher dimensional complex manifolds.

2. For instance, \( H^1(M, \mathcal{O}) = 0 \iff \) for any \((0,1)\)-form \( \omega \), there is an \( \eta \) such that \( \bar{\partial}\eta = \omega \).

Next, we want to generalize the above Dolbeault theorem by replacing \( \mathcal{O}, \Omega^1 \) by \( \mathcal{O}(L), \Omega^1(L) \) where \( L \) is a line bundle, respectively.

Let \((L, \pi, M)\) be a holomorphic line bundle over \( M \). Let
\[
\Omega^0(L) = \mathcal{O}(L) = \text{sheaf of germs of } L\text{-valued holomorphic functions (cf. page 51)},
\]
\[
\Omega^1(L) = \text{sheaf of germs of } L\text{-valued holomorphic } 1\text{-forms (cf. page 51)} ,
\]
\[
\Omega^1(L)(W) = \Gamma(W, L \otimes T^{* (0,1)}) = \{ s = \sum_j \omega_j \otimes s_j \text{ with } \omega_j \in \Omega^1(W), \text{ and } s_j \in \Gamma(W, L) \}.
\]

Now, we can define \( \bar{\partial} - \text{ operator} \):

\[
\bar{\partial} : \mathcal{A}^0(L) \rightarrow \mathcal{A}^{0,1}(L),
\]

\[
\bar{\partial} : \mathcal{A}^{1,0}(L) \rightarrow \mathcal{A}^{1,1}(L),
\]

where \( \mathcal{A}^0(L) \) is the sheaf of germs of smooth sections of \( L \), and \( \mathcal{A}^{1,0}(L) \) is the sheaf of germs of smooth sections of \( L \otimes T^{* (1,0)}(M) \).

Let \( L \leftrightarrow \{ W_\alpha, f_\alpha^\beta \} \). On each \( W_\alpha \), we take a holomorphic section \( e_\alpha \in \Gamma(W_\alpha, L) \), such that \( e_\alpha \neq 0 \). For any holomorphic section \( s_\alpha \in \Gamma(W_\alpha, L) \), \( s_\alpha = h_\alpha e_\alpha \) with \( h_\alpha \in \text{Hol}(W_\alpha) \). If \( s \in \Gamma(W_\alpha \cup W_\beta, L) \), then \( s = h_\alpha e_\alpha \) over \( W_\alpha \), and \( s = h_\beta e_\beta \) over \( W_\beta \), and we have \( h_\beta = f_\beta^\alpha h_\alpha \) where \( f_\beta^\alpha \) is the transition function of \( L \).

Similarly if \( e_\alpha \in \Gamma_\infty(W_\alpha, L) \) is a \( C^\infty \) section, we can write \( s_\alpha = h_\alpha e_\alpha \) with \( h_\alpha \in C^\infty(W_\alpha) \) is a \( C^\infty \) function. If \( s \in \Gamma_\infty(W_\alpha \cap W_\beta, L) \) then \( s = h_\alpha e_\alpha \) over \( W_\alpha \), and \( s = h_\beta e_\beta \) over \( W_\beta \), and \( h_\beta = f_\beta^\alpha h_\alpha \) where \( f_\beta^\alpha \) is the transition function of \( L \).

More generally, if \( s \in \mathcal{A}^{(p,q)}(L)(W_\alpha \cup W_\beta) \), then \( s = \sigma_\alpha e_\alpha \) over \( W_\alpha \), and \( s = \sigma_\beta e_\beta \) over \( W_\beta \), with \( \sigma_\alpha \in A^{(p,q)}(W_\alpha) \), and \( \sigma_\beta \in A^{(p,q)}(W_\beta) \). Moreover, \( \sigma_\beta = f_\beta^\alpha \sigma_\alpha \).
2.7. HERMITIAN METRIC AND CONNECTION

Now, we define
\[ \bar{\partial}s := \bar{\partial}\sigma_\alpha e_\alpha \text{ over } W_\alpha, \text{ and } \bar{\partial}s := \bar{\partial}\sigma_\beta e_\beta \text{ over } W_\beta. \]

Since \( f^\alpha_\beta \) are holomorphic, \( \bar{\partial}\sigma_\beta = \partial( f^\alpha_\beta \sigma_\alpha) = f^\alpha_\beta \bar{\partial}\sigma_\alpha \) on \( W_\alpha \cap W_\beta \), so that we see the above definition is well-defined.

Moreover, \( \bar{\partial}^2 = 0 \). Hence, we get the following exact sequence
\[
0 \to \mathcal{O}(L) \to \mathcal{A}^0(L) \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,1)}(L) \xrightarrow{\bar{\partial}} \mathcal{A}^{(0,2)}(L) \to \cdots,
\]
\[
0 \to \Omega^1 \to \mathcal{A}^{1,0}(L) \xrightarrow{\bar{\partial}} \mathcal{A}^{(1,1)}(L) \xrightarrow{\bar{\partial}} \mathcal{A}^{(1,2)}(L) \to \cdots.
\]

One can also see that the above exact sequences give the fine sheaf decomposition \( \mathcal{O}(L) \) and \( \Omega^1(L) \), hence, by the Dolbeault lemma, we obtain

**Theorem 2.12** (Dolbeault theorem) Let \( M \) be a Riemann surface with \( L \) a holomorphic line bundle over \( M \). Then for any \( p, q \),
\[
H^q(M, \Omega^p(L)) = \frac{\bar{\partial} \text{ closed } L\text{-valued } (p, q)\text{-forms}}{\bar{\partial} \text{ exact } L\text{-valued } (p, q)\text{-forms}}.
\]

Notation:
\[
H^q(M, \Omega^p(L)) = H^p_{\bar{\partial}}(M, L).
\]

**Corollary 2.13**
\[
\ell(D) \cong H^0(M, \mathcal{O}(L(D))),
\]
\[
i(D) \cong \Gamma(T^*_h - L(D)) \cong H^0(M, \Omega^1(L(-D))) = H^0(M, \Omega^1 \otimes L^*(D)).
\]

**Remark:** Dolbeault theorem also holds on higher dimensional complex manifolds.

### 2.7 Hermitian metric and connection

Before we can compute the cohomology groups by algebraic method, we need to know if \( H^p(M, L) \) is of finite dimension for \( q \geq 1 \). This cannot be seen directly from the Dolbeault theorem. This is equivalent to prove the \( \bar{\partial}u = \omega \) is solvable up to a finite dimensional obstructions. It can be done by using the Hodge theorem through the elliptic method.

**Definition** Let \( (L, \pi, M) \) be a holomorphic line bundle over \( M \). A **Hermitian metric** on \( L \) is an assignment \( \{ \langle \cdot, \cdot \rangle_x : x \in M \} \), where \( \langle \cdot, \cdot \rangle_x \) is a Hermitian inner product on \( \pi^{-1}(x) \) such
that $\langle \, , \, \rangle_x$ depends smoothly on $x$. Namely, for any $s, s' \in \Gamma_\infty(W_\alpha, L)$, $x \mapsto \langle s, s' \rangle_x$ is a smooth function over $M$.

Let $L \leftrightarrow \{ W_\alpha, f_\alpha \beta \}$, and $\langle \, , \, \rangle$ be a Hermitian metric on $L$. Let $\psi$ be a trivialization:

\[ \psi_\alpha : \pi^{-1}(W_\alpha) \to W_\alpha \times \mathbb{C}, \]

and

\[ e_\alpha(x) := \psi_\alpha^{-1}(x, 1). \]

Write

\[ g_\alpha(x) := |e_\alpha(x)|^2 = \langle e_\alpha(x), e_\alpha(x) \rangle. \]

Since $e_\alpha = f_\alpha \beta e_\beta$ on $W_\alpha \cap W_\beta$, we get

\[ g_\alpha = |f_\alpha \beta|^2 g_\beta. \tag{2.13} \]

For any holomorphic section $s \in \Gamma(M, L)$, we can define the metric

\[ |s|^2 = \langle s, s \rangle := \langle s_\alpha e_\alpha, s_\alpha e_\alpha \rangle = |s_\alpha|^2 g_\alpha, \text{ on } W_\alpha. \tag{2.14} \]

It is independent of choice of $W_\alpha$. In fact, $e_\alpha(x) = \Psi_\alpha^{-1}(x, 1)$ so that $s := \Psi_\alpha^{-1}(x, s_\alpha) = s_\alpha e_\alpha(x)$ by linearity of the trivialization. Since $s_\alpha = f_\alpha \beta s_\beta$ and $g_\alpha = |f_\alpha \beta|^2 g_\beta$, we have

\[ |s_\alpha|^2 g_\alpha = |f_\alpha \beta s_\beta|^2 |f_\alpha \beta|^2 g_\beta = |s|^2 g_\beta \]

because $f_\alpha \beta f_\beta \alpha = 1$.

Let $L \leftrightarrow \{ W_\alpha, f_\alpha \beta \}$, and $\langle \, , \, \rangle$ be a Hermitian metric on $L$. Then

\[ \langle \, , \, \rangle \leftrightarrow \exists \{ g_\alpha \in C_\infty(W_\alpha), \; g_\alpha > 0 \} \text{ such that } g_\alpha = |f_\alpha \beta|^2 g_\beta. \]

Conversely, if we have $\{ g_\alpha \in C_\infty(W_\alpha), \; g_\alpha > 0 \}$ such that $g_\alpha = |f_\alpha \beta|^2 g_\beta$, we can find a Hermitian metric $\langle \, , \, \rangle$ on $L$ with $|e_\alpha|^2 = g_\alpha$.

**Definition** Let $(L, \pi, M, \langle \, , \, \rangle)$ be a Hermitian line bundle. A connection $D$ over $L$ is an operator

\[ D : \Gamma_\infty(L) \to A^1 \otimes \Gamma_\infty(L) = A^1(L) \]

such that

\[ D(V + V') = D(V) + D(V'), \; D(fV) = df \otimes V + f \; DV. \]

**Remark** For $X \in T_pM, D_XV = X \cdot DV = \langle DV, X \rangle$, let $\{ e_\alpha \}$ be as before, write $De_\alpha = \theta_\alpha e_\alpha$ with $\theta_\alpha$ a 1-form. For $\sigma \in A^p(L)(M), \sigma = \omega_\alpha e_\alpha$ on $W_\alpha$ with $\omega_\alpha \in A^p(W_\alpha)$. Then we define $D\sigma = d\omega_\alpha e_\alpha + (-1)^p \omega_\alpha \wedge \theta_\alpha e_\alpha$, or $D\sigma = (d\omega_\alpha + (-1)^p \omega_\alpha \wedge \theta_\alpha)e_\alpha$. One can verify the above is well-defined.
Theorem 2.14 (Existence of the Hermitian connections) Let \((L, \pi, M, \langle \ , \rangle)\) be a Hermitian line bundle. Then there is a unique connection \(D\) such that

1. \(d\langle V, V' \rangle = \langle DV, V' \rangle + \langle V, DV' \rangle, \forall V, V' \in A^0(L),\)
2. \(D\) is of type \((1,0)\), namely, \(De_\alpha = \theta_\alpha e_\alpha\) with \(\theta_\alpha\) a \((1,0)\)-form.

Proof: Since \(d\langle e_\alpha, e_\alpha \rangle = \langle De_\alpha, e_\alpha \rangle + \langle e_\alpha, De_\alpha \rangle\) and \(De_\alpha = \theta_\alpha e_\alpha\), we get

\[dg_\alpha = \theta_\alpha g_\alpha + \bar{\theta}_\alpha g_\alpha,\]

so that \(\partial g_\alpha = \theta_\alpha g_\alpha\), i.e.,

\[\theta_\alpha = g_\alpha^{-1}\partial g_\alpha = \partial \log g_\alpha,\]

or

\[D e_\alpha = (\partial \log g_\alpha) e_\alpha. \quad (2.15)\]

This proves the uniqueness. We can verify that (2.15) also well define a connection over \(L\). \(\square\)

\(\Theta_\alpha = d\theta_\alpha\) is called the curvature form with respect to the hermitian metric. (i.e., \(\Theta_\alpha = d\theta_\alpha - \theta_\alpha \wedge \theta_\alpha = d\theta_\alpha\)). This is a global defined \((1,1)\)-form:

\[\Theta_\alpha = \bar{\partial} \partial \log g_\alpha, \text{ on } W_\alpha.\]

In fact, on \(W_\alpha \cap W_\beta\), we find

\[\Theta_\beta = \bar{\partial} \partial \log g_\beta = \bar{\partial} \partial \log(g_\alpha |f_{\alpha\beta}|^2) = \bar{\partial} \partial \log g_\alpha + \bar{\partial} \partial \log |f_{\alpha\beta}|^2 = \bar{\partial} \partial \log g_\alpha = \Theta_\alpha.\]

Suppose that \(L = T^{1,0}M\) and let \(\langle \ , \rangle\) be a metric over \(T^{1,0}M\). Write the metric as

\[\sigma = r_\alpha dz_\alpha \otimes d\bar{z}_\alpha\]

where \(r_\alpha = (\frac{\partial}{\partial z_\alpha}, \frac{\partial}{\partial \bar{z}_\alpha})\).

Then

\[\Omega = \Omega_\alpha = r_\alpha dx_\alpha \wedge dy_\alpha = r_\alpha(\frac{i}{2}dz_\alpha \wedge d\bar{z}_\alpha), \text{ on } W_\alpha,\]

is the well-defined volume form. Let \(\Theta_0\) be the curvature form of the metric \(\sigma\). Since \(\Omega\) never vanish (i.e., \(r_\alpha\) never vanish), we can write \(\Theta_0 = \frac{K_0}{i} \Omega\). Then

\[K_0 = -\frac{i \Theta_0}{\Omega}\]
is called the \textit{Gauss curvature function} of the metric $\sigma$. $K_0$ is a globally defined function over $M$. By direct calculation,

$$K_0 = -\frac{2}{r_\alpha} \frac{\partial^2 \log g_\alpha}{\partial z_\alpha \partial \bar{z}_\alpha}.$$ 

In general, if $L$ is a line bundle over $M$ with a metric $H = \{g_\alpha\}$. Denote by $\Theta$ the curvature form with respect to $(L, H)$. Then there is a $C^\infty$ function $K$ such that

$$\Theta = K \omega.$$

We have

$$K = -\frac{2}{r_\alpha} \frac{\partial^2 \log g_\alpha}{\partial z_\alpha \partial \bar{z}_\alpha}$$

which is called the \textit{Gauss curvature} with respect to $(L, H)$.

$$\text{Ricc} := -i \partial \bar{\partial} \log g_\alpha$$

is called the \textit{Ricci curvature form} of $(L, H)$.

\section{2.8 Statement of Hodge Theorem}

Now, we let $(L, \pi, M)$ be a holomorphic line bundle over $M$. $H = r_\alpha dz_\alpha \otimes d\bar{z}_\alpha$ a Hermitian metric on $M$ and $G = r_\alpha (dx_\alpha \otimes dx_\alpha + dy_\alpha \otimes dy_\alpha)$ a Riemann metric on $M$. Then

$$G(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\alpha}) = r_\alpha = G(\frac{\partial}{\partial y_\alpha}, \frac{\partial}{\partial y_\alpha}), \quad G(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial y_\alpha}) = 0.$$

$$\left\{ \frac{\partial}{\sqrt{r_\alpha} \partial x_\alpha}, \frac{\partial}{\sqrt{r_\alpha} \partial y_\alpha} \right\}$$

is an orthonormal basis of $TM$. $\left\{ \sqrt{r_\alpha} dx_\alpha, \sqrt{r_\alpha} dy_\alpha \right\}$ is an orthonormal basis of $T^*M$. Hence, we get

$$G(dz_\alpha, dz_\alpha) = \frac{1}{r_\alpha}, \quad G(d\bar{z}_\alpha, d\bar{z}_\alpha) = \frac{1}{r_\alpha}.$$

$$G(dz_\alpha \wedge d\bar{z}_\alpha, d\bar{z}_\alpha \wedge d\bar{z}_\alpha) = \frac{1}{r_\alpha^2}.$$

Let $\Omega = r_\alpha dx_\alpha \wedge dy_\alpha = \frac{i r_\alpha}{2} dz_\alpha \wedge d\bar{z}_\alpha$ be the volume form. Then

$$G(\Omega, \Omega) = 1.$$

Denote $A^p = \mathcal{A}^p(M) = \{C^\infty p - \text{forms on } M\}$ and $A^{p,q} = \mathcal{A}^{p,q}(M) = \{C^\infty (p, q) - \text{forms on } M\}$. 


Lemma 2.15 There exists an operator $*: A^{p,q} \to A^{1-q,1-p}$ (and hence $*: A^k \to A^{2-k}$) such that $\forall \varphi, \psi \in A^{p,q}$ the following holds:

1. $G(\varphi, \psi)\Omega = \varphi \wedge \ast \psi$,
2. $\ast \ast : A^{p,q} \to A^{p,q}, \ast \ast \varphi = (-1)^{p+q} \varphi$,
3. $\ast \ast : A^p \to A^p, \ast \ast \varphi = (-1)^p \varphi$.

(4) $\ast$ is real, $\ast \overline{\varphi} = \overline{\ast \varphi}$. In the local coordinate, $\ast$ is defined as follows:

$$\ast 1 = \Omega, \ast \Omega = 1, \ast d z_\alpha = -id z_\alpha, \ast d \bar{z}_\alpha = -id \bar{z}_\alpha.$$ 

Now we can define an inner product for $A(L)$. For $\sigma_1, \sigma_2 \in A^{p,q}(L)$, locally $\sigma_j = \omega_j^{(a)} s_j^{(a)}$ on $W_\alpha, j = 1, 2$. Let $\{\chi_\alpha\}$ be a partition of the unit subordinate to $\{W_\alpha\}$. We define

$$\langle \sigma_1, \sigma_2 \rangle := \sum_\alpha \int_M \chi_\alpha \langle s_1^{(a)}, s_2^{(a)} \rangle_H \langle \omega_1^{(a)}, \omega_2^{(a)} \rangle_G \Omega = \sum_\alpha \int_M \chi_\alpha \langle s_1^{(a)}, s_2^{(a)} \rangle_H \omega_1^{(a)} \wedge \ast \omega_2^{(a)}.$$ 

We can show that it is independent of choice of $\{W_\alpha\}$ and $\{\chi_\alpha\}$. Then $(\cdot, \cdot)$ induces an inner product on $A(L) = \oplus_{p,q} A^{p,q}(L)$.

Definition Let $T_1, T_2 : A(L) \to A(L)$ be two linear operators such that

$$\langle T_1 \sigma, \eta \rangle = \langle \sigma, T_2 \eta \rangle, \forall \sigma, \eta \text{ with compact support}.$$

We call $T_1, T_2$ are adjoint to each other. We write $T_2 = T_1^*$, or $T_1 = T_2^*$.

[Example] $\ast$ and $\ast^{-1}$ are adjoint to each other.

Theorem 2.16

$$\overline{\partial}^* = -\ast D^1_*,$$

where on each $W_\alpha$

$$D^1 : A^{p,q}(L) \to A^{p,q+1}(L), \quad \sigma = \omega_\alpha e_\alpha \mapsto (\omega_\alpha + (-1)^{p+q} \omega_\alpha \wedge \theta_\alpha) e_\alpha.$$ 

(2.16)
CHAPTER 2. PROOF OF THE RIEMANN-ROCH THEOREM

Proof: \( \forall \sigma_1 = \omega_1 e_\alpha \in A^{p,q-1}(L), \forall \sigma_2 = \omega_2 e_\alpha \in A^{p,q}(L) \), to show:

\[
(\bar{\partial} \sigma_1, \sigma_2) - (\sigma_1, \bar{\partial} \sigma_2) = 0,
\]
i.e.,

\[
(\bar{\partial} \sigma_1, \sigma_2) - (\sigma_1, -*D_1 * \sigma_2) = 0.
\]

Now

\[
(\bar{\partial} \sigma_1, \sigma_2) = \int_M \langle e_\alpha, e_\alpha \rangle \bar{\partial} \omega_1 \wedge \overline{\omega_2} = \int_M g_\alpha \bar{\partial} \omega_1 \wedge \overline{\omega_2}.
\]

Since \(-*D_1 * \sigma_2 = -* (\partial * \omega_2 + (-1)^{p+q} * \omega_2 \wedge \theta_\alpha) e_\alpha\), we have

\[
(\sigma_1, -*D_1 * \sigma_2) = -\int_M \langle e_\alpha, e_\alpha \rangle \omega_1 \wedge **(\partial * \omega_2 + (-1)^{p+q} * \omega_2 \wedge \theta_\alpha)
\]
\[
= -\int_M g_\alpha \omega_1 \wedge (-1)^{p+q+1} (\partial * \overline{\omega_2} + (-1)^{p+q} * \overline{\omega_2} \wedge \overline{\theta_\alpha})
\]
\[
= -\int_M g_\alpha \omega_1 \wedge (-1)^{p+q+1} (\partial * \overline{\omega_2} + (-1)^{p+q} * \overline{\omega_2} \wedge g_\alpha^{-1} \overline{\partial} g_\alpha)
\]
\[
= -\int_M (-1)^{p+q-1} \omega_1 \wedge (\partial * \overline{\omega_2}) g_\alpha - \omega_1 \wedge \overline{\omega_2} \wedge \overline{\partial} g_\alpha.
\]

Here we used the facts that \(* * \varphi = (-1)^{p+q} \varphi\) and \(* \overline{\varphi} = \overline{\varphi} \).

Continue the calculation:

\[
(\bar{\partial} \sigma_1, \sigma_2) - (\sigma_1, -*D_1 * \sigma_2)
\]
\[
= \int_M \left( \bar{\partial} \omega_1 \wedge \overline{\omega_2} g_\alpha + (-1)^{p+q-1} \omega_1 \wedge (\partial * \overline{\omega_2}) g_\alpha - \omega_1 \wedge \overline{\omega_2} \wedge \overline{\partial} g_\alpha \right)
\]
\[
= \int_M \overline{\partial} (\omega_1 \wedge \overline{\omega_2} g_\alpha) = \int_M d(\omega_1 \wedge \overline{\omega_2} g_\alpha) = 0
\]
by Stokes theorem. \( \square \)

Now

\[
\bar{\partial}: A^{p,q}(L) \to A^{p,q+1}(L), \quad \bar{\partial}^* : A^{p,q+1}(L) \to A^{p,q}(L), \quad \bar{\partial}^2 = 0, \quad \bar{\partial}^{*2} = 0.
\]

Let

\[
\square := \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial} = (\bar{\partial} + \bar{\partial}^*)^2.
\]

\( \square \) is called the Laplacian operator with respect to \((L, \pi, M, H, G)\).
2.8. STATEMENT OF HODGE THEOREM

We remark that if $L = \mathcal{O}$ be the trivial line bundle with the trivial metric, then

$$\Box = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = 2 \frac{\partial^2}{\partial z \partial \bar{z}}.$$

That is why we call $\Box$ Laplacian operator.

We have

$$(\Box \sigma_1, \sigma_2) = (\sigma_1, \Box \sigma_2), \ \forall \sigma_1, \sigma_2 \in A(L).$$

**Lemma 2.17**

$$\Box \varphi = 0 \iff \overline{\partial} \varphi = 0, \text{ and } \overline{\partial}^* \varphi = 0.$$  

**Proof:**

$$(\Box \varphi, \varphi) = (\overline{\partial} \overline{\partial}^* \varphi, \varphi) + (\overline{\partial}^* \varphi, \overline{\partial} \varphi) = ||\overline{\partial} \varphi||^2 + ||\overline{\partial}^* \varphi||^2.$$

Write

$$\Box_0 = -\frac{2}{r_0} \left( \frac{\partial^2}{\partial z_0 \partial \bar{z_0}} \right) + \frac{\partial \log g_0}{\partial z_0} \frac{\partial}{\partial \bar{z}_0}.$$  

Let $f = f_\alpha \varphi_\alpha e_\alpha$, where $f_\alpha \in C^\infty(W_\alpha)$, and

$$\varphi_\alpha = \begin{cases} 
1, & \text{if } (p, q) = (0, 0), \\
\bar{z}_\alpha, & \text{if } (p, q) = (1, 0), \\
\bar{z}_\alpha, & \text{if } (p, q) = (0, 1), \\
\Omega, & \text{if } (p, q) = (1, 1).
\end{cases}$$

Here $\varphi_\alpha e_\alpha$ is a basis of $A(L)$ over $W_\alpha$.

By direct computation, we have the following formulas:

For $f \in A^0(L)$,

$$\Box f = (\Box_0 f_\alpha) \varphi_\alpha e_\alpha.$$  

For $f \in A^{1,0}$,

$$\Box f = \left( (\Box_0 f_\alpha + \frac{2}{r_\alpha} \frac{\partial \log r_\alpha}{\partial z_\alpha} \frac{\partial}{\partial \bar{z}_\alpha}) f_\alpha \right) \varphi_\alpha e_\alpha.$$  

For $f \in A^{0,1}$,

$$\Box f = \left( (\Box_0 f_\alpha + \frac{2}{r_\alpha} \frac{\partial \log r_\alpha}{\partial z_\alpha} \frac{\partial}{\partial \bar{z}_\alpha} + [K + \frac{2}{r_\alpha} \frac{\partial \log r_\alpha}{\partial z_\alpha} \frac{\partial \log g_\alpha}{\partial \bar{z}_\alpha}] f_\alpha \right) \varphi_\alpha e_\alpha,$$
For $f \in A^{1,1}(P)$,
\[ \square f = (\square_0 + K)f_\alpha \varphi_\alpha e_\alpha. \]

Then for any $f = f_\alpha \varphi_\alpha e_\alpha$,
\[ \square f = \tilde{f}_\alpha \varphi_\alpha e_\alpha, \]
where
\[ \tilde{f}_\alpha = -\frac{2}{r_\alpha} \left( \frac{\partial^2}{\partial z_\alpha \partial \bar{z}_\alpha} + k_1 \frac{\partial}{\partial z_\alpha} + k_2 \frac{\partial}{\partial \bar{z}_\alpha} + k_3 \right) f_\alpha. \]

Since $-\frac{2}{r_\alpha} < 0$, $\square$ is an elliptic operator. Write
\[ H^{p,q}(L) = \{ f \in A^{p,q}(L) \mid \square f = 0 \}. \]

$H^{p,q}(L)$ is called the space of harmonic $(p,q)$-forms. Denote
\[ \mathcal{H}(L) = \oplus_{p,q} H^{p,q}(L). \]

**Theorem 2.18** (Hodge theorem) Let $(L, \pi, M, H)$ be a Hermitian line bundle over a Hermitian Riemann surface $(M, G)$. Let $\square$ and $\mathcal{H}(L)$ be defined as above. Then

1. $\mathcal{H}(L)$ is a finite dimensional space.
2. There is an operator $G$, called the Green operator of $\square$, $G : A(L) \to A(L)$ such that $\ker(G) = \mathcal{H}(L), G(A^{p,q}) \subset A^{p,q}$, $G$ commutes with $\bar{\partial}, \bar{\partial}^\ast$, $\square G(\omega) = G(\square \omega), \forall \omega \in \mathcal{H}^\perp$, and
   \[ \| \omega \|_2 \lesssim \| G \omega \| + \| \omega \|. \]
3. $A(L) = \mathcal{H}(L) \oplus \square G A(L) = \mathcal{H}(L) \oplus G \square A(L).$ \(^2\)

**Corollary 2.19**

\[ H^q(M, \Omega^p(L)) = \mathcal{H}^{p,q}(L). \]

**Proof:** By Dolbeault theorem,

\[ H^q(M, \Omega^p(L)) \cong \frac{\overline{\partial} \text{ closed } L \text{-valued } (p,q) \text{-forms}}{\overline{\partial} \text{ exact } L \text{-valued } (p,q) \text{-forms}}. \]

When $q = 0$, by above, $H^0(M, \Omega^p(L)) \cong \{ f \in A^{p,q}(L) \mid \bar{\partial} f = 0 \}$. In this case, for $f \in A^{p,0}(L), \bar{\partial}^\ast f \in A^{p,-1}(L) = \{ 0 \}$ so that $\bar{\partial}^\ast f = 0$. By Lemma 2.17, $H^0(M, \Omega^p(L)) \cong \mathcal{H}^{p,0}(L).$

\(^2\) $A(L) = \mathcal{H}(L) \oplus \bar{\partial}(A(L)) \oplus \bar{\partial}^\ast (A(L)).$
2.9. Serre Duality Theorem

When \( q = 1 \), by Dolbeault theorem,
\[
H^1(M, \Omega^p(L)) \cong A^{p,1}(L)/\overline{\partial}A^{p,0}(L).
\]
Notice any \( f \in A^{p,1}(L) \) must be \( \overline{\partial} \)-closed by consideration of degree. By Hodge theorem,
\[
A^{p,1}(L) = \mathcal{H}^{p,1}(L) \oplus G(\overline{\partial} \overline{\partial} + \partial \partial^* )A^{p,1}(L)
\]
\[= \mathcal{H}^{p,1}(L) \oplus G \overline{\partial} \overline{\partial} A^{p,1}(L) = \mathcal{H}^{p,1}(L) \oplus \overline{\partial} \overline{\partial} G A^{p,1}(L) \subset \mathcal{H}^{p,1}(L) \oplus \overline{\mathcal{A}}^{p,0}(L)
\]
because \( \overline{\partial} G A^{p,1}(L) \subset \overline{\partial} A^{p,1}(L) \subset A^{p,0}(L) \). Since \( \mathcal{H}^{p,1}(L) \oplus \overline{\mathcal{A}}^{p,0}(L) \subset A^{p,1}(L) \), we have
\[
A^{p,1}(L) = \mathcal{H}^{p,1}(L) \oplus \overline{\mathcal{A}}^{p,0}(L).
\]
Then \( H^1(M, \Omega^p(L)) \cong \mathcal{H}^{p,1}(L) \) holds. □

Recall that for a divisor \( D \) over \( M \),
\[
\ell(D) \cong H^0(M, \mathcal{O}([D])),
\]
\[
i(D) \cong H^0(M, \Omega^1(−[D])).
\]
By Hodge theorem,
\[
\dim \ell(D), \dim i(D) < \infty.
\]

2.9 Serre Duality Theorem

Theorem 2.20 (Serre Duality) Let \( L \) be a line bundle over a compact Riemann surface \( M \). Then
\[
H^q(M, \Omega^p(L)) \cong H^{1−q}(M, \Omega^{1−p}(−L)).
\]

Remark Serre Duality Theorem is valid for higher dimensional complex manifolds. We’ll use Hodge theorem to prove it.

Consider \( L \leftrightarrow \{W_\alpha, f_{\alpha\beta}\} \) and \( −L \leftrightarrow \{W_\alpha, f_{\beta\alpha} = \frac{1}{f_{\alpha\beta}}\} \). Let \( H \leftrightarrow \{g_\alpha\} \) be a metric on \( L \), and \( \{\frac{1}{g_\alpha}\} \) gives a metric on \( −L \). On each \( W_\alpha \), we take \( e_\alpha \in \Gamma(W_\alpha, L) \), \( e_\alpha \not\equiv 0 \), and \( \tilde{e}_\alpha \in \Gamma(W_\alpha, −L) \), \( \tilde{e}_\alpha \not\equiv 0 \). Then
\[
e_\alpha = f_{\beta\alpha}e_\beta, \quad \tilde{e}_\alpha = f_{\alpha\beta}\tilde{e}_\beta, \quad \text{on } W_\alpha \cap W_\beta.
\] (2.17)
∀φ ∈ A^{p,q}(L), on each W_α, φ = ω_α e_α where ω_α ∈ A^{p,q}(W_α). On W_α ∩ W_β ≠ ∅, from φ = ω_α e_α = ω_α f_{βα} e_β = ω_β e_β, it implies

ω_α f_{βα} = ω_β.  \hspace{1cm} (2.18)

We define a conjugate operator

∼: A^{p,q}(L) → A^{q,p}(-L), \quad φ = ω_α e_α ↦∼ φ := \bar{ω}_α g_α \tilde{e}_α

We need to show that such definition is well defined. In fact, on any W_α ∩ W_β ≠ ∅, write φ = ω_β e_β where ω_β ∈ A^{p,q}(W_β). By (2.17) (2.18) and g_α = |f_{βα}|^2 g_β, we obtain

\bar{ω}_α g_α \tilde{e}_α = \bar{f}_{αβ} \bar{ω}_β |f_{βα}|^2 g_β f_{βα} \tilde{e}_β = \bar{ω}_β g_β \tilde{e}_β so that the operator ∼ is well defined.

Define

∗:= ∗◦∼: A^{p,q}(L) → A^{1−p,1−q}(−L), \quad ω_α e_α ↦ (∗\bar{ω}_α)g_α \tilde{e}_α

which is a conjugate isomorphism. Then we have

∗◦∼ = ∼◦∗.

We can construct an operator \tilde{□} = \nabla \bar{\nabla} + \bar{\nabla} \nabla : A^{p,q}(−L) → A^{p,q}(−L) such that

\tilde{□} ∗ = ∗\tilde{□}.

In fact, by direct computation, \tilde{\nabla} := −∗ \tilde{D}^{1}∗ where D^{1} is defined in (2.16). Hence ker□ ∼ ker\tilde{□}.

**Corollary 2.21** φ ∈ A^{p,q}(L) is harmonic if and only if \tilde{∗}φ ∈ A^{1−p,1−q}(−L) is harmonic.

Therefore

\tilde{∗}: \mathcal{H}^{(p,q)}(L) ∼= \mathcal{H}^{(1−p,1−q)(−L)}.

is an isomorphism. By the corollary of Hodge theorem, Serre duality theorem is proved.
2.10 Proof of the Riemann-Roch theorem

For any sheaf $F$ over $M$, if $H^q(M, F)$ is of finite dimension for any $q$, we define the Euler number of a sheaf $F$:

$$\chi(F) = \sum_{j=0}^{\infty} (-1)^j \dim H^j(M, F)$$

provided $\dim H^q(M, F) = 0$ for $q >> 1$ and $\dim H^q(M, F) < \infty$ for all $q$.

Notice that when $F = \mathbb{C}$, $\chi(\mathbb{C})$ is the regular Euler number and that if $F = \mathcal{O}(L)$ is the sheaf of germs of holomorphic sections of a line bundle $L$, by Hodge theorem and its consequences, the Euler number is well-defined: $\chi(L) < \infty$.

[Examples]

1. Let $L = \mathcal{O}$ be the trivial line bundle. Then the Euler number is

$$\chi(\mathcal{O}) = \dim H^0(M, \mathcal{O}) - \dim H^1(M, \mathcal{O}) + \ldots$$

$$= \dim H^0(M, \mathcal{O}) - \dim H^1(M, \mathcal{O})$$

$$= \dim H^0(M, \mathcal{O}) - \dim H^0(M, \Omega^1) \quad \text{(by Serrer duality theorem)}$$

$$= 1 - g. \quad (H^0(M, \Omega^1) = \{\text{holomorphic 1-forms}\})$$

Here we used the fact that for any $q \geq 2$, and any compact Riemann surface $M$,

$$H^q(M, \mathcal{O}) \cong \frac{\partial \text{closed } (0, q) - \text{forms}}{\partial \text{exact } (0, q) - \text{forms}} = 0$$

by Dolbeault theorem.

2. Let $D$ be a divisor over $M$ and $[D]$ the associated line bundle. Then

$$\chi([\mathcal{O}[D]]) = \dim H^0(M, \mathcal{O}[D]) - \dim H^1(M, \mathcal{O}[D]) + \dim H^2(M, \mathcal{O}[D]) - \ldots$$

$$= \dim H^0(M, \mathcal{O}[D]) - \dim H^1(M, \mathcal{O}[D])$$

$$= \dim H^0(M, \mathcal{O}[D]) - \dim H^0(\Omega^1(\mathcal{O}[-D])) \quad \text{(By Serrer duality theorem)}$$

$$= \dim \ell(D) - \dim i(D).$$

Here we used the fact that for any $q \geq 2$, and any compact Riemann surface $M$,

$$H^q(M, \mathcal{O}(L)) \cong \frac{\partial \text{closed } L - \text{valued } (0, q) - \text{forms}}{\partial \text{exact } L - \text{valued } (0, q) - \text{forms}} = 0$$

by Dolbeault theorem.
3. Let \( D = \sum_p n(p) p \geq 0 \) be an effective divisor. Consider the short exact sequence:

$$0 \to J_D \to \mathcal{O} \to S_D \to 0$$

where \( J_D \) is given by

\[
J_D(W) = \{ s \in \mathcal{O}(W) \mid (s) - D \geq 0 \}, \quad \forall \text{ open subset } W,
\]

and \( S_D = \mathcal{O}/J_D = \bigoplus n(p)S_p \) is the skyscraper sheaf:

\[
S_D|_p = \begin{cases} 
0, & p \text{ with } n(p) = 0 \\
\oplus_{n(p)\mathbb{C}}, & p \text{ with } n(p) > 0.
\end{cases}
\]

\( S_D \) is a fine sheaf so that \( H^q(M, S_D) = 0, \forall q \geq 1 \).

Also \( \dim H^0(M, S_D) = \sum_p n(p) = \deg(D) \).

Then

\[
\chi(S_D) = \dim H^0(M, S_D) - \dim H^1(M, S_D) = \dim H^0(M, S_D) - 0 = \deg(D).
\]

**Lemma 2.22** Let \( 0 \to F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} F_3 \to 0 \) be a short exact sequence. Assume that \( \dim H^q(M, F_j) < \infty \) and \( H^q(M, F_j) = 0, \forall q > 1 \). Then

\[
\chi(F_2) = \chi(F_1) + \chi(F_3).
\]

**Proof:** We have the long exact sequence

\[
0 \to H^0(M, F_1) \xrightarrow{\alpha^*} H^0(M, F_2) \xrightarrow{\beta^*} H^0(M, F_3) \xrightarrow{\delta^*} H^1(M, F_1) \to ....
\]

Consider

\[
H^{j-1}(M, F_3) \xrightarrow{\delta^*} H^j(M, F_1) \xrightarrow{\alpha^*} H^j(M, F_2) \xrightarrow{\beta^*} H^j(M, F_3) \xrightarrow{\delta^*} H^{j+1}(M, F_1).
\]

Then

\[
\beta^* H^j(M, F_2) \cong H^j(M, F_2)/\alpha^* H^j(M, F_1)
\]

and hence

\[
\dim H^j(M, F_2) = \dim \alpha^* H^j(M, F_1) + \dim \beta^* H^j(M, F_2).
\]

Similarly,

\[
\dim H^j(M, F_1) = \dim \delta^* H^{j-1}(M, F_3) + \dim \alpha^* H^j(M, F_1).
\]
2.10. PROOF OF THE RIEMANN-ROCH THEOREM

\[ \dim H^j(M, F_3) = \dim \delta^*H^j(M, F_3) + \dim \beta^*H^j(M, F_2). \]

It follows

\[ \chi(F_2) - \chi(F_1) - \chi(F_3) = \sum_{j=0}^{\infty} (-1)^j \dim \alpha^*H^j(M, F_1) + (-1)^j \dim \beta^*H^j(M, F_2) \]
\[ + \sum_{j=0}^{\infty} (-1)^{j+1} \dim \delta^*H^{j-1}(M, F_3) + (-1)^{j+1} \dim \alpha^*H^j(M, F_1) \]
\[ + \sum_{j=0}^{\infty} (-1)^{j+1} \dim \delta^*H^j(M, F_3) + (-1)^{j+1} \dim \beta^*H^j(M, F_2) \]
\[ = 0. \]

This gives \( \chi(F_2) = \chi(F_1) + \chi(F_3). \) \( \square \)

Proof of the Riemann-Roch Theorem: First consider a special case \( D = \sum n_j p_j \geq 0. \)

Consider the short exact sequence:

\[ 0 \to \mathcal{O}(L - [D]) \to \mathcal{O}(L) \to \mathcal{S}_D \to 0 \]

By the above lemma, we have

\[ \chi(\mathcal{O}(L)) = \chi(\mathcal{O}(L - [D])) + \chi(\mathcal{S}_D). \]

Take \( L = [D]. \) We have

\[ \chi(\mathcal{O}[D]) = \chi(\mathcal{O}) + \chi(\mathcal{S}_D). \]

From the above examples, we know

\[ \chi(\mathcal{O}) = 1 - g, \quad \chi(\mathcal{O}[D]) = \dim \ell(D) - \dim i(D), \quad \chi(\mathcal{S}_D) = \deg(D). \]

We obtain

\[ \dim \ell(D) - \dim i(D) = \chi(\mathcal{O}[D]) = \chi(\mathcal{O}) + \chi(\mathcal{S}_D) = 1 - g + \deg(D). \]

In general, we write \( D = D_1 + D_2 \) with \( D_1 \geq 0 \) and \( D_2 \geq 0. \) Then we have the following short exact sequence:

\[ 0 \to \mathcal{O}([D]) \to \mathcal{O}([D_1]) \to \mathcal{S}_{D_2} \to 0, \]
where
\[ O([D])(U) = \{ h \in \mathcal{M}(U) \mid (h) + D_1 - D_2 \geq 0 \}, \]
\[ O([D_1])(U) = \{ h \in \mathcal{M}(U) \mid (h) + D_1 \geq 0 \}. \]

Then we obtain
\[ \chi(O([D_1])) = \chi(O([D])) + \deg(D_2). \]

Since \( D_1 \geq 0 \), from above we know that \( \chi(O([D_1])) = 1 - g + \deg(D_1) \) so that
\[ \chi(O([D])) = 1 - g + \deg(D). \]

The proof of the Riemann-Roch theorem is complete. □
Chapter 3

Proof of Hodge Theorem

In this chapter, we present a proof of Hodge theorem.

3.1 Sobolev spaces

Let \( \Omega \subset \mathbb{R}^n \) be open subset. For each \( s \in \mathbb{Z} \) with \( s \geq 0 \), we define

\[
A_s(\Omega) := \{ f \in C^\infty(\Omega) \mid |f|^2_s < \infty \}
\]

where \( f \) are complex valued functions and

\[
|f|^2_s := \sum_{|\alpha| \leq s} \int_\Omega |D^\alpha f|^2 dx.
\]

For any \( f, g \in A_s(\Omega) \), we define the inner product

\[
(f, g)_s := \sum_{|\alpha| \leq s} \int_\Omega D^\alpha f D^\alpha g dx, \quad \text{and} \quad |f|^2_s := (f, f)_s.
\]

Its finiteness follows by Schwarz inequality:

\[
|(f, g)_s| \leq |f|_s |g|_s.
\]

The space \( A_s(\Omega) \) with the norm \( \cdot |_s \) is a pre-Hilbert space. The complete extension of \( (A_s(\Omega), |\cdot|_s) \), denoted by \( H_s(\Omega) \), is called the Sobolev space.
We introduce the concept of “weak derivatives” as follows. For any $f \in C^\infty(\Omega)$ and $\varphi \in C_0^\infty(\Omega)$, by the integral by parts, we have
\[
\int_\Omega f \overline{D^\alpha \varphi} = (-1)^{|\alpha|} \int_\Omega (D^\alpha f) \overline{\varphi},
\]
i.e.,
\[
(f, D^\alpha \varphi)_0 = (-1)^{|\alpha|} (D^\alpha f, \varphi)_0.
\]
**Definition** \(\forall f \in H_0(\Omega), \exists h \in H_0(\Omega)\) such that
\[
(f, D^\alpha \varphi)_0 = (-1)^{|\alpha|} (h, \varphi)_0, \quad \forall \varphi \in C_0^\infty(\Omega),
\]
then we call \(h\) the *weak derivative* of \(f\) of order \(\alpha\). We denote it as \(D^\alpha f = h\) weakly.

**Lemma 3.1** If \(f \in H_s(\Omega)\), then \(\forall \alpha \) with \(|\alpha| \leq s\), the weak derivative \(D^\alpha f \in H_0(\Omega)\) exists.

**Proof:** By the definition of \(H_s(\Omega)\), there is a sequence \(\{f_j\} \subset A_s(\Omega)\) such that
\[
|f - f_j|_s \to 0, \quad \text{as } j \to \infty.
\]
Then \(|f_j - f_k|_s \to 0\) as \(j, k \to \infty\), and hence for any \(\alpha\) with \(|\alpha| \leq s\), we have
\[
|D^\alpha f_j - D^\alpha f_k|_0 \to 0, \quad \text{as } j, k \to \infty.
\]
by the definition of \(|\cdot|_s\). Since \(H_0(\Omega)\) is complete, there is \(h \in H_0(\Omega)\) such that
\[
|D^\alpha f_j - h|_0 \to 0, \quad \text{as } j \to \infty.
\]
Therefore
\[
(f, D^\alpha \varphi)_0 = \lim_j (f_j, D^\alpha \varphi)_0 = \lim_j (-1)^{|\alpha|} (D^\alpha f_j, \varphi)_0 = (-1)^{|\alpha|} (h, \varphi),
\]
i.e., \(D^\alpha f = h\). \(\square\)

Let us recall the following classical analysis without proof.

**Lemma 3.2** Let \(\chi \in C_0^\infty(\mathbb{R}^n)\), \(\chi \geq 0\) with \(\text{supp}(\chi) \subset B^n(0, 1)\) and \(\int_{\mathbb{R}^n} \chi = 1\). For any \(\epsilon > 0\), we define \(\chi_{\epsilon}(x) = \frac{1}{\epsilon^n} \chi\left(\frac{x}{\epsilon}\right)\). For any \(f \in L^2(\Omega)\), we define convolution
\[
(f * \chi_{\epsilon})(x) := \int_{\mathbb{R}^n} f(y) \chi_{\epsilon}(x - y) dy.
\]
Then the following hold:
3.1. SOBOLEV SPACES

1. \( \text{dist}(\text{supp}(f), \mathbb{R}^n - \text{supp}(f \ast \chi_\epsilon)) \leq \epsilon. \)

2. \( L^2(\Omega) \rightarrow C^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n) \)
   \[ f \rightarrow f \ast \chi_\epsilon \]
   is a map such that with the norm on \( L^2(\mathbb{R}^n), \)
   \[ |f - f \ast \chi_\epsilon|_0 \rightarrow 0, \text{ as } \epsilon \rightarrow 0. \]

3. If \( \text{dist}(\text{supp}(f), \mathbb{R}^n - \Omega) \geq 2\epsilon, \) then the weak derivative \( D^\alpha f \) exists such that
   \[ D^\alpha(f \ast \chi_\epsilon) = (D^\alpha f) \ast \chi_\epsilon. \]

Lemma 3.3

\[ H_s(\Omega) = \{ f \in L^2(\Omega) \mid \forall \alpha, \text{ with } |\alpha| \leq s, \text{ weak derivative } D^\alpha f \text{ exists} \}. \]

Proof: \( H_s(\Omega) \) belongs to the space at the right hand side follows by Lemma 3.1. Conversely, we'll show that the space at the right hand side belongs to \( H_s(\Omega) \). Namely, \( \forall f \in H_0(\Omega) \) such that \( D^\alpha f \) exists, \( \forall \alpha \) with \( |\alpha| \leq s \), we want to show that \( \exists \) a sequence \( \{ f_j \} \subset A_s(\Omega) \) such that

\[ |f_j - f|_s \rightarrow 0, \text{ as } j \rightarrow \infty. \]

Let \( \{ W_\alpha \} \) be a locally finite open covering of \( \Omega \) with \( W_\alpha \subset \subset \Omega \), and \( \{ \eta_\alpha \} \) a partition of unity subordinate to \( \{ W_\alpha \} \).

For each \( \alpha \), \( \text{supp}(\eta_\alpha f) \subset \subset \Omega \). By Lemma 3.2, we can take a sequence \( \delta(n) \searrow 0 \), such that for any \( \delta(n) > 0 \), \( \exists \epsilon(\alpha, n) > 0 \) such that

\[ \left| D^\beta(\eta_\alpha f) \ast \chi_{\epsilon(\alpha, n)} - D^\beta(\eta_\alpha f) \right|_0 < \frac{\delta(n)}{2|\beta|+1}, \forall \beta \text{ with } |\beta| \leq s. \]

Let \( f_\delta(n) := \sum_\alpha (\eta_\alpha f) \ast \chi_{\epsilon(\alpha, n)} \). We have \( f_\delta(n) \in C^\infty(\Omega) \) and \( \forall \beta, |\beta| \leq s, \)

\[ \left| D^\beta f_\delta(n) - D^\beta f \right|_0 = \left| D^\beta(\sum_\alpha (\eta_\alpha f) \ast \chi_{\epsilon(\alpha, n)}) - D^\beta(\sum_\alpha (\eta_\alpha f)) \right|_0 \]

\[ = \sum_\alpha \left| D^\beta((\eta_\alpha f) \ast \chi_{\epsilon(\alpha, n)}) - D^\beta(\eta_\alpha f) \right|_0 \]

\[ \leq \sum_\alpha \left| [D^\beta(\eta_\alpha f) \ast \chi_{\epsilon(\alpha, n)}) - D^\beta(\eta_\alpha f)] \right|_0 \]

\[ \leq \sum_\alpha \frac{\delta(n)}{2|\alpha|} \leq \delta(n). \]
Then $|f_{\delta(n)} - f|_s \to 0$, as $n \to \infty$. □

**Corollary 3.4**

1. $H_s(\Omega) \subset H_{s-1}(\Omega) \subset \ldots \subset H_1(\Omega) \subset H_0(\Omega)$, which is called the **Sobolev chain**.

2. Let $f \in H_t(\Omega)$. Suppose that $\forall \alpha$ with $|\alpha| \leq t$, we have $D^\alpha f \in H_s(\Omega)$, then $f \in H_{s+t}(\Omega)$.

Now we define Sobolev space over Riemann surface $M$. Let $\{W_\alpha\}$ be a locally finite open covering. Let $(M, G)$ be a Hermitian Riemann surface which is locally written as $r_\alpha dz_\alpha d\overline{z}_\alpha$. Let $\Omega$ be the volume form which is locally written as $\Omega_\alpha = r_\alpha (i^2 dz_\alpha \wedge d\overline{z}_\alpha)$.

Let $(L, H)$ be a Hermitian line bundle over $M$. Let $\{W_\alpha, f_{\alpha\beta}\} \leftrightarrow L$. Let $e_\alpha = \psi^{-1}_\alpha (\cdot, 1) \in \Gamma_h(W_\alpha, L)$ and $g_\alpha = \langle e_\alpha, e_\alpha \rangle$. We define Laplacian operator $\Box = \partial^* \partial + \partial \partial^*$ : $A^{p,q}(L) \to A^{p,q}(L)$. On each $W_\alpha$,

$$\Box = \frac{2}{r_\alpha} \left( \frac{p^2}{\partial z_\alpha \partial \overline{z}_\alpha} + \text{the first order derivatives} \right)$$

which is an elliptic operator. Let $\{\chi_\alpha\}$ be a fixed partition of unity subordinate to $\{W_\alpha\}$. Let

$$\omega_\alpha = \begin{cases} 
1, & (p, q) = (0, 0), \\
\Omega_\alpha, & (p, q) = (1, 1), \\
\sqrt{r_\alpha} dz_\alpha, & (p, q) = (1, 0), \\
\sqrt{r_\alpha} d\overline{z}_\alpha, & (p, q) = (0, 1).
\end{cases}$$

Then $\langle \omega_\alpha, \omega_\alpha \rangle = 1$. For $\varphi \in \Gamma(M, A^{p,q}(L))$, we write $\varphi = h_\alpha \omega_\alpha e_\alpha$ as before. We say that $\varphi \in L^2(M, A^{p,q}(L))$ if $h_\alpha \in L^2(W_\alpha)$, and define

$$|\varphi|^2 = \sum_\alpha \chi_\alpha \int_{W_\alpha} |h_\alpha|^2 g_\alpha \Omega_\alpha, \quad g_\alpha = \|e_\alpha\|^2.$$  

$L^2(M, A^{p,q}(L))$ is a Hilbert space completed by $A^{p,q}(L)$ with the above norm.

For $s \geq 1$, $\varphi \in L^2(M, A^{p,q}(L))$, we say that $\varphi \in H_s(M, A^{p,q}(L))$ if $h_\alpha \in H_s(W_\alpha)$ and in this case, we define the **Sobolev norm**:

$$|\varphi|_s := \sum_{\alpha, |\beta| \leq s} \chi_\alpha \int_{W_\alpha} |D^\beta h_\alpha|^2 g_\alpha \Omega_\alpha.$$  

Then $H_s(M, A^{p,q}(L)) \subset L^2(M, A^{p,q}(L))$ is a Hilbert space completed by $A^{p,q}(L)$ with the Sobolev $|:\_s$-norm.
3.2 Three theorems

In order to prove Hodge theorem, it needs to prove the following three theorems.

**Theorem I.** (Garding inequality about $\Box$) There are constants $C_1, C_2 > 0$ such that

$$(\Box f, f) \geq C_1 |f|_1^2 - C_2 |f|_0^2, \quad \forall f \in A(L).$$

**Theorem II.** (Regularity of $\overline{\partial} + \partial^*$) If $f \in H_0(L)$, $g \in A(L)$, and $(\overline{\partial} + \partial^*) f = g$ weakly, then $f \in A(L)$.

**Theorem III.** If $\{f_j\}$ is a sequence of $A(L)$ that is bounded with respect to the norm $|\cdot|_1$, then there is a subsequence $\{f_{j_k}\}$ such that it is a Cauchy sequence with respect to the norm $|\cdot|_0$.

By assuming these three theorems, let us give some consequences and prove Hodge Theorem in this section.

**Lemma 3.5** (Regularity of $\Box$) If $f \in H_1(L)$, $g \in A(L)$ with $\Box f = g$ weakly, then $f \in A(L)$.

**Proof:** If $f \in H_1(L)$, then $\phi := (\overline{\partial} + \partial^*) f \in H_0(L)$ by Lemma 3.1. Then applying Theorem II to $\phi$, since $\Box f = (\overline{\partial} + \partial^*) \phi = g$ weakly, we conclude $\phi \in A(L)$, i.e., $(\overline{\partial} + \partial^*) f \in A(L)$. Applying Theorem II again, it follows $f \in A(L)$. \qed

**Lemma 3.6** Let $\mathcal{H}^\perp$ be the orthogonal complement of $\mathcal{H}(L)$ in $A(L)$ with respect to the inner product $\langle \cdot, \cdot \rangle$, i.e.,

$$\mathcal{H}^\perp = \{ f \in A(L) \mid \langle f, v \rangle = 0, \forall v \in A(L) \text{ with } \Box u = 0 \}.$$

Then there exists a constant $C_0$, such that

$$|f|^2_1 \leq C_0(\Box f, f), \quad \forall f \in \mathcal{H}^\perp.$$

**Proof:** Suppose that the inequality does not hold. Then there is a sequence $\{f_n\} \subset \mathcal{H}^\perp$ with $|f_n|_1 = 1$, but $\langle \Box f_n, f_n \rangle \to 0$ as $n \to \infty$. By Theorem III and the fact that $|f_n|_1 = 1$, there exists a subsequence $\{f_{n(j)}\}$ and $F \in H_0(L)$ such that

$$|f_{n(j)} - F| \to 0 \quad \text{as } n(j) \to \infty.$$
On the other hand, by $(\square f_n, f_n) \to 0$, it means $|(\overline{\partial} + \overline{\partial}^*) f_n| \to 0$. Then for any $\varphi \in A(L)$,

$$|\langle F, (\overline{\partial} + \overline{\partial}^*) \varphi \rangle| = \lim_{n(j) \to \infty} |\langle f_{n(j)}, (\overline{\partial} + \overline{\partial}^*) \varphi \rangle|$$

$$= \lim |\langle (\overline{\partial} + \overline{\partial}^*) f_{n(j)}, \varphi \rangle| \leq \lim |(\overline{\partial} + \overline{\partial}^*) f_{n(j)}| \cdot |\varphi| = 0,$$

i.e., $\langle F, (\overline{\partial} + \overline{\partial}^*) \varphi \rangle = 0, \forall \varphi \in A(L)$. In other words,

$$(\overline{\partial} + \overline{\partial}^*) F = 0, \text{ weakly.}$$

By Theorem II, $F \in A(L)$. Since $\square F = (\overline{\partial} + \overline{\partial}^*)^2 F = 0$, it implies $F \in \mathcal{H}(L)$. Also, $F \in \mathcal{H}^\perp$ holds because for any $\varphi \in \mathcal{H}(L)$, $\langle F, \varphi \rangle = \lim_{n \to \infty} \langle f_n, \varphi \rangle = 0$. Therefore, $F \in \mathcal{H}(L) \cap \mathcal{H}^\perp = \{0\}$ so that $F = 0$.

By applying Garding inequality in Theorem I,

$$|\langle (\overline{\partial} + \overline{\partial}^*) f_n \rangle|^2 \geq C_1 |f_n|^2 - C_2 |f_n|_0^2$$

for some constants $C_1 > 0$ and $C_2 > 0$. Recall that $|f_n|_1 = 1, \forall n$, that $|f_n|_0 \to |F|_0$ as $n \to \infty$, and that $|(\overline{\partial} + \overline{\partial}^*) f_n| \to 0$. The above inequality becomes

$$0 \geq C_1$$

but it is a contradiction. The proof of the lemma is complete. $\square$

**Lemma 3.7** $\square : \mathcal{H}^\perp \to \mathcal{H}^\perp$ is bijective.

**Proof:** **Step 1.** We show that $\square(\mathcal{H}^\perp) \subset \mathcal{H}^\perp$.

Let $\omega \in \mathcal{H}^\perp$, then

$$\langle \square \omega, \eta \rangle = \langle \omega, \square \eta \rangle = 0, \forall \eta \in \mathcal{H}(L).$$

This implies that $\square \omega \in \mathcal{H}^\perp$.

**Step 2.** We show that the map $\square$ is injective. In fact, if $\alpha, \beta \in \mathcal{H}^\perp$ with $\square(\alpha - \beta) = 0$, then $\alpha - \beta \in \mathcal{H}(L)$ and hence $\alpha - \beta \in \mathcal{H}(L) \cap \mathcal{H}^\perp$ so that $\alpha - \beta = 0$, i.e., $\alpha = \beta$.

**Step 3.** We prove that the map $\square$ is surjective: if $f \in \mathcal{H}^\perp$, there exists $u \in \mathcal{H}^\perp$ such that $\square u = f$.

Let $B_1$ be the closure of $\mathcal{H}^\perp$ in $H_1(L)$ with respect to the norm $|\cdot|_1$. By Lemma 3.5, it suffices to prove that if $u \in B_1$ such that $\square u = f$ weakly.
3.2. THREE THEOREMS

Let us define inner product in $\mathcal{H}^\perp$:

$$[\varphi, \psi] := (\Box \varphi, \psi), \quad \text{and} \quad ||\varphi||^2 := [\varphi, \varphi].$$

We claim that the norm $|| \cdot ||$ is equivalent to the norm $| \cdot |_1$. In fact, by the inequality in Lemma 3.6, we have

$$|\varphi|_1^2 \leq C_0 ||\varphi||^2, \quad \forall \varphi \in \mathcal{H}^\perp.$$ 

On the other hand, since $\overline{\partial} + \overline{\partial}^*$ is a differential operator of order 1, by the definition of $| \cdot |_1$, it implies that

$$|(\overline{\partial} + \overline{\partial}^*) \varphi|^2 \leq C'_0 |\varphi|_1^2$$

where $C'_0$ is a constant independent of $\varphi$. Hence

$$||\varphi||^2 = |(\overline{\partial} + \overline{\partial}^*) \varphi|^2 \leq C'_0 |\varphi|_1^2, \quad \forall \varphi \in \mathcal{H}^\perp.$$ 

Our claim is proved. Now the norm $|| \cdot ||$ is equivalent to the norm $| \cdot |_1$ so that $B_1$ is a Hilbert space with respect to the norm $[ \cdot , \cdot ]$.

Define a linear functional

$$\Phi : \mathcal{H}^\perp \rightarrow \mathbb{C}$$

$$\varphi \mapsto (\varphi, f)$$ 

Then

$$|\Phi(\varphi)| \leq |f||\varphi| \leq (a|f|)|\varphi|_1 \leq (a\sqrt{C_0}|f|)||\varphi||.$$ 

Here $a > 0$ is a constant such that $|\psi| \leq a|\psi|_0 \leq a|\psi|_1$, $\forall \psi \in A(L)$. Thus $\Phi$ is bounded with respect to the norm $|| \cdot ||$.

Then $\Phi$ uniquely extends to a linear functional

$$\Phi : B_1 \rightarrow \mathbb{C}.$$ 

which is bounded with respect to the norm $|| \cdot ||$. By Riesz representation theorem, there exists $u \in B_1$ such that

$$\Phi(\varphi) = [\varphi, u], \quad \forall \varphi \in B_1.$$ 

In particular, the above identity holds for all $\varphi \in \mathcal{H}^\perp$, i.e.,

$$(\varphi, f) = (\Box \varphi, u), \quad \forall \varphi \in \mathcal{H}^\perp.$$ 

For any $\varphi \in \mathcal{H}(L)$, since $f \in \mathcal{H}^\perp$, the left hand side of the above identity is zero; while the right hand side of the above identity also zero because $\Box \varphi = 0$. Then we have proved

$$(\varphi, f) = (\Box \varphi, u), \quad \forall \varphi \in A(L).$$
This is: $\Box u = f$ weakly. □

**Hodge Theorem**  
(a) The dimension of the space of all harmonic forms $\dim \mathcal{H}(L) < \infty$.  
(b) There is a compact operator $G : A(L) \rightarrow A(L)$ such that

1. $G|\mathcal{H}(L) = 0$.  
2. $G : \mathcal{H}^\perp \rightarrow \mathcal{H}^\perp$ is a linear isomorphism.  
3. $G\overline{\Box} = \overline{\Box} G, G\overline{\Box} = \overline{\Box} G$.  
4. $A(L) = \mathcal{H}(L) \oplus G\Box A(L)$.  

Here $G$ is called the Green operator.

**Proof:**  
(a) Suppose $\mathcal{H}(L)$ is infinite dimensional. Then we can find an infinite orthonormal basis $\{\omega_1, \omega_2, \ldots\}$ with respect to the norm $| \cdot |_0$. By Theorem I, Garding inequality, we have

$$|\omega_j|_1^2 \leq C_1^{-1} \left( (\Box \omega_j, \omega_j) + C_2 |\omega_j|_0^2 \right) = C_1^{-1} C_2.$$  

Thus $\{\omega_j\}$ is a bounded set with respect to the norm $| \cdot |_1$. By Theorem III, $\{\omega_j\}$ has a Cauchy subsequence $\{\omega_{j(k)}\}$ with respect to the norm $| \cdot |_0$. But this is impossible because

$$|\omega_{j(k)} - \omega_{j(l)}|_0^2 = 2, \quad \forall j(k) \neq j(l).$$

We have proved that $\dim \mathcal{H}(L) < \infty$.

(b) By Lemma 3.7, $\Box : \mathcal{H}^\perp \rightarrow \mathcal{H}^\perp$ is bijective. Now we define a linear operator $G : A(L) \rightarrow A(L)$ as follows:

$$G|\mathcal{H}(L) := 0$$

$$G|\mathcal{H}^\perp := (\Box|\mathcal{H}^\perp)^{-1}.$$  

Now we prove: $A(L) = \mathcal{H}(L) \oplus G\Box A(L)$. Let $H_{\text{proj}} : A(L) \rightarrow \mathcal{H}(L)$ be the projection operator. Then for any $f \in A(L)$,

$$f - H_{\text{proj}} f \in \mathcal{H}^\perp.$$  

Then

$$f - H_{\text{proj}} f = G\Box (f - H_{\text{proj}} f) \quad (\text{because } G|\mathcal{H}^\perp := (\Box|\mathcal{H}^\perp)^{-1})$$

$$= G\Box f - 0 \quad (\text{because } H_{\text{proj}} f \text{ is harmonic form})$$

$$= G\Box f.$$
$G \partial f = \partial G f,$ $\forall f \in A(L).$

Since $A(L) = \mathcal{H}(L) \oplus \mathcal{H}^\perp$ and $f = f_1 + f_2$ where $f_1 \in \mathcal{H}(L)$ and $f_2 \in \mathcal{H}^\perp$, we can prove the above equality by separating two cases: $f = f_1$ and $f = f_2$.

For any $f_1 \in \mathcal{H}(L)$, we have

$$G \partial f_1 = \partial G f_1 = 0$$

because $f_1 \in \mathcal{H}(L)$ iff $\partial f_1 = 0$ and $\bar{\partial} f_1 = 0$, and the restriction of $G$ on $\mathcal{H}(L)$ is zero.

For any $f_2 \in \mathcal{H}^\perp$,

$$G \bar{\partial} f_2 = \bar{\partial} G f_2$$

because $f_2 \in \mathcal{H}^\perp$ iff $\partial f_2 = 0$ and $\bar{\partial} f_2 = 0$, and the restriction of $G$ on $\mathcal{H}^\perp$ is zero.

3.3 Garding Inequality

Now we prove:

**Theorem I.** (Garding inequality about $\square$) There are constants $C_1, C_2 > 0$ such that

$$(\square f, f) \geq C_1 |f|_1^2 - C_2 |f|_0^2, \quad \forall f \in A(L).$$

**Proof:** Step 1. It suffices to estimate the lower bound of $|\bar{\partial} f|_0^2$. Let us prove it for $f \in A^{0,1}(L)$; other cases are proved similarly.
Let \( \{W_\alpha\} \) be locally finite covering of \( M \). Since \( M \) is compact, we assume that \( \{W_\alpha\} \) is a finite set. Namely, \( M = W_1 \cup ... \cup W_k, \ z \in \mathbb{Z^+} \). Let \( \{\eta_\alpha\} \) be a partion of \( C^\infty \) unity subordinate to \( \{W_\alpha\} \). Denote
\[
Z_\alpha := \text{supp}(\eta_\alpha) \subseteq W_\alpha.
\]

On each \( W_\alpha \), \( f \) can be written as
\[
f = f_\alpha dz_\alpha e_\alpha
\]
where \( f_\alpha \in C^\infty(W_\alpha) \) and \( e_\alpha(x) = \psi_\alpha^{-1}(x, 1) \in \Gamma(W_\alpha, L) \) as before. Since \( f \) is \((0,1)\)-form, \( \overline{\partial} f = 0 \) so that
\[
(\Box f, f) = ((\overline{\partial}\overline{\partial} + \overline{\partial}\partial)f, f) = (\overline{\partial}\overline{\partial}^* f, f) = (\overline{\partial}^* f, \overline{\partial}^* f) = |\overline{\partial}^* f|^2.
\]
Recall that there exist constants \( a_0, a_1 > 0 \) such that
\[
a_0|h|^2_0 \leq |h|^2 \leq a_1|h|^2_0, \quad \forall h \in A(L).
\]

To prove Garding inequality, it suffices to show:
\[
|\overline{\partial}^* f|^2_0 \geq C_1|f|^2_1 - C_2|f|^2_0, \quad \forall f \in A(L).
\]

**Step 2. Calculation of \(|\overline{\partial}^* f|^2_0\)** We claim
\[
\overline{\partial}^* f = -\frac{2}{r_\alpha} \left( \frac{\partial f_\alpha}{\partial z_\alpha} + f_\alpha \frac{\partial \log g_\alpha}{\partial z_\alpha} \right) e_\alpha, \quad \text{on } W_\alpha.
\]
where \( g_\alpha = \langle e_\alpha, e_\alpha \rangle \), the Hermitian metric on \( M \) is \( r_\alpha dz_\alpha d\overline{z}_\alpha \), and \( De_\alpha = \theta_\alpha e_\alpha \) on \( W_\alpha \). In fact, we calculate on \( W_\alpha \)
\[
\overline{\partial}^* f = -* D' * (f_\alpha dz_\alpha e_\alpha) \\
= -i * D'(f_\alpha dz_\alpha e_\alpha) \quad \text{(because } * d\overline{z}_\alpha = i d\overline{z}_\alpha \text{)} \\
= -i * (\partial f_\alpha \wedge d\overline{z}_\alpha + f_\alpha \theta_\alpha \wedge d\overline{z}_\alpha) e_\alpha \\
= -i * \left( \frac{\partial f_\alpha}{\partial z_\alpha} + f_\alpha \frac{\partial \log g_\alpha}{\partial z_\alpha} \right) dz_\alpha \wedge d\overline{z}_\alpha e_\alpha \\
= -\frac{2}{r_\alpha} \left( \frac{\partial f_\alpha}{\partial z_\alpha} + f_\alpha \frac{\partial \log g_\alpha}{\partial z_\alpha} \right) e_\alpha.
\]
By the definition of \(| \cdot \cdot |_0\),
\[
|\overline{\partial}^* f|^2_0 = \sum_\alpha \int_{W_\alpha} \left( \frac{2}{r_\alpha} \right)_0^2 \eta_\alpha^2 \left| \frac{\partial f_\alpha}{\partial z_\alpha} + f_\alpha \frac{\partial \log g_\alpha}{\partial z_\alpha} \right|^2 dx_\alpha dy_\alpha.
\]
Step 3. An elementary inequality  
Let \( a, b \in \mathbb{C} \). Then
\[
|a + b|^2 \geq (|a| - |b|)^2 = |a|^2 + |b|^2 - 2|a||b|.
\]
For any \( \epsilon > 0 \),
\[
0 \leq \left( \epsilon |a| - \frac{1}{\epsilon} |b| \right)^2 = \epsilon^2 |a|^2 + \frac{1}{\epsilon^2} |b|^2 - 2|a||b|.
\]
Then
\[
|a + b|^2 \geq (1 - \epsilon^2)|a|^2 - \left( \frac{1}{\epsilon^2} - 1 \right)|b|^2.
\]

Step 4. Lower bound of \( |\bar{\nabla} f|^2 \)  
Let \( a = \eta_\alpha \frac{\partial f_\alpha}{\partial z_\alpha}, b = \eta_\alpha f_\alpha \frac{\partial \log g_\alpha}{\partial z_\alpha} \) and \( \epsilon = \frac{1}{\sqrt{2}} \), we use Step 3 to get
\[
|\bar{\nabla} f|^2 = \sum_\alpha \int_{W_\alpha} \left( \frac{2}{r_\alpha} \right)^2 \eta_\alpha \frac{\partial f_\alpha}{\partial z_\alpha} + \eta_\alpha f_\alpha \frac{\partial \log g_\alpha}{\partial z_\alpha} \right|^2 \ dx_\alpha dy_\alpha
\]
\[
\geq \sum_\alpha \int_{W_\alpha} \frac{2}{r_\alpha^2} \left( \eta_\alpha \frac{\partial f_\alpha}{\partial z_\alpha} \right)^2 dx_\alpha dy_\alpha - \sum_\alpha \int_{W_\alpha} \left( \frac{2}{r_\alpha} \right)^2 \left( \frac{\partial \log g_\alpha}{\partial z_\alpha} \right)^2 \ dy_\alpha
\]
\[
\geq A_1 \sum_\alpha \int_{W_\alpha} \left( \eta_\alpha \frac{\partial f_\alpha}{\partial z_\alpha} \right)^2 dx_\alpha dy_\alpha - A_2 \sum_\alpha \int_{W_\alpha} |\eta_\alpha f_\alpha|^2 dx_\alpha dy_\alpha
\]
\[
= A_1 \sum_\alpha \int_{W_\alpha} \left( \frac{\partial (\eta_\alpha f_\alpha)}{\partial z_\alpha} - f_\alpha \frac{\partial \eta_\alpha}{\partial z_\alpha} \right)^2 dx_\alpha dy_\alpha - A_2 |f|^2_0
\]
where
\[
A_1 = \min_{1 \leq \alpha \leq k} \left( \min_{z_\alpha} \frac{2}{r_\alpha^2} \right), \quad A_2 = \max_{1 \leq \alpha \leq k} \left( \max_{z_\alpha} \frac{2}{r_\alpha} \left( \frac{\partial \log g_\alpha}{\partial z_\alpha} \right)^2 \right)
\]
are constants independent of \( f \).
Again we set
\[
a = \frac{\partial (\eta_\alpha f_\alpha)}{\partial z_\alpha}, \quad b = -f_\alpha \frac{\partial \eta_\alpha}{\partial z_\alpha}, \quad \epsilon = \frac{1}{\sqrt{2}}.
\]
By Step 3,
\[
|\bar{\nabla} f|^2 \geq A_1 \sum_\alpha \int_{W_\alpha} \left( \frac{\partial (\eta_\alpha f_\alpha)}{\partial z_\alpha} \right)^2 dx_\alpha dy_\alpha - A_2 \sum_\alpha \int_{W_\alpha} |\eta_\alpha|^2 dx_\alpha dy_\alpha - A_2 |f|^2_0.
\]
Let
\[ A_3 := \max_{\alpha=1,2,\ldots,k} \left\{ \max_{\partial \eta} \left| \frac{\partial \eta}{\partial z} \right|^2 \right\}, \]
which is a positive constant independent of \( f \). Then
\[
- \sum_{\alpha} \int_{W_\alpha} \left| \frac{\partial (\eta \alpha)}{\partial z} \right|^2 |f_\alpha|^2 dx_\alpha dy_\alpha \geq -A_3 \sum_{\alpha} \int_{Z_\alpha} |f_\alpha|^2 dx_\alpha dy_\alpha.
\]
Recall \( *dz_\alpha = -idz_\alpha \). We have
\[
|f|^2 = \int_M \langle e_\alpha, e_\alpha \rangle |f_\alpha|^2 (dz_\alpha \wedge *dz_\alpha) \geq \int_{Z_\alpha} \langle e_\alpha, e_\alpha \rangle |f_\alpha|^2 (dz_\alpha \wedge *dz_\alpha)
\]
\[= \int_{Z_\alpha} 2g_\alpha |f_\alpha|^2 dx_\alpha dy_\alpha \geq A_4 \int_{Z_\alpha} |f_\alpha|^2 dx_\alpha dy_\alpha
\]
where
\[ A_4 := \min_{\alpha=1,2,\ldots,k} \left\{ \min_{Z_\alpha} 2g_\alpha \right\}. \]
Then by (3.1), we obtain
\[- \int_{Z_\alpha} |f_\alpha|^2 dx_\alpha dy_\alpha \geq -\frac{a_0}{A_4} |f|^2_0. \]
Since the number \( \# \{ W_\alpha \} = k \), from above, we have
\[
|\overline{\partial} f|^2 \geq \frac{A_1}{2} \sum_{\alpha} \int_{W_\alpha} \left| \frac{\partial (\eta \alpha f_\alpha)}{\partial z} \right|^2 dx_\alpha dy_\alpha - \left( \frac{a_0 A_1 A_3 k}{A_4} + A_2 \right) |f|^2_0.
\]
By \( (\square f, f) = |\overline{\partial} f|^2 \) in Step 1, we get
\[
(\square f, f) \geq A_5 \sum_{\alpha} \int_{W_\alpha} \left| \frac{\partial (\eta \alpha f_\alpha)}{\partial z} \right|^2 dx_\alpha dy_\alpha - A_6 |f|^2_0,
\]
where \( A_5 \) and \( A_6 \) are positive constant that are independent of \( f \).

Now for any \( \varphi \in C_0^\infty (W_\alpha) \), we use integral by parts to obtain
\[
\int_{W_\alpha} \left| \frac{\partial \varphi}{\partial z} \right|^2 dx_\alpha dy_\alpha = \int_{W_\alpha} \frac{\partial \varphi}{\partial z} \cdot \overline{\frac{\partial \varphi}{\partial z}} dx_\alpha dy_\alpha = -\int_{W_\alpha} \frac{\partial^2 \varphi}{\partial z \partial \overline{z}} \cdot \overline{\varphi} dx_\alpha dy_\alpha
\]
\[= -\int_{W_\alpha} \frac{\partial^2 \varphi}{\partial z \partial \overline{z}} \cdot \overline{\varphi} dx_\alpha dy_\alpha = \int_{W_\alpha} \frac{\partial \varphi}{\partial z} \cdot \overline{\frac{\partial \varphi}{\partial z}} dx_\alpha dy_\alpha = \int_{W_\alpha} \left| \frac{\partial \varphi}{\partial z} \right|^2 dx_\alpha dy_\alpha.
\]
Notice that in (3.2), $\eta f$ not only has compact support, but also
\[
\sum_\alpha \int_{W_\alpha} \left| \frac{\partial (\eta f)}{\partial z_\alpha} \right|^2 dx_\alpha dy_\alpha = \frac{1}{2} \sum_\alpha \int_{W_\alpha} \left\{ \left| \frac{\partial (\eta f)}{\partial z_\alpha} \right|^2 + \left| \frac{\partial (\eta f)}{\partial z_\alpha} \right|^2 \right\} dx_\alpha dy_\alpha = \frac{1}{2} (|f|^2 - |f|_0^2).
\]
Combining (3.2),
\[
(\Box f, f) \geq \frac{A_5}{2} |f|^2 - (A_6 + \frac{A_5}{2}) |f|_0^2. \qquad \Box
\]

## 3.4 Relich Lemma and the proof of Theorem III

The reason why Sobolev spaces are useful because they are all Hilbert spaces so that we can treat them by Hilbert space theory. Relich lemma is an important example.

Recall that $H_s(\Omega)$ is a Hilbert space which is the completion of $A_s(\Omega)$ with respect to the norm $|\cdot|_s$:
\[
|f|_s^2 = \sum_{|\alpha| \leq s} \int_\Omega |D^\alpha f|^2 dx.
\]
We can define $H^\circ_s(\Omega)$ to be the Hilbert space that is completion of $A_s(\Omega) \cap C^\infty(\Omega)$ with respect to the norm $|\cdot|_s$. Then for integers $s, t$ with $0 \leq s < t$, we have inclusions:
\[
i : H_t(\Omega) \rightarrow H_s(\Omega), \quad i : H^\circ_t(\Omega) \rightarrow H^\circ_s(\Omega).
\]
Recall in the theory of functional analysis, a compact operator is a linear operator $L$ from a Banach space $X$ to another Banach space $Y$, such that the image under $L$ of any bounded subset of $X$ is a relatively compact subset of $Y$. Such an operator is necessarily a bounded operator, and so continuous. A bounded operator $L$ is compact if and only if for any sequence $\{x_n\}$ from the unit ball in $X$, the sequence $\{L(x_n)\}$ contains a Cauchy subsequence.

**Theorem 3.8 (Relich lemma)** Let $\Omega \subset \mathbb{R}^n$ be an open subset. Let $0 \leq s < t$. Then $i : H^\circ_t(\Omega) \rightarrow H^\circ_s(\Omega)$ is a compact operator.

The proof is skipped here, which can be found in many Real Analysis textbooks (cd. Real Analysis - modern techniques and their applications, G.B. Folland, p.305). Here we discuss the following example which is the model case for Relich lemma.

**[Example]** Let $C^1[a, b]$ be the space of all $C^1$-smooth functions defined on $[a, b]$. Define two norms
\[
\|f\|_0 := \max_{x \in [a, b]} |f(x)|,
\]
...
and
\[
\|f\|_1 := \max_{x \in [a,b]} |f(x)| + \max_{x \in [a,b]} |f'(x)|.
\]

Let \( S_0 \) and \( S_1 \) be the completions of \( C^1[a,b] \) with respect to the norms \( \| \cdot \|_0 \) and \( \| \cdot \|_1 \), respectively. Then \( S_1 \subset S_0 \subset C[a,b] \). We claim that the inclusion map
\[
i : S_1 \to S_0 \text{ is a compact operator.}
\]

To show: \( \forall \{f_n\} \subset S_1 \text{ with } \|f\|_1 \leq 1, \exists \text{ a subsequence } \{f_{n_j}\} \text{ such that the subsequence is } \] 

a Cauchy sequence with respect to the norm \( \| \cdot \|_0 \). By the Ascoli-Arzela theorem, it suffices to prove that \( \{f_n\} \) is uniformly bounded and equicontinuous. In fact, since \( \|f\|_0 \leq \|f\|_1 \leq 1 \).

We know that \( \{f_n\} \) is uniformly bounded. Also by the mean value theorem,
\[
|f_n(x) - f_n(y)| \leq |f'(z)||x - y| \leq |x - y|, \text{ for some } z \in [a,b]
\]
so that \( \{f_n\} \) is equicontinuous. Our claim is proved. \( \Box \)

**Proof of Theorem III:** If a sequence \( \{f_n\} \subset A(L) \) is bounded with respect to the norm \( |\cdot|_1 \), to show the existence of a Cauchy subsequence with respect to the norm \( |\cdot|_0 \).

Let \( \{W_\alpha\} \) be a finite open covering which is used to define the norm \( |\cdot|_s \). Let \( \{\eta_\alpha\} \) be a \( C^\infty \) partition of unity subordinate to \( \{W_\alpha\} \). Then for each \( \alpha \),
\[
|\eta_\alpha f_n| \leq E
\]
where \( E \) is a constant. Regarding \( \eta_\alpha f_n \) as an element of \( C^\infty(W_\alpha) \), by applying Rellich Lemma, there exists a subsequence \( \{\eta_\alpha f_{n_j}\} \) such that it is a Cauchy sequence in \( H_0(W_\alpha) \).

Since \( \{W_\alpha\} \) is finite, after a finite number of steps, we can assume that for each \( \alpha \), \( \{\eta\alpha f_{n_j}\} \) is a Cauchy sequence in \( H_0(W_\alpha) \). Then
\[
|f_i - f_j|_0 = \sum_\alpha \eta_\alpha |f_i - f_j|_0 \leq \sum_\alpha |\eta_\alpha f_i - \eta_\alpha f_j|_0 \to 0, \forall i, j \to \infty.
\]

Therefore \( \{f_{n_j}\} \) is a Cauchy sequence with respect to \( |\cdot|_0 \). The proof of Theorem III is complete. \( \Box \)

### 3.5 Sobolev Lemma

While we can study Sobolev spaces by theory of Hilbert spaces, the well-known Sobolev lemma provides a link between Sobolev space and the continuously differentiable spaces \( C^t(\Omega) \), which is the foundation for regularity theorems.
3.5. **SOBOLEV LEMMA**

**Theorem 3.9 (Sobolev lemma)** Let \( \Omega \subset \mathbb{R}^n \) be an open subset. For any integers \( t \in \mathbb{Z}^+ \) and \( s > \frac{n}{2} \), we have

\[ H_{s+t}(\Omega) \subset C^s(\Omega). \]

We first show

**Lemma 3.10 (Sobolev inequality)** Let \( s > \frac{n}{2} \) as above. Let \( K \subset \Omega \) be compact. Then there exists a constant \( C > 0 \) such that \( \forall f \in A_s(\Omega) \), we have

\[ \max_{x \in K} |f(x)| \leq C |f|_s. \]

**Proof of Sobolev inequality:** Since \( K \) is compact in \( \Omega \). There is a constant \( R > 0 \) such that \( \forall x \in K, \mathbb{B}(x, R) \subset \Omega \). Let \( \zeta(r) \in C^\infty([0, R]) \) such that \( \text{supp}(\zeta) \subset [0, R) \) and \( \zeta|_{[0,R/2]} \equiv 1 \). For any \( x \in K \), \( \zeta \) defines a function \( \zeta(|y-x|) \) of \( y \) in \( \mathbb{R}^n \). Denote it by \( \zeta \).

Let \( f \in A_s(\Omega) \). Recall \( A_s(\Omega) = \{ f \in C^\infty(\Omega) \mid |f|_s < \infty \} \). For any \( x \in K \), we need to estimate \( |f(x)| \). Assume \( x = 0 \); otherwise the proof is similar.

To estimate \( |f(0)| \). By the integral by parts, we have

\[ f(0) = -\int_0^R \frac{\partial(\zeta f)}{\partial r} dr = (-1)^{s+1} \int_0^R r^{s-1} \left( \frac{\partial^s(\zeta f)}{\partial r^s} \right) dr \]

On \( \mathbb{R}^n \), the volume element is \( dL = r^{n-1} dr d\sigma \), \( d\sigma \) is the area element on the unit Sphere. Denote by \( \sigma \) the volume of the unit sphere. Applying Schwarz inequality, we obtain

\[ |\sigma f(0)| = \left| \int_{\mathbb{B}(0,R)} r^{s-n} \left( \frac{\partial^s(\zeta f)}{\partial r^s} \right) r^{n-1} dr d\sigma \right| \]

\[ \leq \left( \int_{\mathbb{B}(0,R)} r^{2(s-n)} r^{n-1} dr d\sigma \right)^{1/2} \left( \int_{\mathbb{B}(0,R)} \left| \frac{\partial^s(\zeta f)}{\partial r^s} \right| dL \right)^{1/2} \]

\[ \leq C_1 |\zeta f|_s \leq C |f|_s, \]

where \( C > 0 \) is a constant that only depends on \( s \) and \( R \). Here we used the fact that \( r^{2(s-n)} \) is integrable on \( \mathbb{B}(0, R) \) because \( s > \frac{n}{2} \). The proof of Sobolev inequality is complete. \( \Box \)

**Proof of Sobolev Lemma:** If \( f \in H_{s+t}(\Omega) \) with \( s > \frac{n}{2} \) and \( t \geq 0 \), we want to show \( f \in C^t(\Omega) \) (i.e., \( f = g, \text{a.e. for some } g \in C^t(\Omega) \)).
Since it is a local problem, it is sufficient to prove that over any compact subset \( K \subset \Omega \), \( f \in C^t \).

Since \( f \in H_{s+t}(\Omega) \), we can take a sequence \( \{f_j\} \subset A_{s+t}(\Omega) \) with \( |f - f_j|_{s+t} \to 0 \) as \( j \to \infty \). By Sobolev inequality, for \( \forall \alpha \) with \( |\alpha| \leq t \) we have

\[
\max_{x \in K} |D^\alpha f_i(x) - D^\alpha f_j(x)| \leq C|D^\alpha f_i - D^\alpha f_j|_s
\]
\[
\leq C|f_i - f_j|_{s+t} \leq C|f_i - f|_{s+t} + C|f - f_j|_{s+t} \to 0, \quad \text{as } i, j \to \infty.
\]

Then there is a function \( g \in C^t(K) \) such that \( D^\alpha f_j \) uniformly converge to \( D^\alpha g \), \( \forall \alpha \) with \( |\alpha| \leq t \). Then \( f_j \to g \) on \( L^2(K) \) and hence \( f_j \to g \) so that \( f = g \) a.e. on \( K \).

\[ \square \]

**Corollary 3.11** Let \( t \in \mathbb{Z} \) with \( t \geq 0 \) and \( s > \frac{n}{2} \). Then

\[ H_{s+t}(\Omega, \text{loc}) \subset C^t(\Omega). \]

**Corollary 3.12**

\[ C^\infty(\Omega) = \cap_{s \geq 0} H_s(\Omega, \text{loc}). \]

Let us extend Sobolev space \( H_s(\Omega) \) into the case when \( s \) could be negative integers. Let \( \Omega \subset \mathbb{R}^n \) be any open subset, \( s \in \mathbb{Z} \) with \( s \geq 0 \). We define

\[ H_{-s}(\Omega) := \text{the dual space of the Hilbert space } H_s(\Omega), \]

i.e.,

\[ H_{-s}(\Omega) = \{ \varphi - \varphi : H_s(\Omega) \to \mathbb{C} \text{ the bounded linear functionals} \}, \]

As usual, we define the norm on \( H_{-s}(\Omega) \) to be

\[ |\varphi|_{-s} := \inf_f |\varphi(f)| \]

where \( f \in H_s(\Omega) \) with \( |f|_s = 1 \). Since \( H_0(\Omega) = L^2(\Omega) \), when \( s = 0 \), the definition of \( H_{-s}(\Omega) \) coincide with the one of \( H_s(\Omega) \).

Let \( 0 \leq s < t \) be integers. We can have a natural inclusion

\[ H_{-s}(\Omega) \to H_{-t}(\Omega). \]

In fact, if \( \varphi \in H_{-s}(\Omega) \), i.e., \( \varphi : H_s(\Omega) \to \mathbb{C} \) is a bounded linear functional. Since \( s < t \), it implies \( H_t(\Omega) \subset H_s(\Omega) \) so that we can restrict \( \varphi \) to be a linear functional \( \varphi : H_t(\Omega) \to \mathbb{C} \), which is also bounded because

\[ |\varphi(f)| \leq |\varphi|_{-s}|f|_s \leq |\varphi|_{-s}|f|_t, \quad \forall f \in H_t(\Omega). \]
So we have \( H_{-s}(\Omega) \subset H_{-t}(\Omega) \). Then
\[
\ldots \subset H_2(\Omega) \subset H_1(\Omega) \subset H_0(\Omega) \subset H_{-1}(\Omega) \subset H_{-2}(\Omega) \subset \ldots
\]
For any \( f \in H_t(\Omega) \) where \( t \) can be any integer (even negative), we say that the weak derivative \( D^\alpha f \) exists in \( H_u(\Omega) \) if \( \exists g \in H_u(\Omega) \) for some \( u \) such that
\[
(f, D^\alpha \zeta) = (-1)^{|\alpha|}(g, \zeta), \quad \forall \zeta \in C^\infty_0(\Omega).
\]
Here the inner product means: if \( h \in H_t(\Omega) \) and \( \psi \in C^\infty_0(\Omega) \), then
\[
(h, \psi) := \begin{cases} (h, \psi)_0, & \text{if } t \geq 0; \\ h(\overline{\psi}), & \text{if } t < 0. \end{cases}
\]
The next result shows: \( \forall t, \alpha \in \mathbb{Z} \) and \( \forall f = inH_t(\Omega) \), the weak derivative \( D^\alpha f \) always exists.

**Lemma 3.13** Let \( \alpha \) and \( t \in \mathbb{Z} \). \( D^\alpha : H_t(\Omega) \to H_{t-|\alpha|}(\Omega) \) is a bounded linear operator with the norm \( \|D^\alpha\| \leq 1 \).

**Lemma 3.14** (Schwarz inequality) \( \forall \varphi \in H_{-s}(\Omega) \) and \( f \in H_s(\Omega) \), then
\[
|\psi(f)| \leq |\psi|_s|f|_{-s}.
\]
Let \( L \) be a line bundle over a compact Riemann surface \( M \). Similarly, we can define Sobolev spaces \( H_{-s}(L) \):\[
H_{-s}(L) = \text{the dual space of the Hilbert space } H_s(L),
\]
and
\[
\ldots \subset H_2(L) \subset H_1(L) \subset H_0(L) \subset H_{-1}(L) \subset H_{-2}(L) \subset \ldots
\]
and for any \( \omega, \eta \in A(L) \),
\[
|\langle \omega, \eta \rangle_0| \leq |\omega|_s|\eta|_{-s}, \quad \forall s.
\]

### 3.6 Proof of Theorem II

Now we prove the following three results which are special cases of Theorem II but are enough to prove the Riemann-Roch Theorem. In particular Corollary 3.17 was used to prove the Riemann-Roch theorem. We shall prove them but skip the proof of Theorem II.
Lemma 3.15 Let $h \in H_0(L)$ with $\bar{\partial}h \in A(L)$ and $\bar{\partial}^*h \in A(L)$. Then $h \in A(L)$.

Corollary 3.16 Let $h \in H_0^{p,q}(L)$ with $(\bar{\partial} + \bar{\partial}^*)h \in A(L)$. Then $h \in A(L)$.

Proof: Since $h \in H_0^{p,q}(L)$, we have $\bar{\partial}h \in H_0^{p,q+1}(L)$ and $\bar{\partial}^*h \in H_0^{p,q-1}(L)$. Then $(\bar{\partial} + \bar{\partial}^*)h \in A(L) \iff \bar{\partial}h \in A(L)$ and $\bar{\partial}^*h \in A(L)$. By applying Lemma 3.15, $h \in A(L)$. □

Corollary 3.17 Let $f \in H_1(L)$ and $g \in A(L)$ with $\Box f = g$ weakly. Then $f \in A(L)$.

Proof: If $f \in H_1(L)$ and $\Box f \in A(L)$, to show $f \in A(L)$. Denote by $f'$ the $(p, q)$ component of $f$, since $\Box$ preserves the form types, $\Box f'$ is of $(p, q)$ type. Therefore, $\Box f \in A(L)$ implies $\Box f' \in A(L)$.

Let $\psi := (\bar{\partial} + \bar{\partial}^*)f'$. Since $\dim \mathbb{C} M = 1$, by consideration of types,

$$\psi = \begin{cases} \bar{\partial}f', & \text{if } q = 0, \\ \bar{\partial}^*f', & \text{if } q = 1. \end{cases}$$

so that $\psi \in H_0^{p,q+1}(L)$ or $H_0^{p,q-1}(L)$, depending on $q = 0$ or 1. By the hypothesis,

$$(\bar{\partial} + \bar{\partial}^*)\psi = \Box f' \in A(L).$$

Then by applying Corollary 3.16, we conclude $\psi \in A(L)$. Again, since

$$(\bar{\partial} + \bar{\partial}^*)f' = \psi \in A(L),$$

by applying Corollary 3.16, we conclude $f' \in A(L)$. Hence $f \in A(L)$. □

Proof of Lemma 3.15: Step 1. First reduction - local problem Take a finite over covering $\{W_\alpha\}$ of $M$. Take any one $W_\alpha$ and a compact subset $W \subset W_\alpha$. It suffices to prove that $h|_W$ is a $C^\infty$-form. By considering componets, we assume that $h$ is of $(p, q)$-type. On $W_\alpha$, we write

$$h = h_\alpha \varphi_\alpha e_\alpha \quad \text{on } W_\alpha$$

where $e_\alpha = \psi_\alpha^{-1}(\cdot, 1) \in \Gamma(W_\alpha, L)$ is a basis of $L$, $\varphi_\alpha$ was defined in p.71:

$$\varphi_\alpha = \begin{cases} 1, & \text{if } (p, q) = (0, 0), \\ dz_\alpha, & \text{if } (p, q) = (1, 0), \\ d\bar{z}_\alpha, & \text{if } (p, q) = (0, 1), \\ \Omega, & \text{if } (p, q) = (1, 1). \end{cases}$$
and $h_\alpha$ is a function. We identify $W$ with an open subset of $\mathbb{C}$. So $h \in H_0(L)$ implies $h_\alpha \in L^2(W)$. We also identify $A(W, L)$ with $C^\infty(W)$. We need to prove a local problem:

$$h \in C^\infty(W).$$

**Step 2. Calculation of $\overline{\partial}$ and $\overline{\partial}^*$** By the definition of $\overline{\partial}$, we have

$$\overline{\partial} h = \frac{\partial h_\alpha}{\partial z_\alpha} (d\bar{z}_\alpha \wedge \varphi_\alpha)e_\alpha.$$

Recall the formulas of $\overline{\partial}^* h$ in several cases:

When $(p, q) = (0, 1)$,

$$\overline{\partial}^* h = -\frac{2}{\tau_\alpha} \left( \frac{\partial h_\alpha}{\partial z_\alpha} + \frac{\partial \log g_\alpha}{\partial z_\alpha} h_\alpha \right) e_\alpha;$$

When $(p, q) = (1, 1)$,

$$\overline{\partial}^* h = i \left( \frac{\partial h_\alpha}{\partial z_\alpha} + \frac{\partial \log g_\alpha}{\partial z_\alpha} h_\alpha \right) dz_\alpha e_\alpha;$$

When $(p, q) = (0, 0)$ or $(1, 0)$,

$$\overline{\partial}^* h = 0.$$

In general, we write

$$\overline{\partial}^* h = \tau(\rho h) \quad (3.3)$$

where $\tau$ is a $C^\infty$ function that never vanish and

$$\rho h := \left( \frac{\partial h_\alpha}{\partial z_\alpha} + \sigma_\alpha^* h_\alpha \right) \psi_\alpha e_\alpha;$$

where $\sigma_\alpha$ is a $C^\infty$ function, $\psi_\alpha = 1$, or 0 or $dz_\alpha$.

**Step 3. Second reduction - applying Sobolev lemma** In order to prove $h \in C^\infty(W)$, by the corollary of Sobolev lemma, it is sufficient to prove

$$h \in H_s(W, \text{loc}), \forall s \in \mathbb{Z} \text{ with } s \geq 0. \quad (3.4)$$

**Step 4. Third reduction - for a fixed $s$** We claim that to prove (3.4), it is sufficient to prove that

if $h \in H_s(W, \text{loc})$ and $\overline{\partial} h \in H_s(W, \text{loc})$ and $\rho h \in H_s(W, \text{loc}) \implies h \in H_{s+1}(W, \text{loc}). \quad (3.5)$
In fact, by the hypothesis,
\[ \partial h \in C^\infty(W), \quad \text{and} \quad \rho h = \frac{1}{\tau}(\vartheta h) \in C^\infty(W) \]
Then
\[ \overline{\partial} h, \rho h \in H_s(W, \text{loc}), \quad \forall s \geq 0, s \in \mathbb{Z}. \]
If (3.5) holds, from \( h \in H_s(W, \text{loc}) \), then
\[ h \in H_{s+1}(W, \text{loc}). \]
This proves our Claim.

**Step 5. Fourth reduction - \( s = 0 \) with compact a support condition**

Let us prove the case of \( s = 0 \); other cases can be proved similarly. Namely, to show:
\[ \text{if } h \in H_0(W, \text{loc}) \text{ and } \overline{\partial} h \in H_0(W, \text{loc}) \text{ and } \rho h \in H_0(W, \text{loc}) \implies h \in H_1(W, \text{loc}). \quad (3.6) \]
We claim that to prove (3.6), it suffices to prove:
\[ \text{if } f, \overline{\partial} f, \rho f \in H_0(W, \text{loc}) \text{ with supp}(f) \subseteq W \implies f \in H_1(W, \text{loc}). \quad (3.7) \]
In fact, when \( s = 0 \), for any \( h \) as in (3.6), we define \( f := \zeta h \) where \( \zeta \in C_0^\infty(W) \). It is enough to prove \( f \in H_1(W) \). Now, \( f \in H_0(W) \) with supp\((f) \subseteq W\),
\[ \overline{\partial} f = \overline{\partial}(\zeta h) = \zeta(\overline{\partial} h) + \frac{\partial \zeta}{\partial z^\alpha} h_\alpha(d\overline{z^\alpha} \wedge \varphi_\alpha) e_\alpha \in H_0(W), \]
and
\[ \rho f = \rho(\zeta h) = \zeta(\rho h) + \frac{\partial \zeta}{\partial z^\alpha} h_\alpha \psi_\alpha e_\alpha \in H_1(W) \subseteq H_0(W). \]
Suppose (3.7) holds, it implies \( f \in H_1(W) \). By choosing different \( \zeta \), it implies \( h \in H_1(W, \text{loc}) \). Our claim is proved.

**Step 6. Proof of (3.7)**
Now we prove (3.7). Since \( \text{supp}(f) \subseteq W \), \( f \ast \chi_\epsilon \in C_0^\infty(W) \). We want to prove that there is a Cauchy sequence chosen from \( \{ f \ast \chi_\epsilon \} \) in \( H_1(W) \), i.e.,
\[ | f \ast \chi_\epsilon - f \ast \chi_\delta |_1 \to 0, \quad \text{as } \epsilon, \delta \to 0. \quad (3.8) \]
In fact, suppose (3.8) is true. We denote \( \tilde{f} \in H_1(W) \) be the limit of \( \{ f \ast \chi_\epsilon \} \). Recall \( f \in H_0(W) \) and hence \( f \ast \chi_\epsilon \to f \). Also notice \( |\tilde{f} - f \ast \chi_\epsilon|_0 \leq |\tilde{f} - f \ast \chi_\epsilon|_1 \to 0 \). By the uniqueness of limit, we must have \( f = \tilde{f} \in H_1(W) \).
3.6. PROOF OF THEOREM II

It remains to prove (3.8).
Recall Garding inequality:

\[ |\varphi|^2 \leq C(|\varphi|^2 + |\bar{\partial}\varphi|^2 + |\overline{\partial}^* \varphi|^2), \quad \forall \varphi \in A(L). \]

If, in addition, supp(\varphi) \subseteq W, then we have

\[ |\overline{\partial}^* \varphi|^2 \leq C^2_1|\rho\varphi|^2 \]

where \( C_1 := \max_W |\tau| \) because \( \overline{\partial} \varphi = \tau(\rho\varphi) \) from (3.3) in Step 2. In other words, the following inequality should hold:

\[ |\varphi|^2 \leq C(|\varphi|^2 + |\bar{\partial}\varphi|^2 + |\rho\varphi|^2), \quad \forall \varphi \in A(L) \text{ with supp}(\varphi) \subseteq W. \tag{3.9} \]

To prove (3.8), we let

\[ \varphi := f * \chi_\epsilon - f * \chi_\delta. \]

Then supp(\varphi) \subseteq W holds. Therefore, to prove (3.8), by (3.9), it suffices to prove

\[ |\varphi|^2 + |\bar{\partial}\varphi|^2 + |\rho\varphi|^2 \to 0, \quad \text{as } \epsilon, \delta \to 0. \tag{3.10} \]

We’ll consider three terms above separately.

Consider the first term \(|\varphi|^2\). Since \( f \in H_0(W) \), \( f * \chi_\epsilon \to f \) so that

\[ |\varphi|_0 = |f * \chi_\epsilon - f * \chi_\delta|_0 \to |f - f|_0 = 0. \]

Consider the second term \(|\bar{\partial}\varphi|^2\). Since \( \text{supp}(f) \subseteq W \), \( \text{supp}(\overline{\partial}f) \subseteq \text{supp}(f) \subseteq W \). Then \( \overline{\partial}f \in H_0(W) \) and hence \( (\overline{\partial}f) * \chi_\epsilon \in C_0^\infty(W) \). We identify \( f = f_\alpha \varphi_\alpha e_\alpha \) with \( f_\alpha \). Then

\[ |\bar{\partial}\varphi|_0 = |(\overline{\partial}f) * \chi_\epsilon - (\overline{\partial}f) * \chi_\delta|_0 \to 0, \quad \text{as } \epsilon, \delta \to 0. \]

Consider the third term \(|\rho\varphi|^2\). Since \( \text{supp}(\overline{\partial}f) \subseteq \text{supp}(f) \subseteq W \), we see \( \rho f \in H_0(W) \) and hence \( (\rho f) * \chi_\epsilon \in C_0^\infty(W) \). We identify \( f = f_\alpha \varphi_\alpha e_\alpha \) with \( f_\alpha \). Then

\[ |\rho\varphi|_0 = |(\rho f) * \chi_\epsilon - (\rho f) * \chi_\delta|_0 \to 0, \quad \text{as } \epsilon, \delta \to 0. \]

Then (3.10) is proved. \( \square \)
Chapter 4

Positive Closed Currents Theory

4.1 Plurisubharmonic functions

When $n = 1$, for a $C^2$ smooth function $u$ defined an open subset $\Omega \subset \mathbb{C}$, we recall

\[ u \text{ is harmonic on } \Omega \iff \Delta u = 0 \iff \text{locally } u = \text{Re}(f) \text{ for some } f \in \mathcal{O} \]

\[ \forall a \in \omega, \Delta(a, |\zeta|) \in \Omega, \text{ s. t. } \iff dd^c u = 0 \]

\[ u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + \zeta e^{i\theta})d\theta \]

where

\[ d = \partial + \bar{\partial}, \quad d^c = \frac{1}{2\pi i}(\partial - \bar{\partial}). \]

And $u$ is a subharmonic function on $\Omega$ if and only if $u : \Omega \rightarrow [-\infty, +\infty)$ is semicontinuous such that

\[ u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + \zeta e^{i\theta})d\theta. \]

As examples, for any local holomorphic function $f$, $\log |f|$ is subharmonic.

When $n \geq 1$, for any $C^2$ smooth function $u$ defined on an open subset $\Omega \subset \mathbb{C}^n$, we define

\[ u \text{ is harmonic on } \Omega \iff u \in H(\Omega) \iff \Delta u = 0, \]

\[ u \text{ is subharmonic on } \Omega \iff u \in SH(\Omega) \iff \Delta u \geq 0, \]

\[ u \text{ is pluriharmonic on } \Omega \iff u \in PH(\Omega) \iff dd^c u = 0. \]
CHAPTER 4. POSITIVE CLOSED CURRENTS THEORY

u is plurisubharmonic on $\Omega \iff u \in PSH(\Omega) \iff dd^c u \geq 0$.

We have

$$PH(\Omega) \subsetneq H(\Omega) \cap \not\subseteq PSH(\Omega) \subsetneq SH(\Omega) \subset L^1_{loc}(\Omega)$$

**Definition** Let $u : \Omega \subset \mathbb{R}^m \to \mathbb{R}$ be continuous. $u$ is called harmonic if $\forall B(a,r) \in \Omega$, we have

$$u(a) = A(u,a,r) := \frac{1}{\lambda(B(0,1))} \int_{B(a,r)} u(x) d\lambda(x)$$

where $\lambda$ is the Lebesgue measure on $\mathbb{R}^m$. Here we denote $u \in H(\Omega)$.

**Remarks**

1. $C^2$ smoothness condition is not required to define harmonic functions.
2. It is well known that if $u$ is $C^2$ smooth, then $u$ is harmonic if and only if

$$\Delta u = \sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} = 0 \quad \text{on } \Omega.$$

**Definition** Let $u : \Omega \subset \mathbb{R}^m \to [-\infty, \infty)$ be upper semicontinuous. $u$ is called subharmonic if $u \not\equiv -\infty$ on each connected component of $\Omega$, and $\forall B(a,r) \in \Omega$, we have

$$u(a) \leq A(u,a,r) := \frac{1}{\lambda(B(0,1))} \int_{B(a,r)} u(x) d\lambda(x)$$

where $\lambda$ is the Lebesgue measure on $\mathbb{R}^m$. Here we denote $u \in SH(\Omega)$.

**Remarks**

1. $C^2$ smoothness condition is not required to define subharmonic functions.
2. It is well known that if $u$ is $C^2$ smooth, then $u$ is subharmonic if and only if

$$\Delta u = \sum_{j=1}^{m} \frac{\partial^2 u}{\partial x_j^2} \geq 0 \quad \text{on } \Omega.$$
3. For any holomorphic functions $f_1, \ldots, f_m \in \mathcal{O}(\Omega)$ where $\Omega \subset \mathbb{C}^n$, $u = \log(|f_1|^2 + \ldots + |f_m|^2) \in SH(\Omega)$. Notice that $u$ may have singularities.

**Proposition 4.1** Let $u : \Omega \subset \mathbb{R}^m \to [-\infty, 0)$ be a upper semincontinuous and $u \not\equiv -\infty$ on each connected component of $\Omega$. Then

(i) $u \in SH(\Omega) \iff \forall B(a, r + s) \subset \Omega$, 
$$u(x) \leq \frac{r^m}{(r + s)^m} A(u, a, r), \quad \forall x \in B(a, s).$$

(ii) $SH(\Omega) \subset L^1_{loc}(\Omega)$.

(iii) If $u \in SH(\Omega)$, then $E = \{x \in \Omega \mid u(x) = -\infty\}$ has measure zero.

**Remark** The range of $u$ is $[-\infty, 0]$, not $[-\infty, \infty]$.

**Proof** (i) ($\leftarrow$) Take $s = 0$ and $x = a$ and by the definition.

($\Rightarrow$) Since $u \in SH(\Omega)$, we have
$$u(x) \leq A(u, x, r + s). \quad (4.1)$$

Consider
$$(r + s)^m A(u, x, r + s) = \frac{1}{\lambda(B(0, 1))} \int_{B(x, r + s)} u(x) d\lambda(x)$$
$$\leq \frac{1}{\lambda(B(0, 1))} \int_{B(a, r)} u(x) d\lambda(x) \quad \text{(because } B(a, r) \subset B(x, r + s))$$
$$= r^m A(u, a, r).$$

Substituting this into (4.1), we obtain
$$u(x) \leq A(u, x, r + s) \leq \frac{r^m}{(r + s)^m} A(u, a, r).$$

(ii) Since $u \not\equiv -\infty$ on each connected component $U$, for any point $x_0 \in U$, $\exists$ some point $x' \approx x_0$, $x' \in U$ such that
$$-\infty \neq u(x') < \text{constant } A(u, x_0, r) = \text{constant } \int_{B(x_0, r)} u(x) d\lambda(x)$$
by applying (i). Then
\[-\infty < \int_{B(x_0,r)} u(x)d\lambda(x) \leq 0 < \infty.\]
Here the second inequality holds because \(u \leq 0\). It implies \(u \in L^1_{loc}(\Omega)\).

(iii) To show: \(\lambda(E) = 0\).
Suppose \(\lambda(E) > 0\).
\[
\Rightarrow \exists B(a,r+s) \subset \Omega \text{ such that } \lambda(E \cap B(a,r+s)) > 0.
\]
\[
\Rightarrow u(x) \leq \frac{r^n}{(r+s)^m} A(u,a,r) = -\infty, \quad \forall x \in B(a,s).
\]
\[
\Rightarrow u|_{B(a,s)} \equiv -\infty.
\]
which contradicts the fact that \(s \in SH(\Omega) \iff u \in SH(B(a,s)), \forall B(a,s) \subset \Omega\).

**Definition** Let \(u : \Omega \subset \mathbb{C}^n \to \mathbb{R}\) be a \(C^2\) smooth function. \(u\) is called \textit{pluriharmonic} on \(\Omega\) if
\[
\frac{\partial^2 u}{\partial z_j \partial z_k} = 0 \text{ on } \Omega, \quad 1 \leq j, k \leq n.
\]
Here we denote \(u \in PH(\Omega)\).

**Remark:** \(u \in PH(\Omega)\) if and only if
\[
\ddc u = \frac{i}{\pi} \frac{\partial \overline{u}}{\partial z} = \frac{i}{\pi} \sum_{1 \leq j,k \leq n} \frac{\partial^2 u}{\partial z_j \partial z_k} dz_j \wedge d\overline{z}_k = 0.
\]

**Proposition 4.2** Let \(\Omega \subset \mathbb{C}^n\) be an open subset.

(1) \(u \in PH(\Omega) \iff u|_\ell\) is harmonic on any complex line \(\ell\). i.e., \(\forall f(\lambda) = a + \lambda b\) where \(\lambda \in \mathbb{C}\), \(a,b \in \mathbb{C}^n\) with \(b \neq 0\),
\[
\Delta(u \circ f)(a) = \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(a) b_j \overline{b}_k = 0, \quad \forall b \in \mathbb{C}^n.
\]

(2) If \(f \in \mathcal{O}(\Omega)\), then \(Re(f), Im(f) \in PH(\Omega)\). Conversely, if \(u \in PH(B(a,r)), \forall B(a,r) \subset \Omega, \exists v \in PH(B(a,r))\) such that
\[
u + iv \in \mathcal{O}(B(a,r)).
\]
4.1. PLURISUBHARMONIC FUNCTIONS

Definition Let $u : \Omega \subset \mathbb{C}^n \to \mathbb{R}$ satisfying

(i) $u$ is upper semicontinuous.
(ii) $u \not\equiv -\infty$ on each connected component of $\Omega$.
(iii) $u|_{\Omega \cap \ell}$ is subharmonic, or $u|_{\Omega \cap \ell} \equiv -\infty$, $\forall$ complex line $\ell$. Such function $u$ is called plurisubharmonic on $\Omega$. Here we denote $u \in \text{PSH}(\Omega)$.

Remarks

1. $\log|f|^2 \in \text{PSH}(\Omega), \forall f \in \mathcal{O}(\Omega)$. Notice that $\log(|f_1|^2 + ... + |f_m|^2)$ may not be in $\text{PH}(\Omega)$ where $m > 1$. But we always have

$$\log(|f_1|^2 + ... + |f_m|^2) \in \text{PSH}(\Omega) \subset \text{SH}(\Omega) \subset L_{\text{loc}}^1(\Omega).$$

2. If $u \in C^2(\Omega)$, then

$$u \in \text{PSH}(\Omega) \iff \left(\frac{\partial^2 u}{\partial z_j \partial z_k}\right) \text{is a semipositive definite matrix}$$

3. If $\varphi, \psi \in \text{PSH}(\Omega)$, so are $\varphi + \psi$ and $\alpha \varphi$ for any constant $\alpha \geq 0$.

4. If $\psi \in \text{PSH}(\Omega)$, then, given $z \in \Omega$,

$$M(\psi, t) := \frac{1}{\text{vol}(\partial B(z, t))} \int_{\partial B(z, t)} \psi dS$$

is monotone increasing on $t$ with $B(z, t) \subset \Omega$.

5. If $u_k \in \text{PSH}(\Omega), u_k \searrow u$, then

$$u = \lim_{k} u_k \in \text{PSH}(\Omega).$$

6. Let $u \in \text{PSH}(\Omega)$. Then $u \ast \rho_\epsilon \to u$ where $u \ast \rho_\epsilon \in C^\infty(\Omega_\epsilon) \cap \text{PSH}(\Omega_\epsilon)$ and $\Omega_\epsilon = \{x \in \Omega \mid \text{dist}(x, \partial \Omega) > \epsilon\}$.

7. Let $u_1, ..., u_p \in \text{PSH}(\Omega)$ and $\chi : \mathbb{R}^p \to \mathbb{R}$ be a convex function such that $\chi(t_1, ..., t_p) \nearrow$ for each $t_j$, then $\chi(u_1, ..., u_p) \in \text{PSH}(\Omega)$.

8. For any complex manifold $X$, $u \in \text{PSH}(X)$ can be defined. (Defined in each coordinate system and then independent of choice of such system).
4.2 Positive closed currents

Recall that if \( f, g \in C^0[0,1] \), then

\[
 f \equiv g \iff \int_0^1 f(x)\varphi(x)\,dx = \int_0^1 g(x)\varphi(x)\,dx, \quad \forall \varphi \in C_0^\infty[0,1].
\]

Also recall that if \( A, B \subset \mathbb{R} \) be closed intervals, then

\[
 A = B \iff \int_A \varphi(x)\,dx = \int_B \varphi(x)\,dx, \quad \forall \varphi \in C_0^\infty[0,1].
\]

Here “functions” and “subsets” both can be regarded as linear functional from \( C_0^\infty[0,1] \) to \( \mathbb{C} \). In fact, both “function” and “subset” are special cases of or unified by a general concept: “currents.”

Let \( M \) be a real differentiable manifold with \( \dim \mathbb{R} M = m \). Define

\[
 \mathcal{D}^p(M) := \{ C^\infty - p\text{forms on } M \text{ with compact support} \}.
\]

Write \( q = m - p \). Define

\[
 T \in \mathcal{D}'^q(M) = \mathcal{D}'^p(M) = [\mathcal{D}^p(M)]^* = \{ \text{currents of degree } q \}
\]

if and only if \( T : \mathcal{D}^p(M) \to \mathbb{R} \) is a linear functional such that \( \forall \) compact subset \( K \subset M \), \( \exists \) constant \( C_K > 0 \) and \( N \in \mathbb{Z}^+ \) such that

\[
 |T(\varphi)| \leq C_K \sup_K |\varphi|_N, \quad \forall \varphi \in \mathcal{D}^p(M) \supp(\varphi) \subset K.
\]

where \( |\varphi|_N = \sum_{|I| \leq N} |D^I \varphi| \). In this case, we say that \( T \) is of degree \( q \) or of dimension \( p \).

Remarks “Degree” is corresponding to the model case of smooth forms; “dimension” is corresponding to the model case of smooth submanifolds, See examples below.

[Example] Let \( \beta \) be a \( C^\infty \) or \( L_{loc}^1 \) \( q \)-form \( \beta \) on \( M \). Then \( T = [\beta] \in \mathcal{D}'^q(M) \) given by

\[
 T(\varphi) = \int_M \beta \wedge \varphi, \quad \forall \varphi \in \mathcal{D}^p(M), \quad p + q = m.
\]

\( T \) is of degree \( q \).
4.2. POSITIVE CLOSED CURRENTS

[Example] Let $S \subset M$ be a $p$-dimensional orientated submanifold. Then $T := [S] \in \mathcal{D}^q(M) = \mathcal{D}'_p(M)$ given by

$$T(\varphi) = \int_S \varphi, \quad \forall \varphi \in \mathcal{D}^p(M).$$

$T$ is of dimension $p$.

**Definition** For any $T \in \mathcal{D}'^q(M)$, we define $dT \in \mathcal{D}'^{q+1}(M)$ by

$$dT(\varphi) = (-1)^{q+1}T(d\varphi), \quad \forall \varphi \in \mathcal{D}^{m-q-1}(M),$$

where $m = \dim_R(M)$. We say that $T \in \mathcal{D}'^q(M)$ is *closed* if $dT = 0$.

Notice that Stokes’ theorem

$$\int_S d\varphi = \int_{\partial S} \varphi$$

means

$$d[S] = (-1)^{m-p+1}[\partial S], \quad \text{where } m = \dim_R(M).$$

Now we consider a complex manifold $X$ with $n = \dim_C M$. We denote

$$\mathcal{D}^{p,q}(X) = \{ C^\infty (p,q) - \text{forms on } X \text{ with compact support} \}$$

and

$$\mathcal{D}'^{p,q}(X) = \mathcal{D}'_{n-p,n-q}(X) = \{ (p,q) - \text{currents} \}.$$  

**Definition** $T \in \mathcal{D}'_{p,p}(X)$ is called *weakly positive* if $\forall (1,0)$-form $\alpha_1, \ldots, \alpha_p$ on $X$,

$$T \wedge i\alpha_1 \wedge \overline{\alpha_1} \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha_p}$$

is a positive measure.\(^1\) We denote $T \geq 0$.

[Example] If $u \in C^2(\Omega) \cap PSH(\Omega)$ where $\Omega \subset \mathbb{C}^n$, then the matrix

$$\begin{pmatrix} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} \end{pmatrix} \geq 0, \quad \text{i.e., semipositive definite,}$$

$$\iff \sum_{j,k=1}^n \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \zeta_j \zeta_k \geq 0, \quad \forall \zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n,$$

\(^1\)i.e., $(T \wedge i\alpha_1 \wedge \overline{\alpha_1} \wedge \ldots \wedge i\alpha_p \wedge \overline{\alpha_p})(\phi) \geq 0, \quad \forall \phi \in C_0^\infty(X)$ with $\phi \geq 0.$
so that

\[ T = \frac{i}{\pi} \partial \overline{\partial} u \geq 0 \]

is semipositive definite, and hence is a positive current.

[Example] If \( u \in L^1_{\text{loc}}(\Omega) \), then

\[ u \in \text{PSH}(\Omega) \iff T = \frac{i}{\pi} \partial \overline{\partial} u \geq 0 \]

as positive current.

**Theorem 4.3** (1) If \( u \in \text{PSH}(X) \), then \( \frac{i}{\pi} \partial \overline{\partial} u \) is a closed positive \((1, 1)\)-current.

(2) \((\partial \overline{\partial})\)-Poincare lemma) Let \( T \) be a closed positive \((1, 1)\)-current. Then locally

\[ T = \frac{i}{\pi} \partial \overline{\partial} u \]

for some \( u \in \text{PSH}(\Omega) \) where \( \Omega \) is some open subset of \( X \).

**Remark** The important thing is that \( T \) could represents a geometric object, while the function \( u \) represents an analytic object.

**Proof of Theorem:** It suffices to prove (2). Since \( \overline{\partial} T = 0 \) and \( T \) is \((1, 1)\)-current, by Dolbeault’s \( \bar{\partial}\)-Poincare lemma, there is a \((1, 0)\)-current \( \eta \) such that

\[ T = -i \partial \overline{\partial} \eta. \]  

(4.2)

Claim:

\[ \partial \eta \text{ is } d - \text{closed.} \]  

(4.3)

In fact,

\[
\begin{array}{ccc}
\text{d}(\partial \eta) & = & 0 \\
\| \partial \overline{\partial} \eta & = & \text{??} \\
\| \partial + \overline{\partial} \partial \eta & = & \text{??} \\
\| \overline{\partial} \partial \eta & = & T \text{ is } d - \text{closed} \\
\| -\partial \overline{\partial} \eta & = & -i \partial T \\
\end{array}
\]
4.2. POSITIVE CLOSED CURRENTS

Claim (4.3) is proved.

By the $d$-Poincare lemma and by the fact that $\partial \eta$ is $(2,0)$-current, $\partial \eta = d \zeta$ for some $(1,0)$-current $\zeta$. Then $\partial \eta = \partial \zeta$ and $0 = \overline{\partial} \zeta$ (note: if $\overline{\partial} \zeta \neq 0$, then $d \zeta$ will be nonzero $(1,1)$-term). By $0 = \overline{\partial} \zeta$, we write (4.2) as

$$T = -i \partial \eta = -i \partial (\eta - \zeta) = -i \partial \partial \psi. \quad (4.4)$$

Here we used $\partial \eta = \partial \zeta$ to imply that $\eta - \zeta$ is $\partial$-closed. Then by the $\partial$-Poincare lemma, $\eta - \zeta = \partial \psi$ for some $\psi$. □

[Example] If $A \subset \Omega \subset \mathbb{C}^n$ is a complex variety of pure dimension $p$, then $[A] \in \mathcal{D}'_{p,p}(\Omega)$ given by

$$[A](\varphi) = \int_{A_{\text{reg}}} \varphi, \quad \forall \varphi \in \mathcal{D}^{p,p}(\Omega).$$

Proof: Step 1. $[A]$ is well-defined To show: $A_{\text{reg}}$ has locally finite area in a small ball, i.e., $\forall B(a,r) \subset \Omega$,

$$\int_{A_{\text{reg}} \cap B(a,r)} \omega^p < \infty,$$

where

$$\omega = \frac{i}{2} \sum dz_j \wedge d \overline{z_j}$$

is the Euclidean metric and

$$\omega^p = c \sum_{|I|=p} dz_I \wedge d \overline{z_I}$$

where $c = (\frac{i}{2})^p (-1)^{p(p-1)/2} \cdot p!$. Assume $0 \in A_{\text{sing}}$: $a = 0$.

It is sufficient to prove: for each $I$, we have

$$\int_{A_{\text{reg}} \cap B(0,r)} dz_I \wedge d \overline{z_I} < \infty.$$

By the local parametrization theorem and by linear changes of coordinates on $\mathbb{C}^n$, we assume that for each $I$, the projection

$$\pi_I : A \rightarrow \mathbb{C}^p$$

$$(z_1, \ldots, z_n) \mapsto (z_{i_1}, \ldots, z_{i_p})$$

defines a finite ramified covering of $A \cap \Delta$, where $\Delta \subset \mathbb{C}^n$ is a small polydisc containing 0. \footnote{In the real case, real submanifold $A_{\text{reg}}$ may not have local finite volume because it may not have finite branches.}
Then
\[ \int_{A_{\text{reg}}} dz_I \wedge d\overline{z}_I = n_I \int_{\Delta_I} dz_I \wedge d\overline{z}_I = n_I \text{vol}(\Delta_I) < \infty \]
where \( n_I \) = the sheet number of \( A \) at 0, and \( \Delta_I \) is a small polydisc in \( \mathbb{C}^p \).

**Step 2.** \( [A] \geq 0 \) To show by the definition that \( \forall (1, 0) - \text{forms } \alpha_1, ..., \alpha_p \)
\[ [A](i\alpha_1 \wedge \overline{\alpha}_1 \wedge ... \wedge i\alpha_p \wedge \overline{\alpha}_p) \geq ??? \quad 0 \]
\[ \int_{A_{\text{reg}}} i\alpha_1 \wedge \overline{\alpha}_1 \wedge ... \wedge i\alpha_p \wedge \overline{\alpha}_p \]
\[ \int_{A_{\text{reg}}} |\det(\alpha_{jk})|^2 idw_1 \wedge d\overline{w}_1 \wedge ... \wedge idw_p \wedge d\overline{w}_p \geq 0 \]
where \( \alpha_j = \sum_k \alpha_{jk}dw_k \) in terms of local holomorphic coordinate \( (w_1, ..., w_p) \).

**Step 3.** \( [A] \) is closed To show the idea, we consider the following example:
Let \( T = df = f'(x)dx \) where \( f \in C^\infty(\mathbb{R} - \{0\}) \). We extend \( T \) to \( \tilde{T} \in \mathcal{D}'(\mathbb{R}) \) by
\[ \tilde{T}(\varphi) = \lim_{\epsilon \to 0} \left( \int_{-\epsilon}^{\epsilon} \varphi(x)f'(x)dx + \int_{-\infty}^{-\epsilon} \varphi(x)f'(x)dx \right) = \lim_{k \to \infty} \int_{\mathbb{R}} \varphi_k f' dx, \quad \forall \varphi \in C^\infty_0(\mathbb{R}) \]
where \( \varphi_k \in C^\infty_0(\mathbb{R}) \), \( \varphi_k|_{(-1/k, 1/k)} \equiv 0, 0 \leq \varphi_k \leq 1 \) and
\[ \varphi_k(x) \to \begin{cases} 1, & x \in \mathbb{R} - \{0\}, \\ 0, & x = 0. \end{cases} \]
In other words,
\[ \tilde{T} = \lim_{k \to \infty} \varphi_k T \text{ weakly.} \]
To show : \( \tilde{T} \) is \( d \)-closed, we consider
\[ d\tilde{T} = ??? 0 \]
\[ \lim_{k \to \infty} d(\varphi_k T) \]
\[ \lim_{k \to \infty} (d\varphi_k \wedge T + \varphi_k dT) \]
\[ \lim_{k \to \infty} (d\varphi_k \wedge T) = ??? \quad 0 \]
4.2. POSITIVE CLOSED CURRENTS

i.e., it reduces to a problem:

\[
\int_{\mathbb{R}} \varphi d\varphi_k \wedge T \to ???0, \quad \forall \varphi \in C_0^\infty(\mathbb{R}).
\]

The general case can be proved by the similar idea. In fact, it is a consequence of the Skoda-El Mir extension theorem below. \(\square\)

We need the following definition: let \(X\) be a complex manifold. A subset \(E \subset X\) is called a complete pluripolar set if there is an open covering \(\{\Omega_j\}\) of \(X\) and functions \(u_j \in PSH(\Omega_j)\) with \(E \cap \Omega_j = u_j^{-1}(-\infty)\). As an example, any analytic set is complete pluripolar. In fact, if \(A = \text{zero}\{f_1, ..., f_m\}\) where \(f_j\) are holomorphic functions, we put \(v = \log(|f_1|^2 + ... + |f_m|^2)\).

**Theorem 4.4** (Skoda 81, El Mir 84, Sibney 85) Let \(E\) be a closed complex pluripolar set in \(X\). Let \(\Theta\) be a closed positive current on \(X - E\) such that the coefficients \(\Theta_{I,J}\) of \(\Theta\) are measures with totally finite mass near \(E\). Then the trivial extension \(\tilde{\Theta}\) obtained by extending the measures \(\Theta_{I,J}\) by 0 on \(E\) is still closed on \(X\).

**Remark** To prove that \(d[A] = 0\), we simply apply the Skoda-El Mir theorem to the current \(\Theta = [A_{\text{reg}}]\) on \(X - A_{\text{sing}}\).

**Proof:** The problem is local on \(X\). So we may work on a small open set \(\Omega \subset \mathbb{C}^n\) such that \(E \cap \Omega = v^{-1}(-\infty), v \in PSH(\Omega)\) and \(v \leq 0\).

**Step 1. Construction \(v_k\)** By shrinking \(\Omega\), we know \(v * \rho_\epsilon \in C_\infty(\Omega) \cap PSH(\Omega)\) and

\[
v * \rho_\epsilon(x) \to \begin{cases} v(x), & x \notin \Omega - E; \\ -\infty, & x \in E \cap \Omega,
\end{cases}
\]
as \(\epsilon \to 0\).

Let \(\chi : \mathbb{R} \to \mathbb{R}\) be a smooth convex increasing function such that \(\chi(t) = 0\) for \(t \leq -1\) and \(\chi(0) = 1\). Take \(\epsilon_k \to 0\) and let

\[
v_k := \chi\left(\frac{1}{k} v * \rho_{\epsilon_k}\right) \in C_\infty(\Omega) \cap PSH(\Omega).
\]

Here we choose \(\epsilon_k\): For each \(k\), we choose \(\epsilon_k\) sufficiently small such that \(v < -k - 1\) in a neighborhood of \(E \cap \Omega\) (Here we can always shrink \(\Omega\) slightly, so that we can use finite covering argument). Then

\[v_k = 0\] in a neighborhood, which depends on \(k\), of \(E \cap \Omega\). \(\quad (4.5)\)
For any \( x \in \Omega - E \), \( \lim_k v_k = \lim_k \chi(\frac{1}{k} v \ast \rho \epsilon_k) = \lim_k \chi(\frac{v(x) + o(1/k)}{k}) = \chi(0) = 1 \) so that

\[
\lim_k v_k(x) = 1 \quad \text{at every point of} \quad \Omega - E.
\] (4.6)

By shrinking \( \Omega \), there is an integer \( k_0 \) such that

\[
0 \leq v_k \leq 1, \quad \forall k \geq k_0.
\] (4.7)

In particular, there is a neighborhood \( U \) of \( E \) in \( \Omega \) such that

\[
v_k \leq \frac{1}{2}, \quad \text{on} \ U, \quad \text{and} \quad \frac{2}{3} \leq v_k \leq 1, \quad \text{on} \ \Omega - U
\] (4.8)

for all \( k \geq k_1 > k_0 \) for some \( k_1 \).

**Step 2. Construct** \( \theta \)  
Let \( \theta \in C^\infty[0,1] \) be a function such that

\[
\theta = \begin{cases} 
0, & \text{on } [0, \frac{1}{3}]; \\
1, & \text{on } [\frac{2}{3}, 1]; 
\end{cases} \quad \text{and} \quad 0 \leq \theta \leq 1.
\]

Then

\[
\theta \circ v_k \rightarrow 0, \quad \text{near} \ E \cap \Omega, \quad \text{and} \quad \theta \circ v_k \rightarrow 1, \quad \text{on} \ \Omega - E.
\]

By (4.8), we know that the derivative

\[
\theta' \circ v_k = \theta'(v_k) \rightarrow 0, \quad \text{as} \ k \rightarrow \infty.
\] (4.9)

**Step 3. A reduction**

\[
\tilde{\Theta} = \lim_{k \rightarrow \infty} (\theta \circ v_k) \Theta
\]

and

\[
\partial \tilde{\Theta} = \lim_{k \rightarrow \infty} \Theta \wedge \partial(\theta \circ v_k)
\]

in the weak topology of currents. To finish the proof, we need to show:

\[
\Theta \wedge \partial(\theta \circ v_k) \rightarrow 0 \quad \text{weakly.}
\]

Note that \( \overline{\partial} \tilde{\Theta} \) is conjugate to \( \overline{\partial} \tilde{\Theta} \), thus \( \overline{\partial} \tilde{\Theta} = 0 \).

Assume first that \( \Theta \in \mathcal{D}^{n-1,n-1}(\Omega) \). Then we have to show that for any \( \alpha \in \mathcal{D}^{1,0}(\Omega) \),

\[
\langle \Theta \wedge \partial(\theta \circ v_k), \overline{\alpha} \rangle = \langle \Theta, \theta'(v_k) \partial(v_k) \wedge \overline{\alpha} \rangle \rightarrow 0, \quad \text{as} \ k \rightarrow \infty.
\]
4.3. POINCARE-LELONG FORMULA

Step 4. Use Cauchy-Schwarz inequality

As \( \gamma \mapsto \langle \Theta, i^p \gamma \wedge \overline{\gamma} \rangle \) is a non-negative hermitian form on \( \mathcal{D}^{0,1}(\Omega) \), the Cauchy-Schwarz inequality yields

\[
|\langle \Theta, i^p \gamma \wedge \overline{\gamma} \rangle|^2 \leq \langle \Theta, i^p \gamma \wedge \overline{\gamma} \rangle \langle \Theta, i^p \gamma \wedge \overline{\gamma} \rangle, \quad \forall \beta, \gamma \in \mathcal{D}^{1,0}(\Omega).
\]

Let \( \psi \in \mathcal{D}(\Omega) \) with \( 0 \leq \psi \leq 1 \) and \( \psi = 1 \) in a neighborhood of \( \text{supp}(\alpha) \). We find

\[
\left| \langle \Theta, \theta'(v_k) \partial(v_k) \wedge \overline{\alpha} \rangle \right|^2 \leq \langle \Theta, \psi i \partial v_k \wedge \overline{\alpha} \rangle \langle \Theta, \theta'(v_k)^2 i \alpha \wedge \overline{\alpha} \rangle.
\]

For the term \( \langle \Theta, \theta'(v_k)^2 i \alpha \wedge \overline{\alpha} \rangle \). By (4.9) and apply Lebesgue’s dominated convergence theorem to get

\[
\lim_k \langle \Theta, \theta'(v_k)^2 i \alpha \wedge \overline{\alpha} \rangle = \langle \Theta, \lim_k \theta'(v_k)^2 i \alpha \wedge \overline{\alpha} \rangle = 0.
\]

For the term \( \langle \Theta, \psi i \partial v_k \wedge \overline{\partial v_k} \rangle \), because \( v_k \) is psh function, we notice

\[
i \partial \overline{\partial v_k}^2 = 2v_k i \partial \overline{\partial v_k} + 2 \partial v_k \wedge \overline{\partial v_k} \geq 2i \overline{\alpha} \wedge \overline{\partial v_k}
\]

so that

\[
2 \langle \Theta, \psi i \partial v_k \wedge \overline{\partial v_k} \rangle \leq \langle \Theta, \psi i \partial \overline{\partial v_k}^2 \rangle = \langle \Theta, v_k^2 i \partial \overline{\partial v_k} \rangle \leq C \int_{\Omega-E} \|\Theta\| < \infty
\]

where \( C \) is a bound for the coefficients of \( i \partial \overline{\partial v_k} \). Here we used the integral by parts. \(^3\) The proof is complete when \( \Theta \in \mathcal{D}^{n-1,n-1} \).

Step 5 Other cases

In the general case \( \Theta \in \mathcal{D}^{p,p} \), \( p < n \), we simply apply the result already proved to all positive currents \( \Theta \wedge \gamma \in \mathcal{D}^{n-1,n-1} \) where \( \gamma = i^p \gamma_1 \wedge \overline{\gamma_1} \wedge ... \wedge \gamma_{n-p-1} \wedge \overline{\gamma_{n-p-1}} \) runs over a basis of forms of \( A^{n-p-1,n-p-1}(\Omega) \) with constant coefficients. Then \( d(\Theta \wedge \gamma) = 0 \), i.e., \( d\overline{\Theta} \wedge \gamma = 0 \), \( \forall \) such \( \gamma \), hence \( d\overline{\Theta} = 0 \). \( \square \)

4.3 Poincare-Lelong formula

Recall the \( \partial \overline{\partial} \)-Poincare lemma (4.3), locally

\[
T \text{closed, } \geq 0 \text{, } (1,1)\text{-current } \iff T = \frac{i}{\pi} \partial \overline{\partial} u \text{ for some } u \in PSH.
\]

\(^3\)We have the formula: \( \int_{\Omega} f \wedge dd^c g - dd^c f \wedge g = \int_{\partial \Omega} f \wedge d^c g - d^c f \wedge g \). If \( g \in \mathcal{D}^{p,p}(\Omega) \), then \( \int_{\Omega} f \wedge dd^c g = \int_{\Omega} dd^c f \wedge g \).
**Theorem 4.5** (Poincare-Delong formula) Let $X$ be a complex manifold and $f \in \mathcal{O}(X)$ be a holomorphic function. Then

$$\frac{i}{2\pi} \partial \bar{\partial} \log |f|^2 = \text{[zero}(f)] \in D^{1,1}(X).$$

**Proof:** **Step 1. $n=1$** When $n = 1$, it is a local problem so that we can consider $f(z) = z^m g(z)$ where $g$ is a holomorphic function defined in a neighborhood $U$ of 0 in $\mathbb{C}$ such that $g(z) \neq 0, \forall z \in U$. Then $\log |f|^2 = \log |z|^{2m} + \log |g(z)|^2$ and

$$\frac{i}{2\pi} \partial \bar{\partial} \log |f|^2 = \frac{i}{2\pi} \partial \bar{\partial} \log |z|^{2m} + \frac{i}{2\pi} \partial \bar{\partial} \log |g(z)|^2 = \frac{i}{2\pi} \partial \bar{\partial} \log |z|^{2m} + 0$$

so that the proof of (4.10) is equivalent to the proof of the following:

$$\frac{i}{2\pi} \partial \bar{\partial} \log |z|^{2m} = \text{[zero}(z^m)] \in \mathcal{D}^{1,1}(\mathbb{C}).$$

(4.11)

In fact, $\forall \varphi \in \mathcal{D}(\mathbb{C}),$

$$\frac{i}{2\pi} \int_{\mathcal{C}} (\partial \bar{\partial} \log |z|^{2m}) \varphi = \partial \bar{\partial} \varphi = 0$$

and

$$\frac{i}{2\pi} \int_{\mathcal{C}} (\bar{\partial} \log |z|^{2m}) \varphi = \bar{\partial} \varphi = 0$$

and

$$\frac{i}{2\pi} \int_{\mathcal{C}} d(\partial \log |z|^{2m}) \varphi = \partial^2 \varphi = 0$$

and

$$\frac{i}{2\pi} \int_{\mathcal{C}} \partial \log |z|^2 \wedge \partial \varphi = \text{[Cauchy integral formula]}$$

and

$$\frac{i}{2\pi} \int_{\Delta} \frac{\varphi}{|z|^2} dz \wedge \frac{\partial \varphi}{\partial z} dz = \frac{m}{2\pi i} \int_{\Delta} \frac{\partial \varphi}{\partial \bar{z}} dz \wedge \frac{d\bar{z}}{z - 0}$$

Here we used the generalised Cauchy integral formula for smooth functions.  

---

4(Cauchy Integral Formula) Let $\Delta \subset \mathbb{C}$ be a disk, $f \in C^\infty(\Delta), z \in \Delta$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Delta} \frac{f(w)}{w - z} dw + \frac{1}{2\pi i} \int_{\Delta} \frac{\partial f}{\partial \bar{w}} dw \wedge d\bar{w}.$$
4.3. POINCARE-LELONG FORMULA

Step 2. \( n \geq 2 \) and \( x \in \text{zero}(f)_{\text{reg}} \)  

For general \( n \geq 2 \), we still consider it as a local problem.

When \( x \in X - \text{zero}(f) \), the Poincare-Lelong formula holds trivially.

When \( x \in \text{zero}(f)_{\text{reg}} \), \( \text{zero}(f) \) looks like submanifold near \( x \), by changing local coordinates, we can assume that locally

\[
\text{zero}(f) = \{ z_1 = 0 \}.
\]

We can write

\[
f(z) = z_1^m g(z),
\]

where \( z = (z_1, \ldots, z_n) \) and \( g(z) \) is a holomorphic function defined in a neighborhood \( U \) of 0 in \( \mathbb{C}^n \) such that \( g(z) \neq 0, \forall z \in U \). By the same argument in Step 1, it is sufficient to prove:

\[
\frac{i}{2\pi} \partial \ddbar z_1 |^{2m} = [\text{zero}(z_1^m)] \in \mathcal{D}'^{1,1}(\mathbb{C}^n) \tag{4.12}
\]

In fact, \( \forall \varphi \in \mathcal{D}^{-n-n,1}(\Delta) \), where \( \text{supp}(\varphi) \subset \Delta_1 \times \cdots \times \Delta_n \subset \mathbb{C}^n \) and we write \( \varphi = \sum \varphi_{ij} dz_1 \wedge \cdots \wedge \hat{d}z_i \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge d\bar{z}_n \), we have

\[
\frac{i}{2\pi} \partial \ddbar |^{2m}(\varphi) \quad ??? = ? \quad \text{Div}(z_1^m)(\varphi) = m \int_{\{z_1=0\}} \varphi
\]

\[
\frac{\text{im}}{2\pi} \int_{\mathbb{C}^n} \partial \ddbar z_1 |^2 \wedge \left( \sum_{i,j} \varphi_{ij} dz_1 \wedge \cdots \wedge \hat{d}z_i \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge d\bar{z}_n \right)
\]

\[
\frac{\text{im}}{2\pi} \int_{\mathbb{C}^n} \partial \ddbar z_1 |^2 \wedge \left( \varphi_{11} dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n \right)
\]

\[
-\frac{\text{im}}{2\pi} \int_{\mathbb{C}^n} \partial \ddbar z_1 |^2 \wedge \left( \varphi_{11} dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n \right)
\]

\[
\text{integral by parts}
\]

\[
-\frac{\text{im}}{2\pi} \int_{\mathbb{C}^n} \frac{dz_1}{z_1} \wedge \left( \frac{\partial \varphi_{11}}{\partial z_1} d\bar{z}_1 \wedge dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n \right)
\]

\[
-\frac{\text{im}}{2\pi} \int_{\mathbb{C}^n} \frac{1}{z_1} \frac{\partial \varphi_{11}}{\partial z_1} dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n
\]

\[
-\frac{\text{im}}{2\pi} \int_{\mathbb{C}^n} \left( \int_{\Delta_1} \frac{\partial \varphi_{11}(z_1, z_2, \ldots, z_n)}{\partial z_1} \right) dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n
\]

\[
\text{use the generalized Cauchy integral formula}
\]

\[
m \int_{\mathbb{C}^n} \varphi_{11}(0, z_2, \ldots, z_n) dz_2 \wedge \cdots \wedge dz_n \wedge d\bar{z}_2 \wedge \cdots \wedge d\bar{z}_n
\]

\[
m \int_{\{z_1=0\}} \varphi.
\]
Step 3. \( n \geq 2 \) and \( x \in \text{zerp}(f)_{\text{sing}} \) Let 
\[
T := dd^c \log |f|^2 - [\text{Div}(f)]
\]
which is a closed, positive \((1, 1)\)-current with 
\[
\text{supp}(T) \subset \text{zerp}(f)_{\text{sing}}, \text{ where } \text{codim} \left( \text{zerp}(f)_{\text{sing}} \right) \geq 2.
\]
By applying Federer’s Theorem (see below), \( T \equiv 0 \). \( \square \)

A current \( T \) is of order \( p \) if \( T \) is continuous in the \( C^p \)-topology, i.e., \( \forall K \subseteq X, \exists C_K > 0 \) such that 
\[
|T(\varphi)| \leq C_K \sup_K |\varphi|_p, \quad \forall \varphi \in \mathcal{D}_K.
\]

Theorem 4.6 Let \( T \in \mathcal{D}^p(M) \) where \( M \) is a real manifold of dimension \( n \) such that \( T \) and \( dT \) are of order 0. Suppose that \( \text{supp}(T) \subseteq \) a real submanifold \( A \) with \( \dim A < n - p \). Then \( T \equiv 0 \).

Recall Dolbeault cohomology group
\[
H^{p,q}_{\bar{\partial}}(X) = \frac{\ker(\bar{\partial} : C^\infty(X, \Lambda^p\Omega^q(X)) \to C^\infty(X, \Lambda^{p \oplus q + 1}\Omega^q(X)))}{\text{Im}(\bar{\partial} : C^\infty(X, \Lambda^{p \oplus q - 1}\Omega^q(X)) \to C^\infty(X, \Lambda^{p \oplus q}\Omega^q(X)))}
\]
and 
\[
H^{p,q}_{\bar{\partial}}(X) \cong H^q(M, \Omega^p).
\]
Similarly, we define 
\[
H^{p,q}_{\bar{\partial}'}(\mathcal{D}'^{p,\cdot}X) = \frac{\ker(\bar{\partial}' : \mathcal{D}'^{p,q}(X)) \to \mathcal{D}'^{p,q+1}(X)}{\text{Im}(\bar{\partial}' : \mathcal{D}'^{p,q-1}(X) \to \mathcal{D}'^{p,q}(X))}
\]
and 
\[
H^{p,q}_{\bar{\partial}'}(\mathcal{D}'^{p,\cdot}) \cong H^q(M, \Omega^p).
\]
4.4 Wedge product of currents

Let \( f_j \in \mathcal{O}(X) \), \( 1 \leq j \leq p \), \( f_j \neq 0 \). Then by the Poincare-Lelong theorem,

\[
\frac{i}{\pi} \partial \overline{\partial} u_j = [Z_j]
\]

where \( u_j = \log |f_j|^2 \) and \( Z_j = \text{zero}(f_j) \).

Consider \( Z_1 \cap Z_2 = \sum_k m_k [C_k] \) where \( C_k \) are subvarieties of codimension 2. Do we have

\[
\frac{i}{\pi} \partial \overline{\partial} u_1 \wedge \frac{i}{\pi} \partial \overline{\partial} u_2 \ ? = ? \sum_k m_k [C_k].
\]

This raises a question: how to define \( \frac{i}{\pi} \partial \overline{\partial} u_1 \wedge \frac{i}{\pi} \partial \overline{\partial} u_2 \ ? \)? Is it still positive? Or we could ask: \( \forall u \in PSH(X) \) and \( \forall T \), a closed positive current, can we define

\[
\frac{i}{\pi} \partial \overline{\partial} u \wedge T = ?
\]

which is also closed positive current?

In general, the answer is “no.”

If \( T_1 \) and \( T_2 \) are smooth form, which are regarded as currents, their wedge product can be defined as standard wedge product \( T_1 \wedge T_2 \). But, if \( D \) is the exceptional divisor of a blow up in surface, \( [D] \cdot [D] = -1 \).

Nevertheless, it was observed by Bedford-Taylor(1982) that if \( u \) is locally bounded, we have the following definition.

**Definition** Let \( T \) be a positive, closed current on a complex manifold \( X \) and \( u \in PSH(X) \) that is locally bounded. Then

\[
\frac{i}{\pi} \partial \overline{\partial} u \wedge T := \frac{i}{\pi} \partial \overline{\partial} (uT).
\]

**Lemma 4.7** \( \frac{i}{\pi} \partial \overline{\partial} u \wedge T \) is closed, positive current.

**Proof:** Since the problem is a local one, we assume \( X = B(0,r) \subset \mathbb{C}^n \). Let

\[
u := u \ast \rho_{1/v} \in C^\infty(X_{1/v}) \cap PSH(X_{1/v}),
\]

where
Then \( dd^c(uT) = \lim_{v \to u} dd^c(u_v T) \) weakly. \( u_v T \) converges weakly to \( uT \) by Lebesgue’s monotone convergence theorem. □

**Remarks**

1. \( u := \log(|f_1|^2 + ... + |f_m|^2), f_j \in \mathcal{O}(X), \) is not locally bounded if \( \text{zero}(f_1, ..., f_m) \neq \emptyset. \)

2. If the condition “\( u \) is locally bounded” does not hold, the lemma is not true. For example, let \( T = [A] \) where \( A = \text{zero}(f) \) and \( f \in \mathcal{O}(\Omega) \) and \( \Omega \subset \mathbb{C}^n \) is a domain. Then

\[
T(\varphi) := \int_A \varphi, \quad \forall \varphi \in D^{n-1,n-1}(X).
\]

Let \( u = \log|f|^2 \in PSH(\Omega) \) which is not locally bounded. Then

\[
(uT)(\varphi) = \int_A u\varphi = -\infty, \quad \forall \varphi \in D^{n-1,n-1}(\Omega)
\]

so that \((uT)\) is not a current.

3. When \( u \) is \( C^2 \) smooth, we have

\[
\frac{dd^c u \wedge ... \wedge dd^c u}{n \text{ times}} = \det \left( \frac{\partial^2 u}{\partial z_i \partial \overline{z}_j} \right) \frac{n!}{\pi^n} i dz_1 \wedge d\overline{z}_1 \wedge ... \wedge i dz_n \wedge d\overline{z}_n
\]

is called the Monge-Ampere operator.

4. Let \( T \) be a closed positive current on \( X \) and \( u_1, ..., u_q \in PSH(X) \) are locally bounded where \( X \) is a complex manifold. We can define

\[
\frac{dd^c u_1 \wedge ... \wedge dd^c u_q \wedge T}{n \text{ times}} := dd^c(u_1dd^c u_2 \wedge ... \wedge dd^c u_q \wedge T)
\]

by induction on \( q. \)

**Definition** Let \( \Theta \) be a current on \( X \) of order 0 (e.g., positive currents). Let \( K \in X \) be contained in a coordinate patch. We define the mass of \( \Theta \)

\[
\|\Theta\|_K := \int_K \sum_{I,J} |\Theta_{I,J}|
\]

where \( \Theta|_K = \sum_{I,J} \Theta_{I,J} dz^I \wedge d\overline{z}^J \) and \( \Theta_{I,J} \) are distribution and \( |\Theta_{I,J}| = \sup_{|\varphi| \leq 1, \varphi \in D} |\Theta_{I,J}(\varphi)|. \)

**Remarks**
1. If $K$ is not contained in a coordinate patch, we use the partition of unity to define the seminorm $||\Theta||_K$. The finiteness of $||\Theta||_K$ is independent of choice of coordinates patches.

2. If $\Theta \geq 0$, $\exists C, \tilde{C} \geq 0$ such that

$$\tilde{C} \int_K \Theta \wedge \beta^p \leq ||\Theta||_K \leq C \int_K \Theta \wedge \beta^p,$$

where $\beta := \frac{i}{2} \partial \bar{\partial} |z|^2$.

**Theorem 4.8** (Chern-Levine-Nirenberg Inequality) $\forall u_1, ..., u_m \in PSH(X)$ that are locally bounded, $\forall \Theta$ closed, positive current on $X$, $\forall L \subseteq K^c \subseteq K \subseteq X$, $\exists$ constant $C_{KL} > 0$ such that

$$||dd^c u_1 \wedge ... \wedge dd^c u_m \wedge \Theta||_L \leq C_{KL} ||u_1||_{L^\infty(K)}...||u_m||_{L^\infty(K)} ||\Theta||_K.$$

**Proof:** By induction, it is sufficient to prove the case $q = 1$. Taking a covering of $L$ by a family of calls $B'_j \subseteq B_j \subseteq K$, we take $\chi_j \in \mathcal{D}(B_j)$ such that $\chi_j \geq 0$ and $\chi_j|_{B'_j} \equiv 1$. Then

$$||dd^c u_1 \wedge \Theta||_{L \cap B'_j} \leq C \int_{B'_j} dd^c u_1 \wedge \Theta \wedge \beta^{p-1} \quad \text{By Remark (2) above}
\leq C \int_{B_j} \chi dd^c u_1 \wedge \Theta \wedge \beta^{p-1} \quad \because \chi|_{B'_j} \equiv 1
= C \int_{B_j} u_1 \Theta \wedge dd^c \chi \wedge \beta^{p-1} \quad \text{Integral by parts}
\leq \tilde{C} ||u_1||_{L^\infty(K)} ||\Theta||_K \quad \text{By Remark (2) above} \quad \square$$

**Theorem 4.9** (Bedford-Taylor, 1982) Let $T$ be a closed positive current. Let $u_1, ..., u_q$ be locally bounded PSH functions. Let $u^k_1, ..., u^k_q$ be PSH functions such that

$$u^k_j \searrow u_j, \quad \text{as } k \to \infty, 1 \leq j \leq q.$$

Then

$$u^k_1 dd^c u_2 \wedge ... \wedge dd^c u^k_j \wedge T \to u_1 dd^c u_2 \wedge ... \wedge dd^c u_q \wedge T \quad \text{weakly}.$$
If PSH functions $u_1, \ldots, u_q$ are not locally bounded, we cannot define $dd^c u_1 \wedge \ldots \wedge dd^c u_q$ in general. However, if the polar set of $u_j$ are small enough, $dd^c u_1 \wedge \ldots \wedge dd^c u_p$ still can be defined. This is the key to be used to study varieties defined by several holomorphic functions.

**Theorem 4.10** (Demailly) Let $u \in PSH(X)$ and $\Theta$ be a closed positive current of bidimension $(p, p)$. Let $A \subset X$ be an analytic subset of dimension $< p$. Suppose that $u$ is locally bounded on $X - A$. Then $dd^c u \wedge \Theta$ is well-defined, positive, closed current, and

$$dd^c u \wedge \Theta = \lim_{v \to \infty} dd^c u_v \wedge \Theta$$

for any locally bounded PSH functions $u_v \searrow u$.

**Proof:** Assume $A = \{0\}$ (The general case is reduced into this case by slicing). $\forall s \in \mathbb{R}$, let

$$u^{\geq s} = \max\{u, s\}$$

which is PSH and locally bounded. Then

$dd^c u^{\geq s}$ is a closed, positive current.

$\forall r > 0$, let

$$I(s) = \int_{B(0, r)} dd^c u^{\geq s} \wedge \Theta \wedge (dd^c|z|^2)^{p-1}.$$  

We claim that if $s < s' < S(r) := \lim_{|z| \to r} u(z)$, then

$$I(s) = I(s'). \quad (4.13)$$

Assuming this claim, it implies that

$$\int_{B(0, r)} dd^c u \wedge \Theta \wedge (dd^c|z|^2)^{p-1} = \lim_{s \to -\infty} \int_{B(0, r)} dd^c u^{\geq s} \wedge \Theta \wedge (dd^c|z|^2)^{p-1} < \infty$$

So that $dd^c u \wedge \Theta$ has finite mass on $B(0, r) - \{0\}$. Then, by the Skoda-Mir Extension theorem (see § 4.2), this current can extend by trivial extension.
4.5. LELONG NUMBERS

Let us prove the claim (4.13). In fact

\[ I(s) - I(s') = 0 \]
\[ = \int_{B(0,r)} dd^c(u \geq s - u \geq s') \wedge \Theta \wedge (dd^c|z|^2)^{p-1} \]
\[ = \int_{B(0,r)} d\left( dd^c(u \geq s - u \geq s') \wedge \Theta \wedge (dd^c|z|^2)^{p-1} \right) \]
\[ = \int_{\partial B(0,r)} dd^c(u \geq s - u \geq s') \wedge \Theta \wedge (dd^c|z|^2)^{p-1} = 0. \]

Here the last second equality holds because \( u \geq s - u \geq s' = 0 \) in a neighborhood of \( \partial B(0,r) \).

□

**Theorem 4.11** Let \( f_j \in O(X), f_j \not\equiv 0, u_j = \log|f_j|, [Z_j] = dd^c u_j = \text{zero}(f_j), 1 \leq j \leq p, \dim X = n \). Suppose

\[ \dim_{\mathbb{C}} \left( \text{supp}[Z_{j_1}] \cap \ldots \cap \text{supp}[Z_{j_m}] \right) = n - m, \quad \forall j_1 < \ldots < j_m. \]

Then \( [Z_1] \wedge \ldots \wedge [Z_p] := dd^c u_1 \wedge \ldots \wedge dd^c u_p \) satisfies

\[ [Z_1] \wedge \ldots \wedge [Z_p] = \sum_k m_k[C_k] \]

where \( C_k \) are all irreducible components of \( \text{supp}[Z_1] \cap \ldots \cap \text{supp}[Z_m] \) and \( m_j \) are positive integers.

### 4.5 Lelong numbers

For \( f \in O(U) \), where \( U \subset \mathbb{C} \) is an open subset containing 0, with \( f(0) = 0 \), we have

\[ f(x) = z^m g(z) \]

where \( g \in O(U) \) with \( g(z) \neq 0, \forall z \in U \), and \( m \) is the multiplicity (or order of zeros) at 0.

For higher dimensional case, we have
Definition Let $\Theta$ be a closed, positive current of bidegree $(p,p)$ on $\Omega \subset \mathbb{C}^n$. The Lelong number of $\Theta$ at $x \in \Omega$ is defined by

$$\nu(\Theta, x) := \lim_{r \to 0^+} \nu(\Theta, x, r)$$

where

$$\nu(\Theta, x, r) := \frac{1}{r^{2p}} \int_{B(x,r)} \Theta \wedge (dd^c|z|^2)^p.$$ 

Remark In case $\Theta = [A]$ where $A$ is a subvariety, we have interpretation:

$$\nu([A], x, r) := \frac{\text{area of } A \text{ in the ball } B(x, r)}{\text{area of the ball } B(x, r)}.$$ 

Theorem 4.12 (Lelong, 1957) (1) $\nu(\Theta, x, r) \searrow 0$ as $r \searrow 0$ so that the limit $\nu(\Theta, x)$ exists.

(2) $\forall u \in PSH(\Omega)$ and $\Theta = dd^c u$. Then

$$\nu(\Theta, x) = \sup\{\gamma \geq 0 \mid u(z) \leq \gamma \log|z - x| + O(1) \text{ near } x\}.$$ 

In particular, if $u = \log|f|$ for some $f \in O(\Omega)$ and $\Theta = dd^c u = [\text{zero}(f)]$, then

$$\nu(\Theta, x) = \text{ord}_x(f) = \max\{m \in \mathbb{N} \mid D^\alpha f(x) = 0, \ |\alpha| < m\}.$$ 

Theorem 4.13 (Thie’s theorem) Let $A$ be an analytic subset of dimension $p$. Then

$$\nu([A], x) = \text{multiplicity of } A \text{ at } x = \text{the sheet number of } A \text{ at } x.$$
4.5. LELONG NUMBERS

**Theorem 4.14 (Siu’s theorem)** Let \( \Theta \) be a closed, positive current of bidegree \((p,p)\) on a complex manifold \(X\). Then

1. \( \nu(\Theta, x) \) is independent of choice of local coordinates.
2. \( \forall c > 0 \), the sublevel set
   \[ E_c(\Theta) = \{ x \in X \mid \nu(\Theta, x) \geq c \} \]

is an analytic subset of \(X\) of dimension \( \leq p \).

**Definition** (Demailly) Let \( \varphi \) be a PSH function with isolated \(-\infty\) pole at \(x\) such that \(e^\varphi\) is continuous \(^5\). The **generalized Lelong number**

\[ \nu(\Theta, \varphi) := \lim_{t \to -\infty} \nu(\Theta, \varphi, t) \]

where

\[ \nu(\Theta, \varphi, t) := \int_{\varphi(z) < t} \Theta \wedge (dd^c \varphi)^p. \]

**Remarks**

1. Since \( \varphi \) has only isolated singularity, the wedge product of two such currents is well defined.
2. The classical Lelong number satisfies
   \[ \nu(\Theta, x, t) = \nu(\Theta, log|z-x|, log t). \]

In fact, Demailly proved

\[ \int_{\varphi < r} \Theta \wedge (dd^c e^{2r})^p = (2e^{2r})^p \int_{\varphi < r} \Theta \wedge (dd^c \varphi)^p \]

and then we put \( \varphi(z) = log|z-x| \) and \( r = log t \).

\(^5\)E.g., \( \varphi(x) = log \sum_{j=1}^N |f_j(x)|^\gamma \) and \( \{x\} = zero(f_1) \cap ... \cap zero(f_N) \).
Theorem 4.15 (Comparison theorem) Let $\Theta$ be a closed positive current on a complex manifold $X$. Let $\varphi, \psi : X \to [-\infty, \infty)$ be PSH functions with isolated $-\infty$ pole at $x \in X$ such that $e^\varphi, e^\psi$ are continuous. Assume that
\[
\ell = \lim_{z \to x} \frac{\psi(z)}{\varphi(z)} < +\infty.
\]
Then
\[
\nu(\Theta, \psi) \leq \ell^p \nu(\Theta, \varphi)
\]
and the equality holds if and only if $\ell = \lim_{z \to x} \frac{\psi}{\varphi}$. In particular, if $\lim_{z \to x} \frac{\psi}{\varphi} = 1$, then
\[
\nu(\Theta, \psi) = \nu(\Theta, \varphi).
\]

Proof of Siu’s Theorem (1): Take two coordinate systems at $x$:
\[
\tau(z) = (z_1, \ldots, z_n), \quad \tau'(z') = (z'_1, \ldots, z'_n).
\]
Let
\[
\varphi(z) = \log|\tau(z) - \tau(x)|, \quad \psi(z) = \log|\tau'(z) - \tau'(x)|.
\]
Then
\[
\lim_{z \to x} \frac{\psi(z)}{\varphi(z)} = 1.
\]
By Comparison Theorem, $\nu(\Theta, \psi) = \nu(\Theta, \varphi)$. \qed

Let us assume the comparison theorem and prove Siu’s theorem and Thie’s theorem first.

Proof of Thie Theorem: Let
\[
\varphi(z) = \log|z| = \log(|z'|^2 + |z''|^2)^{1/2}, \quad \psi(z) = \log|z'|.
\]
Here $z = (z', z'')$ where $z' = (z'_1, \ldots, z'_p)$ and $A$ is a ramified finite covering space over $\mathbb{C}^p$. Then
\[
\lim_{z \to x} \frac{\psi(z)}{\varphi(z)} = 1.
\]
By Comparison Theorem,
\[
\nu([A], x) = \nu([A], \varphi) = \nu([A], \psi) = m.
\]
The last equality holds by computation. \qed
4.5. LELONG NUMBERS

Proof of Comparison Theorem: Step 1. Assume $\ell < 1$ Since $\Theta$ is of bidimension $p$, by the definition,

$$\nu(\Theta, \lambda \varphi) = \lambda^p \nu(\Theta, \varphi), \quad \lambda > 0.$$ 

By this, we claim that we can assume $\ell < 1$:

$$\lim_{z \to x} \frac{\psi(z)}{\varphi(z)} < 1. \quad (4.14)$$

In fact, consider general case, $\ell = \lim_{z \to x} \frac{\psi(z)}{\varphi(z)}$ and hence for any $\epsilon > 0$,

$$\lim_{z \to x} \frac{\psi(z)}{\ell + \epsilon \varphi(z)} < 1.$$ 

By assuming (4.14), for any $\epsilon > 0$, we have

$$\nu(\Theta, \psi) \leq \nu(\Theta, (\ell + \epsilon) \varphi) = (\ell + \epsilon)^p \nu(\Theta, \varphi).$$

Let $\epsilon \to 0$, we obtain

$$\nu(\Theta, \psi) \leq \ell^p \nu(\Theta, \varphi).$$

Claim (4.14) is proved.

Step 2. Show one inequality We claim

$$\nu(\Theta, \psi) \leq \nu(\Theta, \varphi). \quad (4.15)$$

Another inequality can be proved similarly.

$$\nu(\Theta, \psi) = \nu(\Theta, \psi - c) = \nu(\Theta, u_c) \leq \nu(\Theta, u_c, r) = \nu(\Theta, \varphi, r).$$

We choose $c > 0$ and use $\nu(\Theta, \psi, t) = \int_{\varphi < t} \Theta \wedge (dd^c \psi)^p$.

We choose $c > 0$ and use $\nu(\Theta, \psi, t) = \int_{\varphi < t} \Theta \wedge (dd^c \psi)^p$.

$$\lim_{t \to x} \psi(t) \varphi(t) < 1,$n

$$\Rightarrow \varphi < \varphi, \text{ both go to } -\infty, \text{ as } z \to x \Rightarrow \psi - c > \varphi, \text{ as } z \approx x.$$

$$\Rightarrow \exists t_0 < 0 \text{ such that } u_c = \varphi - c \text{ on } \{u_c < t_0\}$$

$$\Rightarrow \nu(\Theta, u_c, r) \setminus \nu(\Theta, u_c) \text{, as } r \setminus -\infty.$$

Fix $r < 0$. Take $C > 0$ large enough such that $u_c = \varphi$ on a neighborhood of $\varphi^{-1}(r)$. By Stokes’ theorem,

$$\int_{\varphi < r} \Theta \wedge (dd^c u_c)^p = \int_{\varphi < r} \Theta \wedge (dd^c \varphi)^p.$$
Then we have proved
\[ \nu(\Theta, \psi) \leq \nu(\Theta, \varphi, r), \quad \forall r. \]

Let \( r \to -\infty \). We obtain
\[ \nu(\Theta, \psi) \leq \nu(\Theta, \varphi). \quad \Box \]

### 4.6 Singular metric on line bundle

Let \( L \) be a line bundle over a complex manifold \( X \). Let \( \{U_\alpha\} \) be a locally finite open covering of \( X \) and \( \{g_{\alpha}^\beta\} \) the transition functions. Recall that a Hermitian metric \( h \) on \( L \) is a smooth map \( h : L \to (0, \infty) \) such that
\[ h(\lambda v) = |\lambda|^2 h(v), \quad \forall \lambda \in \mathbb{C}, v \in L, \]
or \( h \) is a collection \( \{h_\alpha \in C^\infty(U_\alpha), h_\alpha > 0\} \) such that
\[ h_\alpha |g_{\alpha}^\beta|^2 = h_\beta, \quad \text{on } U_\alpha \cap U_\beta \neq \emptyset. \]

We then have a norm for any holomorphic section \( s = \{s_\alpha \in \mathcal{O}(U_\alpha)\} \):
\[ \|s\| = \sqrt{h_\alpha |s_\alpha|} \]
which is independent of choice of \( \alpha \). The Chern curvature of \( (L, h) \)
\[ i\Theta_h(L)|_{U_\alpha} := -\frac{i}{2\pi} \partial \bar{\partial} \log h_\alpha \]
is a globally defined \((1, 1)\)-form on \( X \).

**Definition** A **Hermitian singular metric** \( h \) on \( L \) is a collection \( \{h_\alpha = e^{-2v_\alpha} \text{ on } U_\alpha \text{ where } v_\alpha \in L^1_{loc}(U_\alpha)\} \) such that
\[ h_\alpha |g_{\alpha}^\beta|^2 = h_\beta, \quad \text{on } U_\alpha \cap U_\beta. \]

The **Chern current** is defined by
\[ i\Theta_h(L)|_{U_\alpha} = -\frac{i}{2\pi} \partial \bar{\partial} \log e^{-2v_\alpha} = \frac{i}{\pi} \partial \bar{\partial} v_\alpha \]
which is a closed, \((1, 1)\)-current.

**Remark** if \( v_\alpha \in PSH(U_\alpha), \forall \alpha \), then \( i\Theta_h(L) \) is a closed, positive, \((1, 1)\)-current on \( X \).
Let \( \sigma_1, \ldots, \sigma_N \in \Gamma(X, L) \) be holomorphic sections. We define a Hermitian singular metric

\[
v_\alpha := \log \left( \sum_{1 \leq j \leq N} |\varphi_\alpha(\sigma_j)|^2 \right)^{1/2}
\]

where \( \varphi_\alpha : L|_{U_\alpha} \to U_\alpha \times \mathbb{C} \) is a trivialization of \( L \). Here we remark that when the sections \( \sigma_1, \ldots, \sigma_N \) have common zeros, \( v_\alpha \) is not smooth so that \( h \) is a singular metric.

We need to verify that the above definition of \( v_\alpha \) is a well defined singular metric. To do that we first define a singular metric on the dual line bundle \( L^* \).

For any \( s^* \in \Gamma(X, L^*) \), we define

\[
\|s^*\|^2(x) := \sum_{1 \leq j \leq N} |\langle s^*, \sigma_j(x) \rangle|^2
\]

by the dual action. Clearly \( \|s^*\| \) is well defined. The above right hand side is actually calculated by

\[
|\langle s^*, \sigma_j \rangle(x)|^2 = |\varphi_\alpha^*(s^*) \cdot \varphi_\alpha(\sigma_j)|^2
\]

which is independent of choice of \( \alpha \). Here the usual product is used because the inner product in \( \mathbb{C} \) is simply usual multiplication.

We can write

\[
\|s^*\|^2 = h^*_\alpha |\varphi_\alpha^*(s^*)|^2
\]
on \( U_\alpha \) and let us find \( h^*_\alpha \):

\[
\|s^*\|^2(x) = \sum_{1 \leq j \leq N} \frac{|s^* \cdot \sigma_j|^2}{|\varphi_\alpha^*(s^*)|^2} |\varphi_\alpha^*(s^*)|^2 = \sum_{j=1}^N \frac{|\varphi_\alpha^*(s^*) \cdot \varphi_\alpha(\sigma_j)|^2}{|\varphi_\alpha^*(s^*)|^2} |\varphi_\alpha^*(s^*)|^2.
\]

Then the metric \( h^* \) of \( L^* \) on \( U_\alpha \) is

\[
h^*_\alpha = \sum_{j=1}^N \frac{|\varphi_\alpha^*(s^*) \cdot \varphi_\alpha(\sigma_j)|^2}{|\varphi_\alpha^*(s^*)|^2} = \sum_{j=1}^N |\varphi_\alpha(\sigma_j)|^2.
\]

We next consider the metric \( h \) on \( L \) which is dual to \( h^* \) on \( L^* \):

\[
\|s\|^2 = |\varphi_\alpha(s)|^2 h_\alpha = |\varphi_\alpha(s)|^2 \frac{1}{h^*_\alpha} = |\varphi_\alpha(s)|^2 \frac{1}{\sum_{j=1}^N |\varphi_\alpha(\sigma_j)|^2}.
\]

Since \( \|s\|^2 = e^{-2\nu_\alpha} |\varphi_\alpha(s)|^2 \), we obtain

\[
\nu_\alpha = \log \left( \sum_{1 \leq j \leq N} |\varphi_\alpha(\sigma_j)|^2 \right)^{1/2}.
\]

We define a meromorphic map

\[
\Phi_{|L|} : X - B_\Sigma \to \mathbb{CP}^{N-1} \quad x \mapsto [\sigma_1(x) : \ldots : \sigma_N(x)].
\]
where $B_{\Sigma} = \cap_{j=1}^{N} \sigma_j^{-1}(0)$ is called the base locus and $\sigma_1, ..., \sigma_N$ from a basis.

$L$ is very ample if the map $\Phi_{|L|}$ is a regular embedding. $L$ is ample if $mL$ is very ample for some integer $m > 0$.

**Theorem 4.16 (Kodaira)** $L$ is ample if and only if $\exists$ a $C^\infty$ Hermitian metric on $L$ such that

$$i\Theta_h(L) \geq \epsilon \omega$$

where $\omega$ is a fixed smooth Hermitian metric and $\epsilon > 0$ is a constant.
Chapter 5

Appendix: Integral Theory

5.1 Tensor product of vector spaces

We have multiplication of integers:

\[ a \cdot b \in \mathbb{Z}, \quad \forall a, b \in \mathbb{Z} \]

and multiplication of matrices:

\[
A \otimes B := \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix} \otimes \begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix} = \begin{pmatrix}
  a_{11} \begin{pmatrix}
    b_{11} & b_{12} \\
    b_{21} & b_{22}
  \end{pmatrix} & a_{12} \begin{pmatrix}
    b_{11} & b_{12} \\
    b_{21} & b_{22}
  \end{pmatrix} \\
  a_{21} \begin{pmatrix}
    b_{11} & b_{12} \\
    b_{21} & b_{22}
  \end{pmatrix} & a_{22} \begin{pmatrix}
    b_{11} & b_{12} \\
    b_{21} & b_{22}
  \end{pmatrix}
\end{pmatrix}
\]

for any \( A = \begin{pmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{pmatrix} \) and \( B = \begin{pmatrix}
  b_{11} & b_{12} \\
  b_{21} & b_{22}
\end{pmatrix} \). We see \( \dim(A \otimes B) = \dim A \cdot \dim B \).

The tensor product of two vector spaces \( V \) and \( W \), denoted \( V \otimes W \) and also called the tensor direct product, is a way of creating a new vector space analogous to multiplication of integers or matrices. For instance,

\[
\mathbb{R}^n \otimes \mathbb{R}^k \cong \mathbb{R}^{nk}.
\]

In particular,

\[
\mathbb{R} \otimes \mathbb{R}^n \cong \mathbb{R}^n
\]

Also, the tensor product obeys a distributive law with the direct sum operation:

\[
U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W).
\]
Algebraically, let $V$ and $W$ be vector spaces over a field $K$, the vector space $V \otimes W$ is spanned by elements of the form $v \otimes w$, and the following rules are satisfied, for any scalar $\alpha$. The definition is the same no matter which scalar field is used.

\[
(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w,
\]

\[
v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2,
\]

\[
\alpha(v \otimes w) = (\alpha v) \otimes w = v \otimes (\alpha w).
\]

One basic consequence of these formulas is that

\[
0 \otimes w = v \otimes 0 = 0.
\]

There are several ways to construct the tensor product $V \otimes W$ of two vector spaces $V$ and $W$ over a field $K$ (they are all equivalent).

Let us introduce one way here. Let $V, W, Z$ be finite dimensional vector spaces over a field $K$. A map $f : V \rightarrow Z$ is called linear if

\[
f(a^1v_1 + a^2v_2) = a^1f(v_1) + a^2f(v_2), \quad \forall v_1, v_2 \in V, a^1, a^2 \in K.
\]

A map $f : V \times W \rightarrow Z$ is called bilinear if

\[
f(a^1v_1 + a^2v_2, w) = a^1f(v_1, w) + a^2f(v_2, w),
\]

\[
f(v, a^1w_1 + a^2w_2) = a^1f(v, w_1) + a^2f(v, w_2),
\]

\[\forall v_1, v_2 \in V, w_1, w_2 \in W, a^1, a^2 \in K.\]

Remark

1. $\langle , \rangle : V \times V^* \rightarrow K$, $(v, v^*) \mapsto v^*(v)$ is a bilinear map.

2. We can define $r$-linear map

\[f : V_1 \times \ldots \times V_r \rightarrow Z\]

where $V_1, \ldots, V_r$ are vector spaces over the field $K$.


4. $L(V; K) := V^* \cong V$. 


For any elements $v^* \in V^*$, $w^* \in W^*$, we define $v^* \otimes w^* \in L(V \times W; K)$ by

$$
V \times W \rightarrow K
(v, w) \mapsto v^*(v)w^*(w).
$$

Then we define

$$
V^* \otimes W^* := \text{the vector space generated by bilinear maps of the form } v^* \otimes w^*.
$$

Clearly, $V^* \otimes W^* \subset L(V \times W; K)$. We claim:

$$
V^* \otimes W^* = L(V, W; K).
$$

(5.1)

In fact, we take basis $v^{*1}, ..., v^{*m}$ for $V^*$ and $w^{*1}, ..., w^{*n}$ for $W^*$, for any bilinear linear functional $f \in L(V, W; K)$, one can show that $f$ can be written as a linear combination of the sum of $v^{*i} \otimes w^{*j}$.

Notice that from the above, we define the tensor product of two vector spaces.

**Definition**

$$
V \otimes W := L(V^*, W^*; K).
$$

(5.2)

**Theorem 5.1** Let $h : V \times W \rightarrow V \otimes W$, $(v, w) \mapsto v \otimes w$ be the bilinear map. Then for any bilinear map $f : V \times W \rightarrow Z$, there exists a unique linear map $g : V \otimes W \rightarrow Z$ such that $f = g \circ h$.

$$
V \times W \xrightarrow{h} V \otimes W \\
\downarrow f \quad \downarrow g \\
Z
$$

**Remarks**

1. It implies uniqueness of $V \otimes W$. In fact, suppose that $H$ be another vector space having the same property as $V \otimes W$. Then by Theorem above, we get a unique linear map $g : V \otimes W \rightarrow H$. Also, Theorem above holds for $H$ so that there exists a unique linear map $\tilde{g} : H \rightarrow V \otimes W$.

2. We could define $V \otimes W$ in another way: $V \otimes W$ is the vector space $Y$ with a bilinear map $h : V \times W \rightarrow Y$ such that for any bilinear map $f : V \times W \rightarrow Z$, there exists a unique linear map $g : Y \rightarrow Z$ such that $f = g \circ h$.

$$
V \times W \xrightarrow{h} Y \\
\downarrow f \quad \downarrow g \\
Z
$$
3. By Theorem above, we obtain

\[ L(V, W, Z) \cong L(V \otimes W, Z) \]

Taking \( Z = K \), the above becomes

\[ V^* \otimes W^* = (V \otimes W)^*. \]

**Proof of Theorem:** Consider basis \( a_i, 1 \leq i \leq n \), for \( V \) and \( b_j, 1 \leq j \leq m \), for \( W \). Define \( g(a_i \otimes b_j) = f(a_i, b_j) \).

\[
\begin{align*}
V \times W &\xrightarrow{h} V \otimes W \\
(a_i, b_j) &\mapsto a_i \otimes b_j \\
\downarrow f &\mapsto Z \\
f(a_i, b_j) &
\end{align*}
\]

Then for any \( v = \sum_{i=1}^{n} u^i a_i \in V \) and any \( w = \sum_{j=1}^{m} v^j b_j \in W \), we have

\[
g(v \otimes w) = g(\sum_{i,j} u^i v^j a_i \otimes b_j) = \sum_{i,j} u^i v^j g(a_i \otimes b_j) = \sum_{i,j} u^i v^j f(a_i, b_j) = f(v, w).
\]

This yields \( f = g \circ h \). Such \( g \) is unique. \( \square \)

### 5.2 \((s,r)\)-tensors

We have proved that \( \forall v^* \in V^* \), \( \forall w^* \in W^* \),

\[ v^* \otimes w^*: V \times W \to K \\
(v, w) \mapsto v^*(v)w^*(w) \]

is a bilinear map. More generally, we have

**Definition** \( \forall f \in L(V_1, ..., V_s; K), \forall g \in L(W_1, ..., W_r; K) \), we define

\[
f \otimes g: V_1 \times \cdots \times V_s \times W_1 \times \cdots \times W_r \to K \\
(v_1, ..., v_s, w_1, ..., w_r) \mapsto f(v_1, ..., v_s)g(w_1, ..., w_r)
\]

is a \( K \)-valued \((r+s)\)-linear functional. Then, we have a bilinear map

\[ \otimes: L(V_1, ..., V_s; K) \times L(W_1, ..., W_r; K) \to L(V_1, ..., V_s, W_1, ..., W_r; K), \ (f, g) \mapsto f \otimes g. \]
Theorem 5.2 \( \forall \varphi \in L(V_1, ..., V_s; K), \forall \psi \in L(W_1, ..., W_r; K), \forall \zeta \in L(Z_1, ..., Z_t; K), \) we have
\[
(\varphi \otimes \psi) \otimes \zeta = \varphi \otimes (\psi \otimes \zeta).
\]
so that \( \varphi \otimes \psi \otimes \zeta \) is well defined.

Proof: Skipped.

\( \forall v \in V, w \in W, z \in Z, \) we regard them as \( K \)-valued linear functionals of \( V^*, W^* \) and \( Z^* \), respectively, we have defined \( v \otimes w \otimes z \). Then
\[
V \otimes W \otimes Z := \text{the vector space generated by } \{v \otimes w \otimes z\}.
\]
Similarly, let \( V_1, ..., V_s \) be vector spaces over \( K \), we can define \( V_1 \otimes ... \otimes V_s \).

Let \( V_j \) have a basis \( \{a_1^{(j)}, ..., a_{n_j}^{(j)}\} \), then \( V_1 \otimes ... \otimes V_s \) has a basis
\[
a_{a_1}^{(1)} \otimes a_{a_2}^{(2)} \otimes ... \otimes a_{a_s}^{(s)}, \quad 1 \leq a_j \leq n_j, \quad 1 \leq j \leq s.
\]
Therefore
\[
\dim(V_1 \otimes ... \otimes V_s) = \dim V_1 \cdot ... \cdot \dim V_s.
\]
Similarly, we have
\[
V_1 \otimes ... \otimes V_s = L(V_1^*, ..., V_s^*; K)
\]
and
\[
V_1^* \otimes ... \otimes V_s^* = L(V_1, ..., V_s; K).
\]

Theorem 5.3 Let \( h : V_1 \times ... \times V_s \to V_1 \otimes ... \otimes V_s, (v_1, ..., v_s) \mapsto v_1 \otimes ... \otimes v_s \) be as before. Then \( \forall f \in L(V_1, ..., V_s; Z) \), there exists a unique linear map \( g \in L(V_1 \otimes ... \otimes V_s; Z) \) such that
\[
f = g \circ h.
\]
Also,
\[
L(V_1, ..., V_s; Z) \cong L(V_1 \otimes ... \otimes V_s; Z).
\]
When \( Z = K \), we have
\[
V_1^* \otimes ... \otimes V_s^* = (V_1 \otimes ... \otimes V_s)^*.
\]
**Definition** Let $V$ be a vector space over a field $K$. Every element of

$$V^r_s := \underbrace{V \times \ldots \times V}_{r \text{ times}} \times \underbrace{V^* \times \ldots \times V^*}_{s \text{ times}}$$

is called a $(r, s)$-tensor, $r$ is called the *covariant order* and $s$ is called the *contravariant order*. In particular, elements in $V^r_0$ are called *$r$-covariant tensors*, and elements in $V^0_s$ are called the *$s$-contravariant tensors*.

We denote $V^0_0 = K$, $V^1_0 = V$, $V^0_1 = V^*$. Elements in $V$ are called *covariant vectors*, and elements in $V^*$ are called *contravariant vectors*.

Recall

$$V^r_s = L(V^*, \ldots, V^*, V, \ldots, V; K).$$

Then any $(r, s)$-tensor is a $K$-valued $(r + s)$-linear functional

$$\underbrace{V^* \times \ldots \times V^*}_{r \text{ times}} \times \underbrace{V \times \ldots \times V}_{s \text{ times}} \to K.$$

Let $\{e_i, 1 \leq i \leq n\}$ and $\{e^*_i, 1 \leq i \leq n\}$ be basis of $V$ and $V^*$, respectively, which are dual to each other. Then

$$e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{*k_1} \otimes \ldots \otimes e^{*k_s}, \quad 1 \leq i_1, \ldots, i_r, k_1, ..., k_s \leq n$$

is a basis of $V^r_s$. Hence

$$\dim V^r_s = n^{r+s}.$$

Any $(r, s)$-tensor $x$ can be written as

$$x = \sum a^{i_1 \ldots i_r}_{k_1 \ldots k_s} e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{*k_1} \otimes \ldots \otimes e^{*k_s},$$

where the coefficient

$$x^{i_1 \ldots i_r}_{k_1 \ldots k_s} = \langle e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{*k_1} \otimes \ldots \otimes e^{*k_s}, x \rangle$$

Let us use Einstein’s conversion:

$$a^i x_i = \sum_j a^i x_i.$$

We can write

$$x = x^{i_1 \ldots i_r}_{k_1 \ldots k_s} e_{i_1} \otimes \ldots \otimes e_{i_r} \otimes e^{*k_1} \otimes \ldots \otimes e^{*k_s}.$$
We notice that a \((r, s)-\text{tensor}\) is defined without using any basis (just like linear transformations). If a basis \(\{e_i\}_{1 \leq i \leq n}\) of \(V\) is changed into another basis \(\{\tilde{e}_i\}_{1 \leq i \leq n}\), then

\[
\tilde{e}_i = \alpha^j_i e_j
\]

where \(\alpha = (\alpha^j_i)\) is a non-singular \(n \times n\) matrix. And

\[
\tilde{e}^j_i = \beta_j^i e^i
\]

where \(\beta = (\beta_j^i) = \alpha^{-1}\).

Denote

\[
x = \tilde{x}^{i_1 \ldots i_r}_{k_1 \ldots k_s} \tilde{e}_{i_1} \otimes \ldots \otimes \tilde{e}_{i_r} \otimes \tilde{e}^{*k_1} \otimes \ldots \otimes \tilde{e}^{*k_s}.
\]

Then

\[
x = \tilde{x}^{i_1 \ldots i_r}_{k_1 \ldots k_s} \alpha_{i_1}^{j_1} \ldots \alpha_{i_r}^{j_r} \beta_{l_1}^{k_1} \ldots \beta_{l_s}^{k_s} e_{j_1} \otimes \ldots \otimes e_{j_r} \otimes e^{*l_1} \otimes \ldots \otimes e^{*l_s}
\]

so that

\[
x^{j_1 \ldots j_r}_{l_1 \ldots l_s} = \tilde{x}^{i_1 \ldots i_r}_{k_1 \ldots k_s} \alpha_{i_1}^{j_1} \ldots \alpha_{i_r}^{j_r} \beta_{l_1}^{k_1} \ldots \beta_{l_s}^{k_s},
\]

which is, indeed, the basic definition of a tensor in the classical tensor theory.

**Definition** Let \(x \in V^{r_1}_{s_1}, y \in V^{r_2}_{s_2}\). Define \(x \otimes y \in V^{r_1+r_2}_{s_1+s_2}\) given by

\[
x \otimes y : \underbrace{V^* \times \cdots \times V^*}_{r_1 + r_2 \text{ times}} \times \underbrace{V \times \cdots \times V}_{s_1 + s_2 \text{ times}} \rightarrow K
\]

\((v^{*1}, \ldots, v^{*r_1+r_2}, v_1, \ldots, v_{s_1+s_2}) \mapsto x(v^{*1}, \ldots, v^{*r_1}, v_1, \ldots, v_{s_1})y(v^{*r_1+1}, \ldots, v^{*r_1+r_2}, v_{s_1+1}, \ldots, v_{s_1+s_2})\)

**Definition**

\[
T^r(V) := V^r_0 = \underbrace{V \otimes \cdots \otimes V}_{r \text{ times}}
\]

\[
T(V) := \bigoplus_{r \geq 0} T^r(V). \quad \text{(direct sume)}
\]

\(\forall x \in T(V)\) can be written as

\[
x = \sum_{r \geq 0} x^r, \quad x^r \in T^r(V),
\]

where all terms except finitely many are zero.
\( T(V) \) is a vector space, \( \dim T(V) = \infty \), and \( T(V) \) has \( \otimes \) product by natural expansion. As a result,

\[
(T(V), +, \cdot, \otimes)
\]

is an algebra, called tensor algebra.

Similarly, we can define \( T(V^*) = \sum_{r \geq 0} V_r^0 \).

### 5.3 Symmetric and antisymmetric tensors

Denote by \( I(r) \) the permutation group of \( \{1, 2, \ldots, r\} \).

[Example] \( I(3) \) consists of six elements:

\[
\begin{bmatrix}
1 & 2 & 3 \\
1 & 2 & 3 \\
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 2 & 3 \\
1 & 3 & 2 \\
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 2 & 3 \\
2 & 1 & 3 \\
\end{bmatrix}, \quad
\begin{bmatrix}
1 & 2 & 3 \\
3 & 2 & 1 \\
\end{bmatrix}
\]

For each \( \sigma \in I(r) \), we define

\[
\text{Sgn}(\sigma) = \begin{cases} 
1, & \text{\( \sigma \) is even permutation;} \\
-1, & \text{\( \sigma \) is odd permutation.}
\end{cases}
\]

We define a self-homomorphism

\[
\sigma : T^r(V) \rightarrow T^r(V) \\
x \mapsto \sigma x
\]

where

\[
\sigma x = \sigma(x) : V^* \times \ldots \times V^* \rightarrow F \\
(v^*_{\sigma(1)}, \ldots, v^*_{\sigma(r)}) \mapsto x(v^*_{\sigma(1)}, \ldots, v^*_{\sigma(r)}).
\]

**Definition** \( \forall x \in T^r(V) \), if

\[
\sigma x = x, \quad \forall \sigma \in I(r),
\]

then we call \( x \) a symmetric \( r \) covariant tensor.

\( \forall x \in T^r(V) \), if

\[
\sigma x = \text{Sgn}(\sigma)x, \quad \forall \sigma \in I(r),
\]

then we call \( x \) an antisymmetric \( r \)-covariant tensor.
We define
\[ P^r(V) := \{ \text{all symmetric } x \in T^r(V) \}, \quad \Lambda^r(V) = \{ \text{all antisymmetric } x \in T^r(V) \}. \]
Both \( P^r(V) \) and \( \Lambda^r(V) \) are subspaces of \( T^r(V) \).

**Definition** \( \forall x \in T^r(V) \), we define
\[ S_r(x) := \frac{1}{r!} \sum_{\sigma \in I(r)} \sigma x, \]
and
\[ A_r(x) = \frac{1}{r!} \sum_{\sigma \in I(r)} \text{Sgn}(\sigma) \sigma x. \]
We claim that we have defined homomorphisms
\[ S_r : T^r(V) \to P^r(V), \]
and
\[ A_r : T^r(V) \to \Lambda^r(V). \]
In fact, \( \forall x \in T^r(V) \), to show: \( S_r(x) \in P^r(V) \), i.e., \( S_r(x) \) is symmetric, i.e.,
\[ \tau(S_r(x)) =?\? S_r(x), \quad \forall \tau \in I(r) \]
\[ \tau \left( \frac{1}{r!} \sum_{\sigma \in I(r)} \sigma(x) \right) = \frac{1}{r!} \sum_{\sigma \in I(r)} \sigma x \]
\[ \frac{1}{r!} \sum_{\sigma \in I(r)} (\tau \circ \sigma)(x) \]
\[ \frac{1}{r!} \sum_{\tau \sigma \in I(r)} (\tau \circ \sigma)(x) = \text{the same} \]
Next we show \( A_r(x) \in \Lambda^r(V) \) for any \( x \in T^r(V) \), i.e., \( \forall \tau \in I(r) \),
\[ \tau(A_r(x)) =?\? \text{Sgn}(\tau) \cdot A_r(x), \]
\[ \tau \left( \frac{1}{r!} \sum_{\sigma \in I(r)} \text{Sgn}(\sigma) \cdot \sigma(x) \right) = \text{Sgn}(\tau) \cdot \frac{1}{r!} \sum_{\sigma \in I(r)} \text{Sgn}(\sigma) \cdot \sigma(x) \]
\[ \frac{1}{r!} \sum_{\sigma \in I(r)} \text{Sgn}(\sigma) \cdot \tau(x) \]
\[ \frac{1}{r!} \sum_{\tau \sigma \in I(r)} \text{Sgn}(\tau \circ \sigma)(\tau \circ \sigma)(x) \]
\[ \text{Sgn}(\tau) \frac{1}{r!} \sum_{\tau \sigma \in I(r)} \text{Sgn}(\tau \circ \sigma)(\tau \circ \sigma)(x) = \text{the same} \]
Here we used the fact that
\[ Sgn(\tau)Sgn(\tau \circ \sigma) = Sgn(\sigma) \]
because \( Sgn(\tau \circ \sigma) = Sgn(\tau)Sgn(\sigma) \).

## 5.4 Exterior Algebra

Recall we have defined \( \Lambda^r(V) = \{ \text{all anti-symmetric } x \in T^r(V) \} \). Any element \( x \in \Lambda^r(V) \) is called an \( r \)-exterior vector.

**Definition** \( \forall \zeta \in \Lambda^k(V) \) and \( \eta \in \Lambda^l(V) \). We define
\[
\zeta \wedge \eta := A_{k+l}(\zeta \otimes \eta)
\]
where \( A_r(x) = \frac{1}{r!} \sum_{\sigma \in S_r} Sgn(\sigma) \cdot x \sigma \). Such \( \zeta \wedge \eta \) is an \((k+l)\)-exterior vector, called wedge product (or exterior product) of \( \zeta \) and \( \eta \).

**Theorem 5.4** \( \forall \xi, \xi_1, \xi_2 \in \Lambda^k(V), \eta, \eta_1, \eta_2 \in \Lambda^l(V), \zeta \in \Lambda^k(V) \), we have

1. \( (\xi_1 + \xi_2) \wedge \eta = \xi_1 \wedge \eta + \xi_2 \wedge \eta \).
2. \( \zeta \wedge (\eta_1 + \eta_2) = \zeta \wedge \eta_1 + \zeta \wedge \eta_2 \).
3. \( \xi \wedge \eta = (-1)^{kl} \eta \wedge \xi \).
4. \( (\xi \wedge \eta) \wedge \zeta = \xi \wedge (\eta \wedge \zeta) \).

**Proof** Skipped. □

**Remarks**

1. If \( \xi, \eta \in \Lambda^1(V) \), then
\[
\xi \wedge \eta = -\eta \wedge \xi, \quad \xi \wedge \xi = \eta \wedge \eta = 0.
\]
2. Let \( \{e_i\}_{1 \leq i \leq n} \) be a basis of \( V \). Then
\[
e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_n} \neq 0, \quad \forall i_s \neq i_t, s \neq t
\]
because \( e_{i_1} \wedge e_{i_1} \wedge e_{i_3} \wedge ... \wedge e_{i_n} = 0, \) etc.
3. When \( r > n \),
\[ e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_r} = 0, \]
because there are at least two indices \( i_s = i_t, \ s \neq t \), i.e.,
\[ \Lambda^r(V) = 0, \ \forall r > n. \]

4. Let \( \xi \) be \( r \)-exterior vector, i.e.,
\[ \xi = \xi^{i_1...i_r} e_{i_1} \otimes ... \otimes e_{i_r}, \text{ and } A_r \xi = \xi. \]

Then
\[ A_r \xi = \xi^{i_1...i_r} A_r (e_{i_1} \otimes ... \otimes e_{i_r}) = \xi^{i_1...i_r} e_{i_1} \wedge ... \wedge e_{i_r} = \xi. \]

5. \( \{ e_{i_1} \wedge ... \wedge e_{i_r}, 1 \leq i_1 < ... < i_r \leq n \} \) is a basis for \( \Lambda^r(V) \).

6. For any \( v^1, ..., v^r \in V^* \), we have a formula
\[
e_{i_1} \wedge ... \wedge e_{i_r}(v^1, ..., v^r) = \frac{1}{r!} \sum_{\sigma \in I(r)} \operatorname{Sgn}(\sigma) \langle e_{i_1}, v^{\sigma(1)} \rangle ... \langle e_{i_r}, v^{\sigma(r)} \rangle \]
\[ = \frac{1}{r!} \det \begin{pmatrix} \langle e_{i_1}, v^1 \rangle & ... & \langle e_{i_1}, v^r \rangle \\ \langle e_{i_2}, v^1 \rangle & ... & \langle e_{i_2}, v^r \rangle \\ ... & ... & ... \\ \langle e_{i_r}, v^1 \rangle & ... & \langle e_{i_r}, v^r \rangle \end{pmatrix} \]
In particular,
\[
e_{i_1} \wedge ... \wedge e_{i_r}(e^{*j_1}, ..., e^{*j_r}) = \frac{1}{r!} \det \left( \langle e_{i_1}, e^{*j_1} \rangle, ..., \langle e_{i_r}, e^{*j_r} \rangle \right) = \frac{1}{r!} \delta^{j_1...j_r}_{i_1...i_r}, \]
where
\[
\delta^{j_1...j_s}_{i_1...i_s} = \begin{cases} 
1, & i_1, ..., i_r \text{ mutually distinct, and} \\
& \{j_1, ..., j_r\} \text{ is even permutation of } \{i_1, ..., i_r\}; \\
-1, & i_1, ..., i_r \text{ mutually distinct, and} \\
& \{j_1, ..., j_r\} \text{ is odd permutation of } \{i_1, ..., i_r\}; \\
0, & \text{otherwise.}
\end{cases}
\]
When $r = n$, it implies
\[ e_1 \wedge ... \wedge e_n(e^{*1}, ..., e^{*n}) = \frac{1}{n!}, \]
and hence
\[ e_1 \wedge ... \wedge e_n \neq 0. \]
Then $e_1 \wedge ... \wedge e_n$ is a basis of $\Lambda^n(V)$, and $\dim \Lambda^n(V) = 1$. \(^1\)

When $r < n$, we can show that
\[ \{e_{i_1} \wedge ... \wedge e_{i_r}, \mid 1 \leq i_1 < i_2 < ... < i_r \leq n\} \]
is linearly independent so that it forms a basis of $\Lambda^r(V)$ with
\[ \dim \Lambda^r(V) = \binom{n}{r} = \frac{n!}{p!(n-r)!}. \]

**Definition** $\Lambda(V) = \bigoplus_{r=0}^{n} \Lambda^r(V)$ is a vector space of dimension $2^n$. \(\forall \zeta = \sum_{r=0}^{n} \zeta^r, \eta = \sum_{s=0}^{n} \eta^s \) where $\zeta^r \in \Lambda^r(V)$ and $\eta^s \in \Lambda^s(V)$, we define
\[ \zeta \wedge \eta := \sum_{r,s=0}^{n} \zeta^r \wedge \eta^s. \]

Then $\left( \Lambda(V), +, \cdot, \wedge \right)$ is an algebra, called the **exterior algebra**. A basis of $\Lambda(V)$ is
\[ \left\{1, e_{i_1}, e_{i_1} \wedge e_{i_2}, ..., e_1 \wedge ... \wedge e_n \right\}, \forall i_1, i_2, .... \]
We call an element in $\Lambda^r(V^*)$ a **$r$-exterior** form of $V$.

**Remark**

1. We define $\Lambda(V^*) = \bigoplus_{r \geq 0} \Lambda^r(V^*)$.

2. The vector spaces $\Lambda^r(V)$ and $\Lambda^s(V^*)$ are dual to each other, i.e., the bases $\{e_{i_1} \wedge ... \wedge e_{i_r}, 1 \leq i_1 < ... < i_r \leq n\}$ and $\{e^{*j_1} \wedge ... \wedge e^{*j_r}, 1 \leq j_1 < ... < j_r \leq n\}$ satisfy
\[ \left\langle e_{i_1} \wedge ... \wedge e_{i_r}, e^{*j_1} \wedge ... \wedge e^{*j_r} \right\rangle = \text{det}(\langle e_{i_\alpha}, e^{*j_\beta} \rangle) = \delta_{i_1...i_r}^{j_1...j_r} \]

---

\(^1\)For any vector bundle of rank $r$, we can define a line bundle $\Lambda^rE$. 


where
\[ \delta^{j_1, \ldots, j_r}_{i_1, \ldots, i_r} = \begin{cases} 1, & \{j_1, \ldots, j_r\} = \{i_1, \ldots, i_r\}, \\ 0, & \{j_1, \ldots, j_r\} \neq \{i_1, \ldots, i_r\}, \end{cases} \]
i.e., these bases are dual to each other.

Let \( f : V \rightarrow W \) be a linear map between vector spaces. It induces a linear map
\[
f^*: \Lambda^r(W^*) \rightarrow \Lambda^r(V^*)
\]
where the linear map defined
\[
f^*\varphi: V \times \cdots \times V \rightarrow K
\]
\[
(v_1, \ldots, v_r) \mapsto \varphi(f(v_1), \ldots, f(v_r))
\]
can extend as a linear map \( f^*\varphi \in \Lambda^r(V^*) \).

**Theorem 5.5** Let \( f : V \rightarrow W \) be a linear map. Then
\[
f^*(\varphi \wedge \psi) = f^*\varphi \wedge f^*\psi, \quad \forall \varphi \in \Lambda^r(W^*), \psi \in \Lambda^s(W^*).
\]

**Proof:** \( \forall v_1, \ldots, v_{r+s} \in V \), we want to show:
\[
f^*(\varphi \wedge \psi)(v_1, \ldots, v_{r+s}) = \frac{1}{(r+s)!} \sum_{\sigma \in I(r+s)} Sgn(\sigma) \varphi(f(v_{\sigma(1)}), \ldots, f(v_{\sigma(r)})) \cdot \psi(f(v_{\sigma(r+1)}), \ldots, f(v_{\sigma(r+s)}))
\]
\[
= \frac{1}{(r+s)!} \sum_{\sigma \in I(r+s)} Sgn(\sigma) f^*\varphi(v_{\sigma(1)}, \ldots, v_{\sigma(r)}) \cdot f^*\psi(v_{\sigma(r+1)}, \ldots, v_{\sigma(r+s)})
\]
\[
f^*\varphi \wedge f^*\psi(v_1, \ldots, v_{r+s}).
\]

**Theorem 5.6** \( v_1, \ldots, v_r \in V \) are linearly dependent \( \iff \) \( v_1 \wedge \cdots \wedge v_r = 0 \).
Proof \((\Rightarrow)\) If \(v_1, ..., v_r\) are linearly dependent and if we write, say, \(v_r = a_1v_1 + ... + a_{r-1}v_{r-1}\), then

\[
v_1 \wedge v_2 \wedge ... \wedge v_{r-1} \wedge v_r = v_1 \wedge w_2 \wedge ... \wedge v_{r-1} \wedge (a_1v_1 + ... + a_{r-1}v_{r-1}) = 0.
\]

\((\Leftarrow)\) If \(v_1, ..., v_r\) are linearly independent, then \(v_1 \wedge ... \wedge v_r \neq 0\). In fact, the set \(\{v_1, ..., v_r\}\) can be extended into a basis of \(V\): \(v_1, v_2, ..., v_r, v_{r+1}, ..., v_n\).

\(\therefore v_1 \wedge v_2 \wedge ... \wedge v_n \neq 0. \therefore v_1 \wedge ... \wedge v_r \neq 0. \quad \square\)

**Theorem 5.7** (Cartan’s lemma) Let \(v_1, ..., v_r, w_1, ..., w_r \in V\) such that

\[
\sum_{\alpha=1}^{r} v_\alpha \wedge w_\alpha = 0.
\]

If \(v_1, ..., v_r\) are linearly independent, then each \(w_\alpha\) can be written as

\[
w_\alpha = \sum_{\beta=1}^{r} c_{\alpha \beta} v_\beta, \quad 1 \leq \alpha \leq r
\]

where \(c_{\alpha \beta} = c_{\beta \alpha}\).

**Proof: \(\therefore v_1, ..., v_r\) are linearly independent.**

\(\therefore\) It can be extended into a basis of \(V\): \(\{v_1, ..., v_r, v_{r+1}, ..., v_n\}\). So

\[
w_\alpha = \sum_{\beta=1}^{r} c_{\alpha \beta} v_\beta + \sum_{i=r+1}^{n} c_{\alpha i} v_i.
\]

To show: \(c_{\alpha i} = 0, \forall i\).

Since \(\sum_{\alpha=1}^{r} v_\alpha \wedge w_\alpha = 0\), it implies

\[
\sum_{\alpha=1}^{r} v_\alpha \wedge \left( \sum_{\beta=1}^{r} c_{\alpha \beta} v_\beta + \sum_{i=r+1}^{n} c_{\alpha i} v_i \right) = 0
\]

\[
\sum_{\alpha=1}^{r} \sum_{\beta=1}^{r} c_{\alpha \beta} v_\alpha \wedge v_\beta + \sum_{\alpha=1}^{r} \sum_{i=r+1}^{n} c_{\alpha i} v_\alpha \wedge v_i
\]

\[
\sum_{1 \leq \alpha < \beta \leq r} (c_{\alpha \beta} - c_{\beta \alpha}) v_\alpha \wedge v_\beta + \sum_{\alpha=1}^{r} \sum_{i=r+1}^{n} c_{\alpha i} v_\alpha \wedge v_i.
\]
Then \( c_{\alpha\beta} - c_{\beta\alpha} = 0 \) and \( c_{\alpha i} = 0 \) so that

\[
w_\alpha = \sum_{\beta=1}^{r} c_{\alpha\beta} v_\beta
\]

with \( c_\alpha = c_{\beta\alpha} \). □

**Example** Let \( G(k, n) \) be the set of all \( k \) dimensional subspace \( L^k \subset V \). \( G(k, n) \) has natural differential structure so that \( G(k, n) \) is a manifold called *Grassmann manifold*.

Take a basis \( \{a_1, a_2, ..., a_n\} \) of \( V \). For any \( L^k \in G(k, n) \), we take \( v_1, ..., v_k \) such that \( \text{span}\{v_1, ..., v_k\} = L^k \).

Then

\[
v_1 = \sum_{j=1}^{n} v_{1j} a_j, \quad \vdots \quad v_k = \sum_{j=1}^{n} v_{kj} a_j.
\]

Then

\[
v_1 \wedge ... \wedge v_k = \sum_{1 \leq i_1 < ... < i_k \leq n} p^{i_1...i_k} a_{i_1} \wedge a_{i_2} \wedge ... \wedge a_{i_k}.
\]

Recall a formula that for any \( v_1, ..., v_k \in V \), \( w_\alpha = \sum_{\beta=1}^{k} t^\beta_\alpha v_\beta \), then

\[
w_1 \wedge ... \wedge w_k = \det(t^\beta_\alpha) v_1 \wedge ... \wedge v_k.
\]

Then for any other choice of \( v_1, ..., v_k \), say, \( v'_1, v'_2, ..., v'_k \),

\[
v_1 \wedge ... \wedge v_k \text{ and } v'_1 \wedge ... \wedge v'_k \text{ only differs a nonzero constant.}
\]

Therefore the coefficients \( \{p^{i_1...i_k}\} \) are uniquely determined up to a non zero constant, which is called the *Plucker-Grassman coordinates* \(^2\)

### 5.5 Tensor Fields

Let \( M \) be a real smooth manifold with \( \dim M = m \). For any \( p \in M \), let

\[
T^r_s(p) = \underbrace{T_p(M) \otimes ... T_p(M)}_{r} \otimes \underbrace{T^*_p(M) \otimes T^*_p(M)}_{s}
\]

---

\(^2\)We can compare the fact that each point \( x \) of \( \mathbb{CP}^n \), the homogeneous coordinates \( [x_0 : x_1 : ... : x_n] \) is determined up to a non zero constant.
is a vector space of dim $M^{r+s}$. Define
\[ T^r_s(M) = \bigcup_{p \in M} T^r_s(p). \]
$T^r_s(M)$ has topology, countable basis, Hausdorff space, and has $C^\infty$ differential structure so that $T^r_s(M)$ is a smooth manifold. We call $T^r_s(M)$ the $(r,s)$-type tensor bundle. In particular, $T^0_0(M) = T(M)$ is called the tangent bundle and $T^0_1(M) = T^*(M)$ is called the cotangent bundle.

A map $\tau : M \to T^r_s(M), p \mapsto T^r_s(M)|_p$ is called a $(r,s)$-type tensor field. A $(r,0)$-type tensor section is called a covariant tensor field and a $(0,s)$-type tensor section is called a countercovariant tensor field.

Taking a coordinate system $(U,u)$ of $M$, $\forall p \in U$, $T^r_s(M)|_p$ and $T^*_s(M)|_p$ have the bases:
\[ \left\{ \left( \frac{\partial}{\partial u^1} \right)_p, \ldots, \left( \frac{\partial}{\partial u^m} \right)_p \right\}, \left\{ (du^1)_p, \ldots, (du^m)_p \right\}. \]

Then $T^r_s(M)|_p$ has a basis
\[ \left( \frac{\partial}{\partial u^{i_1}} \right)_p \otimes \ldots \otimes \left( \frac{\partial}{\partial u^{i_r}} \right)_p \otimes (du^{j_1})_p \otimes \ldots \otimes (du^{j_s})_p, \ \forall 1 \leq i_\alpha, j_\beta \leq m. \]

In terms of local coordinates $(U,u)$, a $(r,s)$-type tensor field $\tau$ is written as
\[ \tau|_U = \tau_{j_1 \ldots j_s}^{i_1 \ldots i_r} \frac{\partial}{\partial u^{i_1}} \otimes \ldots \otimes \frac{\partial}{\partial u^{i_r}} \otimes du^{j_1} \otimes \ldots \otimes du^{j_s} \]
where $\tau_{j_1 \ldots j_s}^{i_1 \ldots i_r}$ are functions on $U$. If $\tau_{j_1 \ldots j_s}^{i_1 \ldots i_r}$ are smooth, we call $\tau$ smooth $(r,s)$-type tensor field.

We denote by $T^r_s(M)$ the set of all smooth $(r,s)$-type tensors over $M$. In particular, we denote
\[ \mathfrak{X}(M) := T^0_0(M), \ A^1(M) := T^0_1(M), \ C^\infty(M) = T^0_0(M). \]
$T^r_s(M)$ has operations of addition, scalar multiplication and tensor product, so that it forms a $C^\infty$-module.

Let $f \in C^\infty(M)$. Then $df$ is a smooth $(0,1)$-type tensor field, i.e., a smooth differential form. In fact, in terms of a local coordinate system $(U,u)$,
\[ df|_U = \sum_{j=1}^m \frac{\partial f}{\partial u^j} du^j. \]
5.6. EXTERIOR DIFFERENTIAL FORMS

Theorem 5.8 A smooth \((r, s)\)-type tensor field \(\tau\) can be identified as a \((r + s)\)-linear map

\[
\tilde{\tau}: \underbrace{A^1(M) \times \ldots \times A^1(M)}_{\text{r times}} \times \underbrace{\mathcal{X}(M) \times \mathcal{X}(M)}_{\text{s times}} \rightarrow C^\infty(M),
\]

is defined as follows: \(\forall \alpha^1, \ldots, \alpha^r \in A^1(M)\) and \(X_1, \ldots, X_s \in \mathcal{X}(M)\), \(\forall p \in M\), we have

\[
\left(\tilde{\tau}(\alpha^1, \ldots, \alpha^r, X_1, \ldots, X_s)\right)(p) = \tau(p)(\alpha^1(p), \ldots, \alpha^r(p), X_1(p), \ldots, X_s(p))
\]
such that for each variable, \(\tilde{\tau}\) is \(C^\infty(M)\)-linear.

[Example] Let \(\tau: A^1(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \rightarrow C^\infty(M)\) given by

\[
\tau(\alpha, X, Y) := \alpha[X, Y] = \alpha[XY - YX], \ \forall \alpha \in T^*(M), X, Y \in T(M).
\]

Clearly, \(\tau\) is \(\mathbb{R}\)-linear for each variable and is also \(C^\infty(M)\)-linear for the variable \(\alpha\), but \(\tau\) is not \(C^\infty(M)\)-linear for \(X\) or \(Y\). Therefore, \(\tau\) is not a smooth \((1, 2)\)-type tensor field.

5.6 Exterior differential forms

Recall \(\Lambda^r(V) = \{\text{anti-symmetric tensors}\} \subset T^r(V)\).

We call a smooth \((0, r)\)-type tensor field \(\varphi\) is anti-symmetric if \(\varphi(p)\) is an anti-symmetric tensor for any \(p \in M\).

An anti-symmetric \((0, r)\)-type tensor field is called a differential \(r\)-form (or simply a \(r\)-form). In particular any \(f \in C^\infty(M)\) is called a smooth 0-form.

We denote by \(A^r(M)\) the set of all differential \(r\)-forms on \(M\). In particular \(A^1(M) = T^0_1(M)\) and \(A^0(M) = C^\infty(M)\).

Any \(\varphi \in A^r(M)\) if and only if \(\forall p \in M\), the map

\[
\varphi(p): T_p(M) \times \ldots \times T_p(M) \rightarrow \mathbb{R}
\]
is an anti-symmetric \(r\)-linear functional, which is smooth as \(p\) varies. Equivalently,

\[
\varphi: \mathcal{X}(M) \times \ldots \times \mathcal{X}(M) \rightarrow C^\infty(M)
\]
is smooth anti-symmetric \(r\)-multiple \(C^\infty(M)\)-linear map.
We define addition, scalar multiplication, and wedge product (i.e., exterior product). For example, \( \forall \varphi \in A^r(M), \forall \psi \in A^s(M), \)

\[
(\varphi \wedge \psi)(p) = \varphi(p) \wedge \psi(p).
\]

Let

\[
A(M) = \bigoplus_{r=0}^{m} A^r(M), \quad m = \dim M.
\]

Now we define the exterior differential operator

\[
d : A(M) \rightarrow A(M)
\]

such that \( d(A^r(M)) \subset A^{r+1}(M), \forall \text{ integer } r \geq 0, \) satisfying the following conditions:

1. \( d \) is linear, i.e., \( \forall \varphi, \psi \in A(M), \lambda \in \mathbb{R}, d(\varphi + \lambda \psi) = d\varphi + \lambda d\psi; \)
2. \( \forall \varphi \in A^r(M), \psi \in A(M), \) we have

\[
d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^r \varphi \wedge d\psi.
\]
3. \( \forall f \in A^0(M), df \) is the usual differential of \( f. \)
4. \( d^2 = d \circ d = 0. \)

Let us outline the proof of existence of the exterior differential operator as follows. Let \( U \subset M \) be an open subset. If \( \varphi, \psi \in A^r(M) \) such that \( \varphi|_U = \psi|_U, \) we show

\[
d\varphi|_U = d\psi|_U.
\]

So it is sufficient to define the operator \( d \) for \( A^r(U). \) In other words, it is reduced into a local problem. Writing any \( \varphi \in A^r(U) \) as

\[
\varphi = \varphi_{i_1,...,i_r} dx^{i_1} \wedge ... \wedge dx^{i_r},
\]

we define

\[
d\varphi = d\varphi_{i_1,...,i_r} \wedge dx^{i_1} \wedge ... \wedge dx^{i_r} = \frac{\partial \varphi_{i_1,...,i_r}}{\partial x^i} dx^i \wedge dx^{i_1} \wedge ... \wedge dx^{i_r}.
\]

We can show the above definition is independent of choice of local coordinates system.
Theorem 5.9 Let $\omega \in A^1(M)$, $\forall X, Y \in (X)(M)$, we have
$$d\omega(X, Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$$

Proof: Denote by $\alpha(X, Y) := X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y]).$ The map $\alpha : (X)(M) \times (X)(M) \to C^\infty(M)$ is anti-symmetric bilinear map. For any $f \in C^\infty(M)$, we have
$$\alpha(fX, Y) = fX(\omega(Y)) - Y(f\omega(X)) - \omega(f[X, Y]).$$

Here $\omega(fX) = f\omega(X)$ holds because $\omega$ is $C^\infty(M)$-linear. Also,
$$\alpha(X, fY) = -\alpha(fX, Y) = -f \cdot \alpha(Y, X) = f \cdot \alpha(X, Y).$$

Then $\alpha$ is a 2-exterior differential form. It remains to prove: locally $\alpha \equiv d\omega$.

Take a local coordinates system $(U, x^j)$, write $\omega|_U = \omega_j dx^j$. Then
$$(d\omega)|_U = d(\omega|_U) = \frac{\partial \omega_i}{\partial x^j} dx^j \wedge dx^i = \sum_{j<i} \frac{1}{2} \left( \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) dx^j \wedge dx^i.$$ 

On the other hand, we can calculate
$$\alpha|_U = \frac{1}{2} \sum_{j<i} \left\{ \frac{\partial}{\partial x^j} \left( \omega \left( \frac{\partial}{\partial x^i} \right) \right) - \frac{\partial}{\partial x^i} \left( \omega \left( \frac{\partial}{\partial x^j} \right) \right) - \omega \left( \left[ \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i} \right] \right) \right\} dx^j \wedge dx^i$$

$$= \sum_{j<i} \frac{1}{2} \left( \frac{\partial \omega_i}{\partial x^j} - \frac{\partial \omega_j}{\partial x^i} \right) dx^j \wedge dx^i. \square$$

Similarly, we can prove
Theorem 5.10 Let $\omega \in A^r(M)$. Then for any $X_1, \ldots, X_{r+1} \in \mathfrak{X}(M)$, we have

$$d\omega(X_1, \ldots, X_{r+1}) = \sum_{\alpha=1}^{r+1} (-1)^{\alpha+1} X_\alpha(\omega(X_1, \ldots, \hat{X}_\alpha, \ldots, X_{r+1}))$$

$$+ \sum_{\alpha<\beta} (-1)^{\alpha+\beta} \omega([X_\alpha, X_\beta], X_1, \ldots, \hat{X}_\beta, \ldots, X_{r+1}).$$

Definition Let

$$Z^r(M) := \{ \alpha \in A^r(M) \mid d\alpha = 0\},$$
$$B^r(M) = dA^{r-1}(M) = \{ \alpha \in A^r(M) \mid \alpha = d\beta \text{ for some } \beta \in A^{r-1}(M)\}.$$

Elements in $Z^r(M)$ are called closed $r$-forms; elements in $B^r(M)$ are called exact differential forms. The quotient group

$$H^r(M) := Z^r(M)/B^r(M)$$

are called the de Rham cohomology groups. de Rham cohomology groups are topological invariants.

5.7 Integrals of exterior differential forms

Definition Let $M$ be a real smooth manifold. If $U = \{(U_\alpha, x_\alpha(\mid \alpha \in I)\}$ is a family of local coordinates systems satisfying the following conditions

1. $\cup_{\alpha \in I} U_\alpha = M$;
2. $\forall \alpha, \beta \in I$ with $U_\alpha \cap U_\beta \neq \emptyset$, the determinant of Jacobian matrix

$$\frac{\partial(x^1_\alpha, \ldots, x^m_\alpha)}{\partial(x^1_\beta, \ldots, x^m_\beta)} > 0,$$

then we say that $M$ is orientale.

Remark Some manifolds are not orientable, e.g., the Mobius strip. We cannot define “integrals” over non orientable manifolds.

Let $M$ be a real oriented smooth manifold with dimension $m$. For any $\omega \in A^r(M)$, we define

$$\text{supp}(\omega) := \{p \in M \mid \omega(p) \neq 0\}$$
5.7. INTEGRALS OF EXTERIOR DIFFERENTIAL FORMS

to be its support set. Denote by $A^r_0(M)$ the set of elements in $A^r(M)$ with compact support. We define a linear operator

$$
\int_M : A^m_0(M) \to \mathbb{R}
$$

by

$$
\omega \mapsto \int_M \omega.
$$

Fix a family of coordinates systems $\mathcal{U}$ which defines an orientation of $M$. Take a local coordinate system $(U, \phi) \in \mathcal{U}$. We first define the operator on $U$. Suppose $\omega \in A^m_0(M)$ with $\text{supp}(\omega) \subset U$. Then $\omega|_{M-U} = 0$ and we can write it as

$$
\omega|_U = adx^1 \wedge ... \wedge dx^m
$$

where $a \in C^\infty(U)$, $(x^i)$ are coordinates over $U$ given by the coordinate map $\varphi$. Since $\text{supp}(a) \Subset U$ is compact, the classical Riemann integral

$$
\int_{\varphi(U)} (a \circ \varphi^{-1}) dx^1 ... dx^m < \infty.
$$

Then we define

$$
\int_M \omega := \int_{\varphi(U)} (a \circ \varphi^{-1}) dx^1 ... dx^m < \infty.
$$

Such definition is independent of choice of coordinates systems. In fact, if $(V, \psi)$ is another coordinates system in $U$ such that $\text{supp}(\omega) \subset V$, we write $\omega|_V = bdy^1 \wedge ... \wedge dy^m$ where $b \in C^\infty(V)$. Then on $U \cap V$, we have

$$
adx^1 \wedge ... \wedge x^m = bdy^1 \wedge ... \wedge y^m = b \frac{\partial(y^1, ..., y^m)}{\partial(x^1, ..., x^m)} dx^1 \wedge ... \wedge dx^m.
$$

Then

$$
a \circ \varphi^{-1} = \frac{\partial(y^1, ..., y^m)}{\partial(x^1, ..., x^m)} (b \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1}). \tag{5.3}
$$

Since $(U, \varphi), (V, \psi) \in \mathcal{U}$, we know

$$
\frac{\partial(y^1, ..., y^m)}{\partial(x^1, ..., x^m)} > 0.
$$

Then by the transformation formula in Calculus:

$$
\begin{align*}
\int_{\phi(V \cap U)} adx^1 \wedge ... \wedge dx^m & \equiv ?? \\
\int_{\psi(V \cap U)} bdy^1 \wedge ... \wedge dy^m \\
\int_{\phi(V \cap U)} (a \circ \phi^{-1}) dx^1 ... dx^m & \equiv ?? \\
\int_{\psi(V \cap U)} (b \circ \psi^{-1}) dy^1 ... dy^m \\
by (5.3) & = \int_{\phi(V \cap U)} (b \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1}) \frac{\partial(y^1, ..., y^m)}{\partial(x^1, ..., x^m)} dx^1 ... dx^m
\end{align*}
$$
For the general case, fix $U = \{(U_\alpha, \phi_\alpha); \alpha \in I\}$ as before, let $\{h_\alpha\}$ be the partition of unit subordinate with $\{U_\alpha\}$. Then for any $\omega \in A^m_0(M)$, we have

$$\omega = \omega \sum_\alpha h_\alpha = \sum_\alpha (h_\alpha \cdot \omega). \quad (5.4)$$

Here $h_\alpha \omega \in A^m_0(M)$ and $\text{supp}(h_\alpha \omega) \subset \text{supp}(h_\alpha) \cap \text{supp} \omega \subset U_\alpha$. Since $\text{supp}(\omega)$ is compact, the right hand side of $(5.4)$ has finitely many terms. Then

$$\int_M \omega = \sum_\alpha \int_{U_\alpha} h_\alpha \omega = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (h_\alpha a_\alpha) \circ \phi^{-1}_\alpha dx_1^1 \cdots dx_m^m$$

where $\omega|_{U_\alpha} = a_\alpha dx_1^1 \wedge \cdots \wedge dx_m^m$ and $a_\alpha \in C^\infty(U_\alpha)$. It can be proved that the value of the integral $\int_M \omega$ defined above is independent of choice of the family $U$. $\int_M \omega$ is well defined.

Let $\omega \in A^r_0(M)$ with $r < m$ and $f : N^r \to M^m$ be a smooth embedded submanifold. Then $f^* \omega \in A^r_0(N)$ and we have

$$\int_{f(N)} \omega = \int_N f^* \omega.$$ 

### 5.8 Stokes Theorem

Let $M$ be a smooth manifold with dimension $m$. A domain with boundary is a subset $D \cup \partial D \subset M$ where $D$ is a domain and for any point $p \in \partial D$, there is a local coordinate system $(U; x)$ such that $x(0) = (x^1(0), \ldots, x^m(0)) = (0, \ldots, 0) \in \mathbb{R}^m$ and

$$U \cap D = \{q \in U \mid x^m(q) \geq 0\}. \quad (5.5)$$

It can show that such $\partial D$ is a submanifold of dimension $m - 1$ and that if $M$ is oriented, so is $\partial D$. In particular, if we fix $U = \{U_\alpha, x_\alpha; \alpha \in I\}$ with respect to the orientation and suppose $U$ include all the $(U; x)$ defining the boundary as in $(5.5)$. Then

$$(U \cap \partial D; x^j, 1 \leq j \leq m - 1)$$

is a local coordinate system for $\partial D$ and the orientation on $\partial D$ is given by

$$(-1)^m dx^1 \wedge \cdots \wedge dx^{m-1}.$$

We call such orientation on $\partial D$ the induced orientation from the one on $M$. We notice that the above definitions of orientation on $D$ and on $\partial D$ coincide with the ones for domains $D \subset \mathbb{R}^2$ or $\mathbb{R}^3$ in Calculus.
Theorem 5.11 (Stokes theorem) Let $M$ be a smooth, oriented manifold with dimension $m$. Let $D \cup \partial D \subset M$ be a domain with boundary. Then for any $\omega \in A^{m-1}_0(M)$, we have

$$\int_D d\omega = \int_{\partial D} \omega$$

where $\partial D$ has the induced orientation from $M$.

**Proof:** Skipped.

**Example** Let $D \subset \mathbb{R}^2$ be a bounded domain. Let $\omega \in A^1_0(\mathbb{R}^2)$ which is written as

$$\omega = Pdx + Qdy$$

where $P = P(x,y)$ and $Q = Q(x,y)$ are $C^1$ smooth functions defined on $D \cup \partial D$. By Stokes’ theorem,

$$\int_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy = \int_{\partial D} Pdx + Qdy.$$  

This is the classical Green formula. Here $\partial D$ has the induced orientation from the one from $D$, that is, imaging a man is walking along $\partial D$, the domain $D$ is always located in the left hand side of the man.

**Example** Let $D \subset \mathbb{R}^3$ be a bounded domain. Let $\omega \in A^2_0(\mathbb{R}^3)$ which is written as

$$\omega = Pdy \wedge dx + Qdz \wedge dx + Rdx \wedge dy$$

where $P = P(x,y,z)$, $Q = Q(x,y,z)$ and $R = R(x,y,z)$ are $C^1$ smooth functions defined on $D \cup \partial D$. By Stokes’ theorem,

$$\int_D \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz = \int_{\partial D} Pdydz + Qdzdx + Rdx dy.$$  

This is the classical Gauss (or divergent) formula. Here $\partial D$ has the induced orientation from the one from $D$.

**Example** Let $S \subset \mathbb{R}^3$ be an oriented surface and $\partial S$ be its boundary with induced orientation from the one in $S$. Let $\omega \in A^1_0(\mathbb{R}^3)$ which is written as

$$\omega = Pdx + Qdy + Rsz$$
where $P$, $Q$ and $R$ are $C^1$ smooth functions defined on $S \cup \partial S$. Then

$$d\omega = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy.$$  

By Stokes’ theorem,

$$\int_S \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

$$= \int_{\partial S} P \, dx + Q \, dy + R \, dz.$$  

This is the classical Stokes formula.
Chapter 6

Homework and solutions

6.1 HW 1

HW 1-1. $f : M \to N$ is holomorphic if and only if $f_*T^{1,0}_p(M) \to T^{1,0}_{f(p)}(N)$ for any $p \in M$.

Proof: For any $p \in M$, take coordinate systems $(U,z_U)$ of $M$ with $p \in U$ and $(V,w_V)$ of $N$ with $f(p) \in V$, we have

$$f_*(\frac{\partial}{\partial z}) g = \frac{\partial}{\partial z}(g \circ f) = \frac{\partial g}{\partial w}|_{f(p)} \frac{\partial f}{\partial z}|_p + \frac{\partial g}{\partial w}|_{f(p)} \frac{\partial \overline{f}}{\partial \overline{z}}|_p, \quad \forall g \in C^\infty(\{f(p)\}).$$

Then $f_* : T^{1,0}_p(M) \to T^{1,0}_{f(p)}(N)$ for any $p \in M$ if and only if

$$\frac{\partial g}{\partial w}|_{f(p)} \frac{\partial \overline{f}}{\partial \overline{z}}|_p = 0, \quad \forall g \in C^\infty(\{f(p)\}), \forall p \in M;$$

if and only if

$$\frac{\partial \overline{f}}{\partial \overline{z}}|_p = 0, \text{ i.e., } \frac{\partial f}{\partial z}|_p = 0, \quad \forall p \in M,$$

i.e., $f$ is holomorphic. □

HW 1-2. Prove that $f : M \to N$ is biholomorphic if and only if there are finite sets $A$ and $B$ such that $f : M - A \to N - B$ is biholomorphic.

Proof: ($\Rightarrow$) Trivial.

($\Leftarrow$) Suppose $f : M - A \to N - B$ is biholomorphic. Take any point $p \in A$. Since $N$ is compact, we can take a sequence $z_k \to p$ such that $f(z_k)$ converges to a point $\tilde{p} \in N$. By the injectivity of $f$, $\tilde{p} \in B$. 

153
CHAPTER 6. HOMEWORK AND SOLUTIONS

Take a coordinate system \((\tilde{U}, \tilde{\Phi})\) with \(\tilde{p} \in \tilde{U} \subset N - B\). We claim that there exists a coordinate system \((U, \Phi)\) with \(p \in U \subset M - A\) such that

\[
f(U) \subset \tilde{U}.
\]

(6.1)

Suppose Claim 6.1 is not true. Then there is another sequence \(z'_k \to p\) such that \(f(z'_k) \to \tilde{q}\) where \(\tilde{q} \not= \tilde{p}\). By shrinking \(\tilde{U}\) if necessary, assume \(\tilde{q} \not\in \tilde{U}\).

Since \(\partial\tilde{U}\) is compact, \(f^{-1}(\partial\tilde{U})\) is also compact closed curve. Assume that the sequences \(\{z_k\}\) and \(\{z'_k\}\) are enclosed by \(f^{-1}(\partial\tilde{U})\). Take a curve \(C(t), 0 < t < 1\), also enclosed by \(f^{-1}(\partial\tilde{U})\) such that it passes through the points \(z_k\) and \(z'_k\) and \(C(t) \to p\) as \(t \to 0\). Since \(\tilde{q} \not\in \tilde{U}\), \(f(C) \cap \partial\tilde{U} \neq \emptyset\) but this is a contradiction with the fact that \(C\) is enclosed by \(f^{-1}(\partial\tilde{U})\). Our Claim is proved.

By applying Riemann extension theorem to \(\tilde{\Phi} \circ F \circ \Phi^{-1} : \Phi(U - \{p\}) \to \tilde{\Phi}(\tilde{U} - \tilde{p})\), it implies that \(\tilde{\Phi} \circ F \circ \Phi^{-1}\) extends holomorphically across \(p\).

We have proved that \(f\) extends holomorphically on \(M\). By the same argument, \(f^{-1}\) extends holomorphically on \(N\).

From above process, we know \#\(A\) = \#\(B\) and both \(f, f^{-1}\) are onto. Since \(f|_A : A \to B\) is bijective, this implies that \(f\) is one-to-one.

6.2 HW 2

HW 2-1 Let \(\Lambda = \{m\omega + n \mid m, n \in \mathbb{Z}\}\) be a lattice where \(\omega\) is a complex number such that \(\omega\) and \(1\) are \(\mathbb{R}\)-linearly independent. Prove that the function given by

\[
p(z) := \frac{1}{z^2} + \sum_{w \in \Lambda - \{0\}} \left( \frac{1}{(z-w)^2} - \frac{1}{w^2} \right)
\]

is well defined meromorphic function satisfying

\[
p(z + w) = p(z), \quad \forall w \in \Lambda.
\]

Hence \(p\) is a meromorphic function defined on the complex torus \(\mathbb{C}/\Lambda\).

Proof: See Noguchi’s book, last chapter.
HW 3

HW 3-1 Let $M$ be a compact Riemann surface. Compute $\dim_\mathbb{C} \ell(mK)$.

Solution: We’ll show:

(i) For $g = 0$,
\[
\dim \ell(K) = \begin{cases} 
0, & \text{for } m > 0, \\
1 - 2m, & \text{for } m \leq 0.
\end{cases}
\]

(ii) For $g = 1$, $\dim \ell(K) = 1 \forall m$.

(iii) For $g \geq 2$,
\[
\dim \ell(K) = \begin{cases} 
0, & \text{for } m < 0, \\
1, & \text{for } m = 0, \\
(2m - 1)(g - 1), & \text{for } m > 1.
\end{cases}
\]

Let $D = mK$, by the Riemann-Roch theorem, we have
\[
\dim \ell(mK) - \dim \ell((1 - m)K) = \deg(mK) + (1 - g).
\]

Since $\deg(K) = 2(g - 1)$, the above is written as
\[
\dim \ell(mK) - \dim \ell((1 - m)K) = (2m - 1)(g - 1). \tag{6.2}
\]

We also know $\dim \ell(K) = g$ (see page 31 in the notes).

Proof of (i): Consider $g = 0$.

If $m > 0$, since $\deg(K) = 2g - 2 = -2$, we can find a meromorphic 1-form $\omega$ such that the induced divisor $K = (\omega) = \sum_k n_{pk}p_k - \sum_s n_{qs}q_s$ such that $\deg(\omega) = \sum_k n_{pk} - \sum_s n_{qs} = -2$.

For any $f \in \ell(mK)$, by the definition, $(f) + m \sum_k n_{pk}p_k - m \sum_s n_{qs}q_s \geq 0$ so that

$f$ has zeros at $q_s$ with order $\geq mn_{qs}$, and $f$ has poles at $p_k$ with order $\leq mn_{pk}$.

It implies
\[
\deg(f) \geq m \sum_s n_{qs} - m \sum_k n_{pk} = 2m > 0.
\]

But this contradicts with the fact that $\deg(f) = 0$. Thus such $f$ does not exist. Then
\[
\dim \ell(mK) = 0, \forall m > 0.
\]
If \( m = 0 \), \( \dim \ell(0) = \dim \mathbb{C} = 1 \).

If \( m < 0 \), by (6.2), \( \dim \ell(mK) = \dim \ell((1 - m)K) + 1 - 2m = 0 + 1 - 2m \). Here we used the fact that \( \dim \ell(mK) = 0 \), \( \forall m > 0 \).

Proof of (ii): Consider \( g = 1 \). By Lemma 1.25, there is holomorphic 1-form \( \omega \) such that \( \omega(p) \neq 0 \) for some \( p \in M \). Since \( \deg(K) = d(\omega) = 2g - 2 = 0 \) and \( \omega \) is holomorphic, it implies

\[
\omega(p) \neq 0, \quad \forall p \in M. \tag{6.3}
\]

If \( m > 0 \), for any \( f \in \ell(mK) \), we have \( (f) + m(\omega) \geq 0 \iff (f) \geq 0 \) by (6.3), i.e., \( f \in \text{Hol}(M) \), i.e., \( f = \text{constant} \). Thus

\[
\dim \ell(mK) = 1, \quad \forall m \geq 1.
\]

If \( m = 0 \), \( \dim \ell(0) = \dim \mathbb{C} = 1 \).

If \( m < 0 \), by (6.2) and the fact that \( \dim \ell(mK) = 1, \quad \forall m \geq 1 \), we obtain \( \dim \ell(mK) = 0 \).

Proof of (iii): Consider \( g > 1 \). Since \( \deg(K) = 2g - 2 > 0 \), there is a nontrivial holomorphic 1-form \( \omega \) on \( M \). Write \( (\omega) = \sum n_{p_j}p_j \) with \( n_{p_j} > 0 \) and \( \sum_j n_{p_j} = 2g - 2 \).

If \( m < 0 \), for any \( f \in \ell(mK) \), it satisfies \( (f) + m(\omega) \geq 0 \) so that

\[
f \text{ has zeros at } p_j \text{ with order } \geq |m|n_{p_j}, \text{ and } f \text{ has no pole}
\]

because \( m \) is negative. Then \( \deg(f) \geq |m| \sum_j n_{p_j} > 0 \) but this contradicts with \( \deg(f) = 0 \). Hence such \( f \) does not exists so that \( \dim \ell(mK) = 0, \forall m < 0 \).

If \( m = 0 \), \( \dim \ell(0) = \dim \mathbb{C} = 1 \).

If \( m = 1 \), recall \( \dim \ell(K) = g \).

If \( m > 1 \), by (6.2), \( \dim \ell(mK) - 0 = (2m - 1)(g - 1) \). \( \Box \)

**HW 3-2** If \( \deg(D) = 1 \) and \( g \geq 1 \) on a compact Riemann surface, prove \( \dim \ell(D) \leq 1 \).
Proof: Write $D = p_1 + \sum n_j q_j$. Since $\deg(D) = 1$, we have $\deg(\sum n_j q_j) = 0$ so that we can write $D = p_1 + (g)$ for some $g \in \mathcal{M}(M)$. Then $\ell(D) = \ell((p_1))$. If $\dim \ell((p_1)) \geq 2$, $M$ is biholomorphic to $\mathbb{CP}^1$ but it has $g = 0$. Hence $\dim \ell((p_1)) \leq 1$. □

HW 3-3 If $D_1 \sim D_2$, then $\ell(D_1) \cong \ell(D_2)$ and $i(D_1) \cong i(D_2)$.

Proof: Since $i(D) \cong i(K - D)$, it suffices to prove $\ell(D_1) \cong \ell(D_2)$. Write $D_1 = D_2 + (h)$ where $h \in \mathcal{M}(M)$. We find

$$
\ell(D_1) = \{ f \in \mathcal{M}(M) \mid (f) + D_1 \geq 0 \} = \{ f \in \mathcal{M}(M) \mid (f) + D_2 + (h) \geq 0 \}
$$

$$
= \{ f \in \mathcal{M}(M) \mid (fh) + D_2 \geq 0 \}.
$$

Then we have a map

$$
\ell(D_1) \to \ell(D_2), \ f \mapsto fh
$$

which is an isomorphism. □

6.4 HW 4

HW 4-1 For the sequence of group homomorphism

$$
C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_1} C^2(\mathcal{U}, \mathcal{F}) \xrightarrow{\delta_2} \ldots
$$

Prove that $\delta_q \circ \delta_{q-1} = 0$.

HW 4-2 If

$$
0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{G} \to 0
$$

is a short exact sequence of sheaf maps and if $H^1(M, \mathcal{E}) = 0$, prove that the sequence of homomorphisms of the groups of global sections of the sheaves

$$
0 \to \mathcal{E}(M) \xrightarrow{\alpha} \mathcal{F}(M) \xrightarrow{\beta} \mathcal{G}(M) \to 0
$$

is also exact.
\section{HW 5}

**HW 5-1** Let $L = M \times \mathbb{C}$ be the trivial line bundle over a Riemann surface $M$. Show that there is a Hermitian metric on $L$ such that its curvature $\equiv 0$.

\textbf{Proof:} Since the line bundle is trivial. We can assume $L = M \times \mathbb{C}$ and $\pi : L = M \times \mathbb{C} \to M$, $(x, \lambda) \mapsto x$. In this case, $L \leftrightarrow \{W_\alpha, f_{\alpha\beta}\} = \{M, 1\}$, i.e., $W_\alpha = M$ and one function $f_{\alpha\beta} = 1$. Then $e_\alpha = 1$ and $g_\alpha = \langle e_\alpha, e_\alpha \rangle = 1$. Hence the curvature $\Theta_\alpha = \overline{\partial \partial \log g_\alpha} = 0$. \(\square\)

\textbf{Another proof:} Since $L$ is trivial, we can write $L \leftrightarrow \{W_\alpha, f_{\alpha\beta}\}$ where $f_{\alpha\beta} = \frac{f_\beta}{f_\alpha}$ and $f_\alpha \in \mathcal{O}^*(W_\alpha)$. Then we define a metric $g_\alpha := |f_\alpha|^2$ on $W_\alpha$. It is well-defined because $g_\alpha > 0$ and it satisfies $g_\alpha = |f_{\alpha\beta}|^2 g_\alpha$, $\forall W_\alpha \cap W_\beta$. The curvature $\Theta_\alpha = \overline{\partial \partial \log g_\alpha} = \overline{\partial \partial |f_\alpha|^2} = 0$, on $W_\alpha$ because $f_\alpha$ never vanish. \(\square\)

**HW 5-2** Show: the operators $*$ and $^{-1}$ are adjoint to each other.

\textbf{Proof:} $\forall \phi_j \in A^{p,q}(L)$, write $\phi_j = \omega_j^{(\alpha)} s_j^{(\alpha)}$ on $W_\alpha$, $j = 1, 2$.

\begin{align*}
(*\phi_1, \phi_2) &= \sum_\alpha \int_{W_\alpha} \chi_\alpha \langle s_1^{(\alpha)}, s_2^{(\alpha)} \rangle_H \ast \omega_1^{(\alpha)} \wedge \overline{\omega_2^{(\alpha)}} \\
&= \sum_\alpha \int_{W_\alpha} \chi_\alpha \langle s_1^{(\alpha)}, s_2^{(\alpha)} \rangle_H \ast \omega_1^{(\alpha)} \wedge \overline{\omega_2^{(\alpha)}}
\end{align*}

Here we used the formula $\varphi \wedge \ast \psi = G(\varphi, \psi) \Omega, \forall \varphi, \psi$. Also

\begin{align*}
(\phi_1, \ast^{-1}\phi_2) &= \sum_\alpha \int_{W_\alpha} \chi_\alpha \langle s_1^{(\alpha)}, s_2^{(\alpha)} \rangle_H \omega_1^{(\alpha)} \wedge \ast^{-1} \circ \ast \omega_2^{(\alpha)} \\
&= \sum_\alpha \int_{W_\alpha} \chi_\alpha \langle s_1^{(\alpha)}, s_2^{(\alpha)} \rangle_H \omega_1^{(\alpha)} \wedge \overline{\omega_2^{(\alpha)}}
\end{align*}

Here we used the fact: $G(\varphi, \psi) = G(\ast \varphi, \ast \psi), \forall \varphi, \psi$. \(\square\)
6.6 HW 6

HW 6-1 If \( f_1, \ldots, f_m \in \mathcal{O}(\Omega) \) where \( \Omega \subset \mathbb{C}^n \) is open, and if \( \alpha_1, \ldots, \alpha_m \) are positive real numbers, prove

\[
\log(|f_1|^{\alpha_1} + \ldots + |f_m|^{\alpha_m}) \in PSH(\Omega).
\]

(Hint: Show that \( \chi(t_1, \ldots, t_n) = \log(e^{t_1} + \ldots + e^{t_n}) \) is convex. Then replace \( t_1 = \alpha_1 \log|f_1| \), \( \ldots, t_n = \alpha_n \log|f_n| \) and use Remark 7 in page 105)