Math 1431

Section 14839
M TH 4:00 PM-5:30 PM Online

Susan Wheeler
swheeler@math.uh.edu

Office Hours:
5:30 - 6:15 pm M Th Online or by appointment
Wed 6:00 – 7:00 PM Online
Important Dates This Week

9-8: Quiz 3, HW 2, EMCF 2
* John 6:30-8:30 PM are all at 11:59 PM
9-9: Lab has problem help session online

9-11: Quiz 4 due at 11:59 PM

9-12: Popper 2, Lab Quiz 3
The theorem states: If \( f(x) \) is continuous on the closed interval \([a, b]\) and \( N \) is a real number such that \( f(a) \leq N \leq f(b) \), then there is at least one value \( c \) in \((a, b)\) so that \( f(c) = N \).

Example 1: Use the intermediate value theorem (IVT) to show that there is a solution to the given equation in the indicated interval.

a. \( 204 - 3x = 0 \) on \([2, 4]\)

b. \( 2\tan 1 - x = 0 \) on \([0, 4]\)

We can also use the IVT to prove the existence of roots/zeros/x-intercepts of a function.

If \( f(a) \) is \((+) \) and \( f(b) \) is \((-) \), yes a root on \([a, b]\).

Does \( f(x) \) have a root on \([a, b] \), \( f(x) \) is cont.

If \( f(a) \) is \((-) \) and \( f(b) \) is \((+) \), yes there is a root on \([a, b]\).
The Extreme Value Theorem

If a function $f$ is continuous on a bounded interval $[a,b]$. Then $f$ takes on both a maximum value and a minimum value. \( \text{on } [a,b] \)

If the function in not continuous in the interval, it may or may not have a minimum or maximum value in that interval. \( \Rightarrow \) \text{No guarantee}
The Derivative

Measuring how $f(x)$ changes when $x$ changes.

Section 2.1
Slope of a Secant line — slope between two points — average rate of change

\[
m_s = \frac{f(b) - f(a)}{b - a} = \frac{\Delta y}{\Delta x}
\]

of \( f(x) \) between \( x = a \) and \( x = b \)
Slope of a Tangent line – slope at A point – instantaneous rate of change of $f(x)$ at point A
$y = f(x)$

$m = \text{instantaneous rate of change of } f(x) \text{ at } x = a$
$y = f(x)$

Slope of Secant = $\frac{f(b) - f(a)}{b - a}$
$y = f(x)$

$h = b - a$

$b = a + h$

Secant Line
$y = f(x)$

Secant Line
As \( h \) gets smaller the secant line is becoming a better approximation to the tangent at point \( A \) on \( f(x) \).

\[
m = \frac{f(b) - f(a)}{b - a} = \frac{f(a+h) - f(a)}{h}
\]

\( b = a + h \quad \Rightarrow \quad h = b - a \)
\[ y = f(x) \]

**Tangent Line**

Distance from 
\[ a \rightarrow b \rightarrow 0 \]
\[ b - a \rightarrow 0 \]
\[ h \rightarrow 0 \]

\[ m = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} \]

The same
\[ a+h \]

\[ \frac{f(b) - f(a)}{b - a} \]

Tangent Line
Slope of the Tangent Line at the point $x = a$ is

$$\lim_{{h \to 0}} \frac{f(a + h) - f(a)}{h}$$

Which is

the \textit{instantaneous rate of change} of $f$ at $x = a$

Which is

the \textit{derivative} of $f$ at $x = a$
PreCal Quick Review:

For $f(x) = 2x + 3$

$f(a) = 2a + 3$  
$f(3) = 2(3) + 3 = 9$

$f(a + h) = 2(a + h) + 3$  
$= 2a + 2h + 3$

$f(3 + h) = 2(3 + h) + 3 = 6 + 2h + 3 = 9 + 2h$

For $f(x) = x^2 + 1$

$f(a) = a^2 + 1$  
$f(3) = 3^2 + 1 = 10$

$f(a + h) = (a + h)^2 + 1$  
$= a^2 + 2ah + h^2 + 1$

$f(3 + h) = (3 + h)^2 + 1 = 9 + 6h + h^2 + 1 = 10 + 6h + h^2$
Using what we know, find the slope of the tangent line at \( x = 3 \) if \( f(x) = x^2 + 1 \).

\[
\begin{align*}
f(3) &= 10, \\
f(3 + h) &= (3 + h)^2 + 1, \\
&= h^2 + 6h + 10
\end{align*}
\]

Slope of the tangent line

\[
\text{slope} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

\[
\begin{align*}
&= \lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 6h + 10 - 10}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 6h}{h} \\
&= \lim_{h \to 0} \frac{h(h + 6)}{h} \\
&= \lim_{h \to 0} (h + 6) = 0 + 6 = 6
\end{align*}
\]

\[**\text{Slope of the tangent line at } x = 3**\]
The Definition of the Derivative

A function \( f(x) \) is differentiable at \( x \) if and only if

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

exists. In this case, we denote

\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
\]

and we refer to \( f'(x) \) as the derivative of \( f \) at \( x \).
\( f'(x) \) can be thought of as the slope function. It gives the slope of the graph of \( f(x) \) at any point \( x \).

You may also see \( f'(x) \) written as \( \frac{df}{dx} \).

If \( y = f(x) \) you may see \( f'(x) = y' = \frac{dy}{dx} \).
Find $f'(x)$ where $f(x) = x^2 - 2x$ using the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^2 - 2(x+h) = x^2 + 2xh + h^2 - (2x + 2h)$$

$$f(x+h) = x^2 + 2xh + h^2 - 2x - 2h$$

$$f'(x) = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - (x^2 - 2x)}{h} =$$

$$f'(x) = \lim_{h \to 0} \frac{2xh + h^2 - 2h}{h} = \lim_{h \to 0} \frac{2x + h - 2}{h} =$$

$$= \lim_{h \to 0} 2x + h - 2 = 2x - 2 = f'(x)$$
Use the previous result to give the equation of the tangent line to the graph of \( f(x) = x^2 - 2x \) at \( x = -1 \).

Slope of the tangent at \( x = -1 \):

\[
\begin{align*}
\text{Slope} = f'(x) &= 2x - 2 \\
\end{align*}
\]

\[
\begin{align*}
f'(-1) &= 2(-1) - 2 = -4 = m \\
\end{align*}
\]

Point:

\[
\begin{align*}
\text{Point} &= (-1, f(-1)) = (-1, 3) \\
\end{align*}
\]

\[
\begin{align*}
f(-1) &= (-1)^2 - 2(-1) = 3 \\
\end{align*}
\]

Line:

\[
\begin{align*}
(y - y_1) &= m(x - x_1) \\
y - 3 &= -4(x + 1) \\
y - 3 &= -4x - 4 \\
y &= -4x - 1
\end{align*}
\]

Eqn of Tangent Line at \( x = -1 \):

\[
\begin{align*}
y &= -4x - 1
\end{align*}
\]
Find $f'(x)$ where $f(x) = x^3 - 2$ using the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f(x+h) = (x+h)^3 - 2 = x^3 + 3x^2h + 3xh^2 + h^3 - 2$$

$$f'(x) = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - 2 - (x^3 - 2)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2)}{h}$$

$$f'(x) = \lim_{h \to 0} 3x^2 + 3xh + h^2 = 3x^2 = f'(x)$$
\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

Find \( f'(x) \) where \( f(x) = \frac{1}{1+x} \) using the definition of the derivative.

\[ f(x+h) = \frac{1}{1+x+h} \]

\[ f'(x) = \lim_{h \to 0} \frac{1}{1+x+h} \left( \frac{1+x}{1+x} \right) - \frac{1}{1+x} \left( \frac{1+x+h}{1+x+h} \right) \]

\[ f'(x) = \lim_{h \to 0} \frac{1+x-1-x-h}{(1+x+h)(1+x)} \]

\[ = \lim_{h \to 0} \frac{-h}{(1+x+h)(1+x)} \]

\[ = \lim_{h \to 0} \frac{-1}{(1+x+h)(1+x)} \]

\[ = \frac{-1}{(1+x)^2} \]
If $f$ is differentiable at $x = a$, then $f$ is continuous at $x = a$.

**But**

Not every continuous function is differentiable.

*Just because it's continuous does not mean it's differentiable*

Example: The function $y = |x|$ is continuous but not differentiable at $x = 0$. 

![Graph showing the function $y = |x|$ with $m = -1$ and $m = 1$ at $x = 0$.]
How can the graph of a function be used to determine where a function is not differentiable?

A function is not differentiable at

1. points of discontinuity
2. cusps
3. sharp turns (corners)
Determine if \( f(x) \) is differentiable at \( x = 2 \).

**Yes continuous \( \Rightarrow \) for continuity**

1) \( f(2) = 2^2 + 1 = 5 \)

2) \( \lim_{x \to 2^+} 4x - 3 = 8 - 3 = 5 \)

3) \( f(2) = \lim_{x \to 2} f(x) = 5 \)

Now see if the right sided deriv = left sided deriv

\( f(x+h) = (x+h)^2 + 1 \)

\( f'(x) \) for \( x \leq 2 \) (Left-Sided)

\[
f'(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}
\]

\[
= \lim_{h \to 0^-} \frac{x^2 + 2xh + h^2 + 1 - (x^2 + 1)}{h}
\]

\[
= \lim_{h \to 0^-} \frac{2xh + h^2}{h}
\]

\[
= \lim_{h \to 0^-} (2x + h) = 2x
\]

\( f'(x) \) for \( x > 2 \) (Right-Sided)

\[
= \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}
\]

\[
= \lim_{h \to 0} \frac{4(x+h) - 3 - (4x - 3)}{h}
\]

\[
= \lim_{h \to 0} \frac{4h}{h} = 4
\]
\[
= \lim_{h \to 0} \frac{4x + 4h - 3 - 4x + 3}{h} \\
= \lim_{h \to 0} \frac{4h}{h} = \lim_{h \to 0} 4 = 4
\]

\[
f'_-(x) = 2x \quad \{ x = 2 \} \quad f'_-(2) = 2(2) = 4
\]

\[
f'_+(x) = 4 \quad \Rightarrow \quad x = 2 \quad f'_+(2) = 4
\]

\[
f'_-(2) = f'_+(2)
\]

Differentiable \quad Yes
Determine if \( f(x) \) is differentiable at \( x = 1 \).

\[
f(x) = \begin{cases} 
  x & x \leq 1 \\
  x^2 & x > 1
\end{cases}
\]

\[
f(x) = \begin{cases} 
  x+h & x \leq 1 \\
  (x+h)^2 & x > 1
\end{cases}
\]

1) \( \checkmark \) Continuity
   - \( f'(1^-) = f'(1^+) \)

\[
f'(x) = \lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0^-} \frac{x+h-x}{h} = \lim_{h \to 0^-} \frac{h}{h} = \lim_{h \to 0^-} 1 = 1
\]

\( f'(x) = 1 \)

2) \( \checkmark \) \( f(x) = \lim_{x \to 1} x = 1 \)
   - \( \lim_{x \to 1^-} x^2 = 1^2 = 1 \)
   - \( \lim_{x \to 1^+} x^2 = 1^2 = 1 \)

\( f(x) = \lim_{x \to 1} x^2 = 1 \)

3) \( \checkmark \) \( f(x) = \lim_{x \to 1} f(x) = 1 \)
   - \( f(1) = \lim_{x \to 1} x^2 = 1 \)
   - \( \text{Yes continuous} \)
\[ f'(x) = \lim_{h \to 0^+} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0^+} \frac{2xh + h^2}{h} = \lim_{h \to 0^+} h(2x + h) = \lim_{h \to 0^+} 2x + h = 2x = f'_+(x) \]

\[ f'(x) = \begin{cases} 
1 & x \leq 1 \\
2x & x > 1 
\end{cases} \]

At \( x = 1 \), \( f'_-(1) = 1 \) and \( f'_+(1) = 2 \).

At \( x = 1 \), \( f'_-(1) \neq f'_+(1) \).

\( f(x) \) is \text{ Not Differentiable at } x = 1.
How can we use the derivative to find the slope of the normal line to the graph of \( f(x) \) at \( x = a \)?

The normal line to the graph at \( x = a \) is the perpendicular line to the graph at \( x = a \).

That is:

The normal line is perpendicular to the tangent line at \( x = a \).

\[
m_T = 2
\]

\[
m_N = -\frac{1}{2}
\]
Algebraic Properties of the Derivative

Differentiation Formulas

Section 2.2
If $f$ and $g$ are differentiable and $c$ is a scalar, then $f + g$, $f - g$ and $(c f)$ are differentiable. Furthermore,

Derivative of the sum is the sum of the derivatives.

\[
\frac{d}{dx} \left( f(x) + g(x) \right) = \frac{d}{dx} f(x) + \frac{d}{dx} g(x)
\]

Derivative of the difference is the difference of the derivatives.

\[
\frac{d}{dx} \left( f(x) - g(x) \right) = \frac{d}{dx} f(x) - \frac{d}{dx} g(x)
\]

And the derivative of any scalar times a function is the scalar times the derivative of the function.

\[
\frac{d}{dx} \left( c f(x) \right) = c \frac{d}{dx} f(x)
\]
\[
\frac{d}{dx} 8 = \\
\frac{d}{dx} x = \\
\frac{d}{dx} (5x) = \\
\frac{d}{dx} (5x + 2) = 
\]
Power Rule

\[ \frac{d}{dx} \left( x^n \right) = nx^{n-1}, \ n \neq 0 \]

Find the derivative of each.

\[ f(x) = x^2 \]

\[ f(x) = x^3 \]

\[ f(x) = x^5 - x^2 \]
\[ f(x) = 3x^4 + 2x^3 - 4x \]

\[ f(x) = \sqrt{x} = x^{\frac{1}{2}} \]

\[ f(x) = x^{\frac{9}{7}} + x^{\frac{5}{7}} \]

\[ f(x) = \frac{1}{x^2} \]
Higher Order Derivatives

\[ f'(x), \quad f''(x), \quad f'''(x), \quad f^{(4)}(x) \]

\[ \frac{d}{dx} f(x), \quad \frac{d^2}{dx^2} f(x), \quad \frac{d^3}{dx^3} f(x), \quad \frac{d^4}{dx^4} f(x) \]
Determine $\frac{d^2}{dx^2}(3x^3 - 5x^2 + 2x - 1)$

Determine $\frac{d^3}{dx^3}(3x^8 + 2x^5 - 3x - 5)$
Trig Derivatives:

\[
\frac{d}{dx} \sin x = \cos x \\
\frac{d}{dx} \cos x = -\sin x \\
\frac{d}{dx} \tan x = \sec^2 x \\
\frac{d}{dx} \cot x = -\csc^2 x \\
\frac{d}{dx} \sec x = \sec x \cdot \tan x \\
\frac{d}{dx} \csc x = -\csc x \cdot \cot x
\]

MEMORIZE THESE!