A NEW PROOF OF TORELLI'S THEOREM

by

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Let $X$ be a complete, nonsingular curve of genus $g > 1$. Let $\varphi: X \to J(X)$ be a canonical map of $X$ into its Jacobian variety, $J(X)$. Assume $\varphi$ chosen so that $\varphi(P) = 0$ for some point, $P \in X$.

If $D = \Sigma d_i Q_i$ is a divisor on $X$, we define $\varphi(D) = \Sigma d_i \varphi(Q_i)$. The image under $\varphi$ of the positive divisors of degree $\leq r$ on $X$ will be denoted by $W^r$, and we extend this definition by setting $W^0 = \{0\}$.

It is known[2] that $W^1$ is birationally equivalent to $X$, and that $W^{g-1}$ determines the canonical polarization of $J(X) = W^g$. The object of this paper is to prove that $W^1$ is determined up to a translation and reflection by $J(X)$ and $W^{g-1}$, (i.e., $X$ is determined up to a birational equivalence by the same data).

A classical version of this theorem was proved by Torelli.[4] Weil[5] gave a modern proof, valid in the abstract case, based on an idea of Andreotti. Other abstract proofs were later given by Matsusaka[3] and Andreotti.[1]

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The proof to be given here is based on a modification of
two of Weil's lemmas which enables us to recover
Torelli's theorem as a combinatorial consequence of the
Riemann-Roch theorem and Abel's theorem.

We begin by proving four preliminary lemmas of
which the second and fourth may be characterized as
modifications of Weil's Hilfssätze 3 and 1, respectively.
Lemmas 2, 3, and 4 admit generalizations which, however,
are not needed for our purposes.

We denote, as usual, by \( W^r_a \) the translate of \( W^r \)
by an element \( \alpha \in J(X) \). Following Weil,[5] we denote by \( (W^r_a)^* \)
the image of \( W^r_a \) under the map \( u \mapsto u + \varphi(Z) \) where \( Z \) is a
canonical divisor on \( X \). We recall [2] that the sets \( W^r_a \) and
\( (W^r_a)^* \) are subvarieties of \( J(X) \).

Our first lemma is a known result which we prove
for convenience:

**Lemma 1**

\[ (W^g_{a-1})^* = W^g_{a-1} \]

**Proof:** Given a positive divisor, \( D \), of degree \( (g-1) \), there
exists a positive divisor \( D' \), of degree \( (g-1) \) such that
\( D + D' \sim Z \), where \( \sim \) denotes linear equivalence. By Abel's
theorem

\[ \varphi(D) - a = - \varphi(D') + \varphi(Z) - a \]

As the left-hand side traverses \( W^g_{a-1} \) the right-hand side
traverses \( (W^g_{a-1})^* \), and conversely.
Lemma 2

Let

\[ 0 \leq r \leq g-1. \]

Then

\[ W_a^r \subset W_b^{g-1} \iff a \in W_b^{g-1-r}. \]

Proof: The implication from right to left is trivial.

Assume now that \( W_a^r \subset W_b^{g-1} \). This means that for every positive divisor, \( D \), of degree \( r \), there is a positive divisor, \( \hat{D} \), of degree \( g-1 \), such that \( \phi(D) + a = \phi(\hat{D}) + b \). In particular, there is a positive divisor, \( A \), of degree \( g-1 \), such that \( a = \phi(A) + b \). Hence, \( \phi(D) + \phi(A) = \phi(\hat{D}) \), and, by Abel's theorem

\[ D + A \sim \hat{D} + rP. \]

Let \( A' \) and \( \hat{D}' \) be positive divisors of degree \( g-1 \) such that \( A + A' \) and \( \hat{D} + \hat{D}' \) are canonical divisors. Then

\[ D + \hat{D}' \sim A' + rP. \]

Since an equivalence of this form must hold for all positive divisors, \( D \), of degree \( r \), it follows* that \( \ell(A' + rP) \geq r+1. \)

*By \( \ell(D) \) we denote the dimension of the (linear) space of functions whose divisors are \( \geq -D \).
By the Riemann-Roch theorem it follows that \( \ell(Z - A' + rP) \geq 1 \). Hence, there is a positive divisor, \( \hat{A} \), of degree \( g-1-r \) such that \( A' + rP + \hat{A} \sim Z \), whence \( \varphi(A) = \varphi(\hat{A}) \). But then

\[
a = \varphi(\hat{A}) + b \in W^{g-1-r}_b.
\]

**Lemma 3**

Let

\[
0 \leq r \leq g-1.
\]

Then

\[
W^{g-1-r} = \cap \left\{ W^{g-1}_{-u} : u \in \mathbb{W}^r \right\}
\]

and

\[
(W^{g-1-r})^* = \cap \left\{ W^{g-1}_{+u} : u \in \mathbb{W}^r \right\}.
\]

**Proof:** By Lemma 2,

\[
W^{g-1-r} \subseteq W^{g-1}_{-u} \leftarrow W^{g-1-r}_u \subseteq W^{g-1} \leftarrow u \in \mathbb{W}^r.
\]

Hence

\[
W^{g-1-r} \subseteq \cap \left\{ W^{g-1}_{-u} : u \in \mathbb{W}^r \right\}.
\]

On the other hand, if \( v \in W^{g-1}_v \) for all \( u \in \mathbb{W}^r \), then \( u \in W^{g-1}_{-v} \) for all \( u \in \mathbb{W}^r \), whence \( W^r \subseteq W^{g-1}_v \) and \( v \in W^{g-1-r} \), by Lemma 2. This proves the first formula, and the second formula follows from the equation.
\[ \bigcap \{ w_{+u} : u \in W^r \} = \bigcap \{ (w_{-u}^{-1})^* : u \in W^r \} = \left( \bigcap \{ w_{-u}^{-1} : u \in W^r \} \right)^* . \]

**Lemma 4**

Let

\[ 0 \leq r \leq g-1 . \]

Let a and b be related by an equation, \( b = a + x - y \), where \( x \in W^1 \) and \( y \in W^{g-1-r} \). Then either \( W_a^{r+1} \subset W_b^{g-1} \), or else

\[ W_a^{r+1} \cap W_b^{g-1} = W_a^r \cup S \]

where

\[ S = W_a^{r+1} \cap (W_{y-a}^{g-2})^* . \]

**Proof:** By assumption, \( x = \varphi(R) \), \( y = \varphi(\hat{R}) \) and \( \varphi(R) + a = \varphi(\hat{R}) + b \), where R and \( \hat{R} \) are positive divisors of degrees 1 and \( g-1-r \), respectively. If \( R \) is a point of \( \hat{R} \), we get an equation \( a = \varphi(R') + b \), where \( \deg(R') = g-2-r \). But then \( a \in W_{b}^{g-2-r} \) and \( W_a^{r+1} \subset W_b^{g-1} \). Hence we assume that \( R \) is not a point of \( \hat{R} \).

Let \( u \in W_a^{r+1} \cap W_b^{g-1} \). Then there are positive divisors, D and \( \hat{D} \), of degrees \( r+1 \) and \( g-1 \), respectively, such that \( u = \varphi(D) + a = \varphi(\hat{D}) + b \). Hence

\[ D + \hat{R} \sim \hat{D} + R . \]
If \( D + \mathcal{R} = \mathcal{D} + R \), R must be a point of \( D \) and
\[
u = \varphi(D) + a = \varphi(D') + \varphi(R) + a, \text{ where } \deg(D') = r. \text{ Then} \]
\( u \in W^r_{a+x} \).

If \( D + \mathcal{R} \neq \mathcal{D} + R \), then \( t(D + \mathcal{R}) \geq 2 \), and, given any point, \( Q \in X \), there is a positive divisor, \( \mathcal{Q} \), of degree \( g-1 \), such that \( D + \mathcal{R} \sim \mathcal{Q} + \mathcal{Q} \). Then
\[
u = \varphi(D) + a = \varphi(\mathcal{Q}) + \varphi(\mathcal{Q}) - \varphi(\mathcal{R}) + a,
\]
whence
\[
u \in \left\{ \frac{w^{g-1}_{a-y+v}}{v \in W^1_{a-y}} \right\} = \left( \frac{w^{g-2}}{y-a} \right)^*.
\]

Since
\[
\left( \frac{w^{g-2}}{y-a} \right)^* \subseteq \left( \frac{w^{g-1}}{y-a-x} \right)^* = W^r_b,
\]
the proof is completed.

**Theorem**

Let \( \varphi : X \rightarrow J(X) \) be a canonical map of a complete, nonsingular curve, \( X \), of genus \( g > 1 \), into its Jacobian variety \( J(X) \). Then \( W^1 = \varphi(X) \) is determined up to a translation and reflection by the canonical polarization of \( J(X) \).

**Proof:** By a translation, if necessary, we may normalize \( \varphi \) such that \( \varphi(P) = 0 \) for some point, \( P \in X \). Let \( Y \) be a second curve with the same Jacobian variety, \( J(X) \), and denote by \( V^r \) the image of the set of positive divisors of
degree ≤ r on Y under the (normalized) canonical map ψ: Y → J(X). The theorem will be proved by showing that if y^g−1 is a translate of W^g−1 (i.e., if the canonical polarizations are the same) then V^1 is a translate of W^1 or of (W^1)^∗.

Let r be the smallest integer such that V^1 ⊆ W^{r+1}_a or V^1 ⊆ (W^{r+1}_a)^∗ for some a. The theorem will be proved if we can show that r = 0. Assume to the contrary that r ≥ 1.

(Clearly, r < g−1.) Assume, changing notation if necessary, that V^1 ⊆ W^{r+1}_a. Choose x ∈ W^1, y ∈ W^{g−1−r}, and set b = a+x−y. Then, unless W^{r+1}_a ⊆ W^{g−1}_b, we have

\[ V^1 \cap W^{g−1}_b = V^1 \cap W^{g−1}_b \cap W^{r+1}_a = (V^1 \cap W^r_{a+x}) \cap (V^1 \cap S) \]

in the notation of Lemma 4. Note that, a being given, W^r_{a+x} depends only on the choice of x, and S depends only on the choice of y.

We shall first show that for a fixed x, V^1 ⊆ W^{g−1}_b for almost all choices of y, and hence W^{r+1}_a ⊆ W^{g−1}_b for the same y.

As y varies over W^{g−1−r}, b varies over W^{g−1−r}_-(a+x). By assumption, there is a constant k, such that V^g−1_k = W^{g−1}_k. Hence, V^1 ⊆ W^{g−1}_b ↔ V^1 ⊆ W^{g−1}_b+k ↔ b ∈ W^{g−2}_k. Thus the set of b for which V^1 ⊆ W^{g−1}_b is given by - b ∈ W^{g−2}_-(a+x).
Now, if $V^1 \subseteq W_b^{g-1}$ for all $b$ in $W^{g-1-k}_{(a+x)}$, then $V^1 \subseteq W_{a+x}^r$ by Lemma 3. This contradicts the assumption on $r$. Hence $W^{g-1-k}_{(a+x)} \nsubseteq V^{g-2}_K$, and the intersection of these sets is a lower dimensional subset of $W^{g-1-k}_{(a+x)}$.

We now return to consider the intersection

$$V^1 \cap W_b^{g-1} = (V^1 \cap W_{a+x}^r) \cup (V^1 \cap S).$$

It is well known\[^2\] that if $V^1 \nsubseteq W_b^{g-1}$, then there is a unique positive divisor, $D(b)$, of degree $g$ on $Y$, such that

$$\psi(D(b)) = b+c$$

(1)

where $c$ is a constant, independent of $b$, and the points of $D(b)$ are the preimages of the points of the intersection $V^1 \cap W_b^{g-1}$ under $\psi$.

We show first that $V^1 \cap W_{a+x}^r$ contains at most one point. If not, then as $-b$ varies over almost all points of $W^{g-1-k}_{(a+x)}$ (for fixed $x$), $D(b)$ will contain at least two fixed points, and hence $\psi(D(b))$ varies over a translate of $V^{g-2}$. By Eq. (1) we should then have an inclusion of $(W^{g-1-k})^*$ in a translate of $V^{g-2}$, say $(W^{g-1-k})^* \subseteq V^{g-2}_d$. But then

$$\cap \left\{ V^{g-1}_{K-u}; u \in V^{g-2}_d \right\} \subseteq \cap \left\{ W^{g-1}_{-u}; u \in (W^{g-1-k})^* \right\}.$$
and, using Lemma 3, we get an inclusion of $V^1$ in a translate of $(W^r)^*$, contradicting the assumption on $r$.

Keeping $y$ fixed and varying $x$, we see by Eq. (1) that $V^1 \cap W^r_{a+x}$ must contain at least one point, and hence it contains exactly one point.*

It is now easily seen that we can find $x, x' \in W^1$ and $y \in W^{g-1-r}$ such that $D(a+x-y) = Q+\mathcal{D}$ and $D(a+x'-y) = Q'+\mathcal{D}$ where $Q, Q' \in Y$ and $\mathcal{D}$ is a positive divisor of degree $g-1$ on $Y$ not containing $Q$ or $Q'$. By Eq. (1), $\varphi(Q) - \varphi(Q') = x-x'$, and hence $W^1$ has two distinct points in common with some translate of $V^1$. Now, if $x, x' \in W^1$, then $W^{g-1}_{-x} \cap W^{g-1}_{-x'} = W^{g-2}_{x+x'} \cap (W^{g-2}_{x+x'})^*$ by Lemma 4. By Lemma 3 we now get an inclusion of some translate of $V^{g-2}$ in $W^{g-2}$ or in $(W^{g-2})^*$, whence, again by Lemma 3, we get an inclusion of some translate of $V^1$ in $W^1$ or $(W^1)^*$. This completes the proof.

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* Which, by the preceding argument, occurs in $D(b)$ with multiplicity 1, for almost all choices of $y$. 

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1959, (Ch. 2, Sec. 2).


Nr. 2.