LARGE DEVIATIONS AND CENTRAL LIMIT THEOREMS FOR SEQUENTIAL AND RANDOM SYSTEMS OF INTERMITTENT MAPS

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Abstract. We obtain optimal large deviations estimates for both sequential and random compositions of intermittent maps. We also address the question of whether or not centering is necessary for the quenched central limit theorems (CLT) obtained by Nicol, Török and Vaienti for random dynamical systems comprised of intermittent maps. Using recent work of Abdelkader and Aimino, Hella and Stenlund we extend the results of Nicol, Török and Vaienti on quenched central limit theorems (CLT) for centered observables over random compositions of intermittent maps: first by enlarging the parameter range over which the quenched CLT holds; and second by showing that the variance in the quenched CLT is almost surely constant (and the same as the variance of the annealed CLT) and that centering is needed to obtain this quenched CLT.

1. Introduction

The theory of limit laws and rates of decay of correlations for uniformly hyperbolic and some non-uniformly hyperbolic sequential and random dynamical systems has recently seen major progress. Results in this area include: in [CR07] strong laws of large numbers and centered central limit theorems for sequential expanding maps; in [AHN+15], polynomial decay of correlations for sequential intermittent systems; in [NTV18], sequential and quenched (self-centering) central limit theorems for intermittent systems; in [ANV15], annealed versions of a central limit theorem, large deviations principle, local limit theorem and almost sure invariance principle are proven for random expanding dynamical systems, as well as quenched versions of a central limit theorem, dynamical Borel-Cantelli lemmas, Erdős-Rényi laws and concentration inequalities; in [AA16], necessary and sufficient conditions are given for a central limit theorem without random centering for uniformly expanding maps; and in [BB16b] mixing rates and central limit theorems are given for random intermittent maps using a Tower construction. Recently the preprint [BBR17] considered quenched decay of correlation for slowly mixing systems and the preprint [AM18] used martingale techniques to obtain large deviations for systems with stretched exponential decay rates.

In this article we obtain large deviations estimates for both sequential and random compositions of intermittent maps. We also address the question of whether or not centering is necessary for the quenched central limit theorems (CLT) obtained in [NTV18] for random dynamical systems comprised of intermittent maps. More precisely, we consider in the first instance a fixed deterministically chosen sequence of maps \( \ldots T_{\alpha_n}, \ldots, T_{\alpha_1} \) in the sequential compositions of intermittent maps.
case, or a randomly drawn sequence \( \ldots T_{\omega_n}, \ldots, T_{\omega_1} \) with respect to a Bernoulli measure \( \nu \) on \( \Sigma := \{ T_1, \ldots, T_k \}^N \), where each of the maps \( T_j \) is a Liverani-Saussol-Vaienti \[LSV99\] intermittent map of form

\[
T_{\alpha_j}(x) = \begin{cases} 
  x + 2^{\alpha_j} x^{1 + \alpha_j}, & 0 \leq x \leq 1/2, \\
  2x - 1, & 1/2 \leq x \leq 1 
\end{cases},
\]

for numbers \( 0 < \alpha_j \leq \alpha < 1 \). We consider the asymptotic behavior of the centered (that is, after subtracting their expectation) sums

\[
S_n := \sum_{k=1}^{n} \varphi \circ (T_{\alpha_k} \circ \ldots \circ T_{\alpha_1})
\]

for sufficiently regular observables \( \varphi \).

Denote by \( m \) Lebesgue measure on \( X := [0, 1] \), and by \( m(\varphi) \) the integral of \( \varphi \) with respect to \( m \). We will also consider the measure \( \tilde{m} \) given by \( d\tilde{m}(x) = x^{-\alpha} dm \), where \( 0 < \alpha \leq \alpha < 1 \). The motivation for introduction of this measure is that in the case of a stationary system, if \( \alpha_k = \alpha \) for each \( k \), then a natural and convenient measure to use is the invariant measure \( \mu_\alpha \) for \( T_\alpha \), which behaves near 0 as \( x^{-\alpha} \). In the stationary case large deviation estimates are given with respect to \( \mu_\alpha \) and \( m \) in \[MN08\].

In the sequential case of a fixed realization we are interested in the large deviations of the self-centered sums:

\[
m \left\{ x : \frac{1}{n} \left| \sum_{k=1}^{n} \varphi \circ (T_{\alpha_k} \circ \ldots \circ T_{\alpha_1}) - \sum_{k=1}^{n} m(\varphi \circ T_{\alpha_k} \circ \ldots \circ T_{\alpha_1}) \right| > \epsilon \right\}
\]

for \( \epsilon > 0 \). We also obtain large deviations with respect to \( \tilde{m} \), which are in a sense sharper.

In the quenched case, once again assuming \( \mu(\varphi) = 0 \), we give bounds for \( \nu \)-almost every realization \( \omega \in \Sigma \).

Since the maps we are considering are not uniformly hyperbolic, spectral methods used to obtain limits laws are not immediately available. Our techniques to establish large deviations estimates are based on those developed for stationary systems, in particular the martingale methods of \[MN08\] \[Mel09\].
Using recent work of [AA16] and [HS20] we extend the results of [NTV18] on quenched central limit theorems (CLT) for centered observables over random compositions of intermittent maps in two ways, first by enlarging the parameter range over which the quenched CLT holds and second by showing as a consequence of results in [HS20] that the variance in the quenched CLT is almost surely constant and equal to the variance of the annealed CLT.

We also study the necessity of centering to achieve a quenched CLT using ideas of [AA16] and [ANV15]. The work of [ANV15] together with our observations show that centering is necessary 'generically' (in a sense made precise later) to obtain the quenched CLT in fairly general hyperbolic situations.

1.1. Improvements of earlier results. With this paper we improve some earlier results of [NTV18].

- we show that the sequential CLT in [NTV18, Theorem 3.1], [HL19], holds for the sharp $\alpha < 1/2$ (from $\alpha < 1/9$) if the variance grows at the rate specified.
- we show that the CLT holds not only with respect to Lebesgue measure $m$ but also for $d\tilde{m} = x^{-\alpha} dm$, which scales at the origin as the invariant measure of $T_{\alpha}$.
- in the case of quenched CLT’s of [NTV18, Theorem 3.1], using results of Hella and Stenlund [HS20] we show that the variance $\sigma^2_\omega$ is almost-surely the same for any sequence of maps and equal to the annealed variance $\sigma^2$.

2. Notation and assumptions

Throughout this article, $m$ denotes the Lebesgue measure on $X := [0,1]$ and $\mathcal{B}$ the Borel $\sigma$-algebra on $[0,1]$. We consider the family of intermittent maps given by

\begin{equation}
T_\alpha(x) = \begin{cases} 
x + 2^\alpha x^{1+\alpha}, & 0 \leq x \leq 1/2, \\
2x - 1, & 1/2 \leq x \leq 1, 
\end{cases}
\end{equation}

for $\alpha \in (0,1)$.

For $\beta_k \in (0,1)$ denote by $P_{\beta_k} = P_k : L^1(m) \to L^1(m)$ the transfer operator (or Ruelle-Perron-Frobenius operator) with respect to $m$ associated to the map $T_{\beta_k} = T_k$, defined as the “pre-dual” of the Koopman operator $f \mapsto f \circ T_k$, acting on $L^\infty(m)$. The duality relation is given by

$$
\int_X P_k f g \ dm = \int_X f g \circ T_k \ dm
$$

for all $f \in L^1(m)$ and $g \in L^\infty(m)$ [BG97, Proposition 4.2.6]. For a fixed sequence $\beta_k$ such that $0 < \beta_k \leq \alpha$ for all $k$, define

$$
\mathcal{T}^\infty := \ldots , T_{\beta_n} , \ldots , T_{\beta_1} \quad \mathcal{T}^n_m := T_{\beta_n} \circ \ldots \circ T_{\beta_m} \quad \mathcal{P}^n := \mathcal{P}_1^n
$$

We will often write, for ease of exposition when there is no ambiguity, $T_{\beta_n} \circ \ldots \circ T_{\beta_m}$ as $T_n \circ \ldots \circ T_m$ and $P_\beta \circ \ldots \circ P_\beta$ as $P_n \circ \ldots \circ P_m$.

Since $L^1(m)$ is invariant under the action of the transfer operators, the duality relation extends to compositions

$$
\int_X \mathcal{P}_k^n f g \ dm = \int_X f g \circ \mathcal{T}_k^n \ dm.
$$
We will write $\mathbb{E}_m[\varphi | \mathcal{F}]$ for the conditional expectation of $\varphi$ on a sub-$\sigma$-algebra $\mathcal{F}$ with respect to the measure $m$. To simplify notation we might write $E$ for $\mathbb{E}_m$.

**Remark 2.1.** In [CR07, NTV18] it is shown that

$$E_m[\varphi \circ \mathcal{T}^\ell | \mathcal{T}^{-k} \mathcal{B}] = \frac{P_k \circ \cdots \circ P_{\ell+1}(\varphi \cdot \mathcal{P}^{\ell}(1))}{\mathcal{P}^{\ell}(1)} \circ \mathcal{T}^k$$

for $0 \leq \ell \leq k$.

One of the main tools to study sequential and random systems of intermittent maps is the use of cones (see [LSV99, AHN15, NTV18]). Define the cone $C_2$ by

$$C_2 := \{ f \in C^0((0,1]) \cap L^1(m) \mid f \geq 0, f \text{ non-increasing} , X^{a+1}f \text{ increasing} , f(x) \leq ax^{-\alpha}m(f) \} ,$$

where $X(x) = x$ is the identity function and $m(f)$ is the integral of $f$ with respect to $m$. In [AHN15] it is proven that for a fixed value of $\alpha \in (0,1)$, provided that the constant $a$ is big enough, the cone $C_2$ is invariant under the action of all transfer operators $P_\beta$ with $0 < \beta \leq \alpha$.

**Notation 2.2.** In general we will denote the transfer operator with respect to a non-singular measure $\mu$ (not necessarily Lebesgue measure) by $P_\mu$. Similarly, the (conditional) expectation will be denoted by $\mathbb{E}_\mu$.

Denote the centering with respect to $\mu$ of a function $\varphi \in L^1(X,\mu)$ by

$$[\varphi]_\mu := \varphi - \frac{1}{\mu(X)} \int_X \varphi \, d\mu$$

In particular, for $g(x) := x^{-\alpha}$, denote the measure $g_m$ by $\widetilde{m}$, the corresponding transfer operator by $\tilde{P} := P_{g_m}$, and the (conditional) expectation by $\mathbb{E}_{\widetilde{m}} := \mathbb{E}_{g_m}$.

**Random dynamical systems.** Now we introduce a randomized choice of maps: consider a finite family of intermittent maps of the form (2.1), indexed by a set $\Omega = \{ \beta_1, \ldots, \beta_m \} \subset (0,\alpha)$. Given a probability distribution $\mathbb{P} = (p_1, \ldots, p_m)$ on $\Omega$, define a Bernoulli measure $\mathbb{P}^{\otimes \mathbb{N}}$ on $\Sigma := \Omega^\mathbb{N}$ by $\mathbb{P}^{\otimes \mathbb{N}} \{ \omega : \omega_{j_1} = \beta_{j_1}, \ldots, \omega_{j_k} = \beta_{j_k} \} = \prod_{i=1}^k p_{j_i}$ for every finite cylinder and extend to the sigma-algebra generated by the cylinders of $\Sigma$ by Kolmogorov's extension theorem. This measure is invariant and ergodic with respect to the shift operator $\tau$ on $\Sigma$, $\tau : \Sigma \to \Sigma$ acting on sequences by $(\tau(\omega))_k = \omega_{k+1}$. We will denote $\mathbb{P}^{\otimes \mathbb{N}}$ by $\nu$ from now on.

For $\omega = (\omega_1, \omega_2, \ldots) \in \Sigma$ define $\mathcal{T}_\omega := T_{(\tau_{\omega_1})} \circ \cdots \circ T_{\omega_1} = T_{\omega_n} \circ \cdots \circ T_{\omega_1}$. The random dynamical system is defined as

$$F : \Sigma \times X \to \Sigma \times X$$

$$(\omega, x) \mapsto (\tau \omega, T_{\omega_1} x).$$

The iterates of $F$ are given by $F^n(\omega, x) = (\tau^n(\omega), \mathcal{T}^n(\omega))$.

We will also use $\Omega$-indexed subscripts for random transfer operators associated to the maps $T_{\omega_i}$, so that $P_{\omega_i} := P_{T_{\omega_i}}$. We will also abuse notation and write $P_\omega$ for $P_{\omega_1}$ if $\omega = (\omega_1, \omega_2, \ldots, \omega_k, \ldots)$.

A probability measure $\mu$ on $X$ is said to be stationary with respect to the RDS $F$ if

$$\mu(A) = \int_{\Sigma} \mu(T_{\omega_1}^{-1}(A)) \, d\nu(\omega) = \sum_{\beta \in \Omega} p_\beta \mu(T_{\beta}^{-1}(A))$$

1The measure $\mu$ is non-singular for the transformation $T$ if $\mu(A) > 0 \implies \mu(T(A)) > 0$. 

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for every measurable set $A$, where $p_\beta$ is the $\mathbb{P}$-probability of the symbol $\beta$. This is equivalent to the measure $\nu \otimes \mu$ being invariant under the transformation $F : \Sigma \times X \to \Sigma \times X$.

See Remark 4.3 about the existence and ergodicity of such a stationary measure in our setting.

The annealed transfer operator $P : L^1(m) \to L^1(m)$ is defined by averaging over all the transformations:

$$P = \sum_{\beta \in \Omega} p_\beta P_\beta = \int_\Sigma P_\omega \, d\nu(\omega).$$

This operator is “pre-dual” to the annealed Koopman operator $U : L^\infty(m) \to L^\infty(m)$ defined by

$$(U \varphi)(x) := \sum_{\beta \in \Omega} p_\beta \varphi(T_\beta x) = \int_\Sigma \varphi(T_\omega x) \, d\nu(\omega) = \int_\Sigma F(\tilde{\varphi})(\omega, x) \, d\nu(\omega)$$

where $\tilde{\varphi}(\omega, x) := \varphi(x)$. The annealed operators satisfy the duality relationship

$$\int_X (U \varphi) \cdot \psi \, dm = \int_X \varphi \cdot P \psi \, dm$$

for all observables $\varphi \in L^\infty(m)$ and $\psi \in L^1(m)$.

3. **Background results and the Martingale approximation.**

In this section we describe the main technique used to prove some of the limit law results: the martingale approximation, introduced by Gordin [Gor69]. Since there is no common invariant measure for the set of maps $\{T_k\}$, for a given $C^1$ observable $\varphi$ we center along the orbit by

$$[\varphi]_k(\omega, x) := \varphi(x) - \int_X \varphi \circ T^k_\omega \, dm,$$

with $T^k_\omega = \text{Id}$ for $k = 0$.

This implies that $\mathbb{E}_m([\varphi]_k \circ T^k) = 0$ and consequently the centered Birkhoff sums

$$\hat{S}_n := \sum_{k=1}^n [\varphi]_k \circ T^k,$$

have zero mean with respect to $m$. Following [NTV18], define

(3.1) \quad H_1 := 0 \quad \text{and} \quad H_n \circ T^n := \mathbb{E}_m(\hat{S}_{n-1} | \mathcal{B}_n) \quad \text{for} \quad n \geq 2

and the (reverse) martingale sequence $\{M_n\}$ by

$$M_0 := 0 \quad \text{and} \quad \hat{S}_n = M_n + H_{n+1} \circ T^{n+1},$$

where the filtration here is $\mathcal{B}_n = T^{-n} \mathcal{B}$. Define $\psi_n \in L^1(m)$ by setting

$$\psi_n = [\varphi]_n + H_n - H_{n+1} \circ T_{n+1},$$

then $M_n - M_{n-1} = \psi_n \circ T^n$ and we have that $\mathbb{E}(M_n | \mathcal{B}_{n+1}) = 0$. Thus $\{\psi_n \circ T^n\}$ is a reverse martingale difference scheme. An explicit expression for $H_n$ is given by

(3.2) \quad H_n = \frac{1}{P_n} \left[P_n([\varphi]_{n-1} \mathcal{P}_{n-1} 1) + P_nP_{n-1}([\varphi]_{n-2} \mathcal{P}_{n-2} 1) + \ldots + P_nP_{n-1} \ldots P_1([\varphi]_0 \mathcal{P}_0 1) \right].
Remark 3.1. The formulas derived so far with \( m \) being the Lebesgue measure actually hold for any measure \( \mu \) that is non-singular for the transformations \( T_\beta \) considered. The conditional expectations \( E_\mu \) will be with respect to \( \mu \) and the transfer operator \( P_\mu \) will be with respect to the measure space \( (X, \mu) \). In particular the centering will have the form

\[
[\varphi]_k(\omega, x) := \varphi(x) - \frac{1}{\mu(X)} \int_X \varphi \circ T^k \omega \, d\mu(t),
\]

but all other equations are the same, with the notational changes just described.

We collect and extend some results from \( \text{[NTV18]} \) concerning the properties of \( H_n \), as well as the non-stationary decay of correlations for the sequential system.

Remark 3.2. As in \( \text{[NTV18]} \) for simplicity, in some of the following statements we will use as rate of decay \( n^{-\frac{1}{2}+\epsilon} \) ignoring a \( \log n \) factor. This is correct if we take for \( \alpha \) a slightly higher value and changes none of the results we obtain.

We state first a few formulas for changing from a measure \( m \) to the measure \( g(x) \, dm(x) \) with \( g \in L^1(m) \); for simplicity, we denote this new measure as \( gm \) when there is no possibility of confusion.

Lemma 3.3 (Change of measure). We state this result only for the situation we need, but it holds instead of \( m \) the Lebesgue measure for any measure \( \mu \) non-singular with respect to \( T \), and instead of \( g(x) = x^{-\alpha} \) for any \( g \in L^1(\mu) \), \( g > 0 \).

Note that \( L^1(gm) = g^{-1}L^1(m) \), so all formulas below make sense for \( \varphi \) in the appropriate \( L^1 \)-space.

Then:

\[
m(\varphi) = m(P_m \varphi) \\
P_{gm}(\varphi) = g^{-1}P_m(g \varphi) \\
g \, [\varphi]_{gm} = [g \varphi]_m - \frac{m(g \varphi)}{m(g)} \, [g]_m \\
E_{gm}(\varphi | B) = E_m(g \varphi | B) / \mathbb{E}_m(g | B)
\]

Therefore

\[
(P_{gm})^k(\varphi)^{gm} = g^{-1}(P_m)^k \left( [g \varphi]^m - \frac{m(g \varphi)}{m(g)} \, [g]^m \right)
\]

Proof. The first two properties are standard, follow from the definition of the transfer operator. The third is a direct computation using the notation \((2.3)\).

For the fourth, \( E_{gm}(\varphi | B) \) is the function \( \Phi \) that is \( B \)-measurable and \( \int \Phi \psi \, d(gm) = \int \varphi \psi \, d(gm) \) for each \( \psi \in L^\infty(B) \). Expanding the LHS,

\[
\int \Phi \psi \, d(gm) = \int \Phi \psi g \, dm = \int \Phi \psi \mathbb{E}_m(g | B) \, dm
\]

whereas the RHS becomes

\[
\int \varphi \psi \, d(gm) = \int \varphi \psi g \, dm = \int \mathbb{E}_m(g \varphi | B) \psi \, dm
\]

Thus \( \Phi \mathbb{E}_m(g | B) = \mathbb{E}_m(g \varphi | B) \), as claimed.
Proposition 3.4 ([NTV18]). If \( \varphi, \psi \) are both in the cone \( C_2 \) and have the same mean, \( \int_X \varphi dm = \int_X \psi dm \), then by [NTV18 Theorem 1.2]

\[
\| P^n(\varphi) - P^n(\psi) \|_{L^1(m)} \leq C_\alpha (\| \varphi \|_{L^1(m)} + \| \psi \|_{L^1(m)}) n^{-\frac{1}{p} + 1}(\log n)^{\frac{1}{p}}
\]

Moreover [NTV18 Remark 2.5 and Corollary 2.6], for \( \varphi \in C^1 \), \( h \in C_2 \) and any sequence of maps \( T^\infty \):

\[
\| P^n([h \varphi]^m) \|_{L^1(m)} \leq C_\alpha F(\| \varphi \|_C + m(h)) n^{-\frac{1}{p} + 1}(\log n)^{\frac{1}{p}}
\]

where \( C_\alpha \) depends only on the map \( T_\alpha \), and \( F : \mathbb{R} \to \mathbb{R} \) is an affine function.

The decay result of Proposition 3.4 for products of elements in the cone with \( C^1 \) observables (see also [LSV99 Theorem 4.1]), follows from Lemma 3.5 which was stated in [LSV99 proof of Theorem 4.1]. The proof of Lemma 3.5 is given in the Appendix; a different – less transparent – proof is given in [NTV18 Lemma 2.4].

Lemma 3.5. Suppose \( \varphi \in C^1 \) and \( h \in C_2 \). Then there exist constants \( \lambda, A, B \in \mathbb{R} \) such that \( (\varphi + A + \lambda x)h + B \) and \( (A + \lambda x)h + B \) both are in \( C_2 \) and hence if \( \int \varphi h dm = 0 \) then \( \| P^n(\varphi h) \|_{L^1(m)} \leq C \rho(j) \| \varphi h \|_{L^1(m)} \) where \( \rho(j) \) is the \( L^1(\mathbb{R}) \)-decay for centered functions from the cone \( C_2 \).

Note that in our setting \( \rho(j) = j^{-\frac{1}{p} + 1}(\log j)^{\frac{1}{p}} \).

A consequence of Proposition 3.4 is the non-stationary decay of correlations ([NTV18 Page 1130])

\[
\left| \int_X \varphi \cdot \psi \circ T_{\omega_n} \cdots \circ T_{\omega_1} \cdot dm - m(\varphi) \cdot m(\psi \circ T_{\omega_n} \cdots \circ T_{\omega_1}) \right| \\
\leq \| \psi \|_\infty \| P^m_\varphi(\varphi) - P^m_\varphi(1 \int_X \varphi dm) \|_{L^1(m)}
\]

We derive next decay estimates with respect to the measure \( \tilde{m} \), which are better in \( L^p \), \( p > 1 \), than those for \( m \).

Proposition 3.6. For \( \varphi : [0, 1] \to \mathbb{R} \) bounded, \( h \in C_2 \) and \( 1 \leq p \leq \infty \):

(3.5) \( \| \tilde{P}^n(\varphi) \|_{L^\infty(\tilde{m})} \leq m(g) \| \varphi \|_{L^\infty(\tilde{m})} \)

For \( \varphi \in C^1([0, 1]) \), \( h \in C_2 \) and \( 1 \leq p \leq \infty \):

(3.6) \( \| \tilde{P}^n \left( [(g^{-1}h)\varphi]^\tilde{m} \right) \|_{L^1(\tilde{m})} \leq C_\alpha F_1 (\| \varphi \|_{C^1} + m(h)) n^{-\frac{1}{p} + 1}(\log n)^{\frac{1}{p}} \)

(3.7) \( \| \tilde{P}^n \left( [(g^{-1}h)\varphi]^\tilde{m} \right) \|_{L^p(\tilde{m})} \leq C_\alpha F_p (\| \varphi \|_{C^1} + m(h)) n^{\frac{1}{p}}(\frac{1}{p} + 1)(\log n)^{\frac{1}{p}} \)

where \( C_\alpha \) depends only on \( T_\alpha \) and \( F_p \) are affine functions.

Note that (3.7) can be seen as providing the \( L^1 \) and \( L^\infty \) estimates as well; the \( L^1 \) and \( L^p \) bounds are relevant only for \( \varphi \in C^1 \).

Proof. We note first that the \( L^1 \) and \( L^\infty \) bounds give (3.7), since

\[
\| f \|_{L^p} \leq \| f \|_{L^\infty}^{1 - \frac{1}{p}} \| f \|_{L^1}^{\frac{1}{p}}
\]

because

\[
\int |f|^p \leq \int \| f \|_{L^\infty}^{p-1} |f| = \| f \|_{L^\infty}^{p-1} \| f \|_{L^1}.
\]
To prove the $L^\infty$ estimate (3.3) note that by the invariance of the cone $C_2$, $P^n(g) \in C_2$, so $P^n(g) \leq x^{-\alpha}m(P^n(g)) = x^{-\alpha}m(g)$. That is, using (3.3),

$$\tilde{P}^n(1) = g^{-1}P^n(g) \leq m(g)$$

Since $-\|\varphi\|_{L^\infty} 1 \leq \varphi \leq \|\varphi\|_{L^\infty} 1$ and $\tilde{P}^n$ are positive operators, we obtain (3.5).

For (3.6) assume that $\varphi \in C^1$ (otherwise it is clearly satisfied). In view of (3.4):

$$\|\tilde{P}^n([g^{-1}h]\varphi)^\bar{m}\|_{L^1(\bar{m})} = \|g^{-1}P^n([h\varphi]^m) - \frac{m(g\varphi)}{m(g)}g^{-1}P^n([g]^m)\|_{L^1(m)}$$

$$= \|P^n([h\varphi]^m) - \frac{m(g\varphi)}{m(g)}P^n([g]^m)\|_{L^1(m)}$$

$$\leq \|P^n([h\varphi]^m)\|_{L^1(m)} + \left|\frac{m(g\varphi)}{m(g)}\right|\|P^n([g]^m)\|_{L^1(m)}$$

By [NTV18] Lemma 2.3, there is an affine function $F : \mathbb{R} \to \mathbb{R}$ such that for $\varphi \in C^1([0, 1])$ and $h \in C_2$ can write $[\varphi h]^m = \Psi_1 - \Psi_2$ with $\Psi_1, \Psi_2 \in C_2$ and $\|\Psi_{1,2}\|_{L^1(m)} \leq F(\|\varphi\|_{C^1} + m(h))$. By [NTV18] Theorem 1.2, for an observable $\psi$ in the cone $C_2$ and for any sequence of maps $T^\infty$, we have

$$\int_X |P^n([\psi]^m)|dm \leq C_\alpha \|\psi\|_{L^1(m)n^{-\frac{1}{\alpha} + 1}} (\log n)^{\frac{1}{\alpha}}$$

where $C_\alpha$ depends only on $T_\alpha$. Applying these to (3.9), we obtain (3.6).

\[\square\]

**Lemma 3.7.** For $\varphi \in C^1$, $\alpha < 1$ and $1 \leq p < -\frac{1}{\alpha} + 1$

$$\sup_{n \geq 1} \|H_n \circ T^n\|_{L^p(m)} < \infty$$

$$\sup_{n \geq 1} \|\tilde{H}_n \circ T^n\|_{L^p(\bar{m})} < \infty$$

where $H_n \circ T^n : = \mathbb{E}_m([S_{n-1}]^m | B_n)$, $\tilde{H}_n \circ T^n : = \mathbb{E}_{\bar{m}}([S_{n-1}]^\bar{m} | B_n)$, $B_n : = T^{-n}B$.

**Proof.** We prove the statement for $\tilde{H}_n$. The one for $H_n$ is obtained the same way, using Proposition 3.4 instead of (3.6).

Using the definition of $\tilde{H}_n$:

$$\|\tilde{H}_n \circ T^n\|_{L^p(\bar{m})} = \|\sum_{k=1}^{n-1} \mathbb{E}_{\bar{m}}([\varphi \circ T^k]^\bar{m} | B_n)\|_{L^p(\bar{m})} \leq \sum_{k=1}^{n-1} \|\mathbb{E}_{\bar{m}}([\varphi \circ T^k]^\bar{m} | B_n)\|_{L^p(\bar{m})}$$

We will bound each term of the above sum in both $L^1$ and $L^\infty$, and then use (3.8) to obtain an $L^p$-bound.

In $L^\infty$ we have

$$\|\mathbb{E}_{\bar{m}}([\varphi \circ T^k]^\bar{m} | B_n)\|_{L^\infty(\bar{m})} \leq \|\varphi \circ T^k\|_{L^\infty(\bar{m})} \leq 2\|\varphi\|_{L^\infty(\bar{m})}.$$

In $L^1$ we use (2.2) to compute the conditional expectation. Since the conditional expectation preserves the expected value, one can check that the centering holds as written below

$$2\bar{m}(\varphi, \tilde{P}^k(1)) = \bar{m}(\varphi \circ T^k)$$

because, by the definition of the transfer operator, $\int \varphi \tilde{P}^k(1)dm = \int \varphi \circ T^k \cdot 1 dm$.
We can then use \[3.6\] for the decay, with \(h = \mathcal{P}^k(g)\), because \(\tilde{\mathcal{P}}^k(1) = g^{-1}\mathcal{P}^k(g)\).

\[
\|E_\tilde{m}([\varphi \circ T^k]^{\tilde{m}} | \mathcal{B}_n)\|_{L^1(\tilde{m})} = \| \tilde{P}_n \circ \cdots \circ \tilde{P}_{k+1}(\varphi \cdot \tilde{\mathcal{P}}^k(1)^{\tilde{m}}) \circ T^n\|_{L^1(\tilde{m})}
\]

\[
= \| \tilde{P}_n \circ \cdots \circ \tilde{P}_{k+1}(\varphi \cdot \tilde{\mathcal{P}}^k(1)^{\tilde{m}}) \|_{L^1(\tilde{m})} = \| \tilde{P}_n \circ \cdots \circ \tilde{P}_{k+1}(\varphi \cdot \mathcal{P}^k(g)^{\tilde{m}}) \|_{L^1(\tilde{m})}
\]

\[
\leq C_{\alpha} F_1(\|\varphi\|_{C^1} + m(\mathcal{P}^k(g)))(n - k)^{-\frac{1}{\alpha} + 1}(\log(n - k))^{\frac{1}{\alpha}}.
\]

Note that \(m(\mathcal{P}^k(g)) = m(g)\), so the coefficient above does not depend on \(k\).

Apply now \[3.8\] to obtain for \(1 \leq p \leq \infty\) that

\[
\|E_\tilde{m}([\varphi \circ T^k]^{\tilde{m}} | \mathcal{B}_n)\|_{L^p(\tilde{m})} \leq C_{p,\alpha,\|\varphi\|_{C^1} + m(g)} [(n - k)^{-\frac{1}{\alpha} + 1}(\log(n - k))^\frac{1}{\alpha}]^\frac{1}{p}
\]

which gives the desired uniform bound for \((3.10)\) as long as \(p < \frac{1}{\alpha} - 1\).

A useful remark is the following lower bound for functions in the cone \(C_2\):

**Proposition 3.8** ([LSV99, Lemma 2.4]). For every function \(f \in C_2\) one has

\[
\inf_{x \in [0,1]} f(x) = f(1) \geq \min \left\{ a, \left[ \frac{\alpha(1 + \alpha)}{a^\alpha} \right]^{\frac{1}{1 - \alpha}} \right\} m(f).
\]

Denote the constant in the above expression by \(D_\alpha\). Then \(P^n 1 \geq D_\alpha > 0\) for all \(n \geq 1\).

We will also use Rio’s inequality, taken from [MPU06]. This is a concentration inequality that allows us to bound the moments of Birkhoff sums.

**Proposition 3.9** ([MPU06, Rio17]). Let \(\{X_i\}\) be a sequence of \(L^2\) centered random variables with filtration \(\mathcal{F}_i = \sigma(X_1, \ldots, X_i)\). Let \(p \geq 1\) and define

\[
b_{i,n} = \max_{i \leq u \leq n} \|X_i \sum_{k=i}^n \mathbb{E}(X_k | \mathcal{F}_i)\|_{L^p},
\]

then

\[
\mathbb{E}|X_1 + \ldots + X_n|^{2p} \leq \left( 4p \sum_{i=1}^n b_{i,n} \right)^p.
\]

4. Polynomial large deviations estimates.

4.1. Sequential dynamical systems. Recall we fixed a sequence \(\mathcal{T}^\infty = \ldots T_{a_n}, \ldots, T_{a_1}\) where each of the maps is of the form

\[
T_{a_j}(x) = \begin{cases} x + 2^{\alpha_j} x^{1+\alpha_j}, & 0 \leq x \leq 1/2, \\ 2x - 1, & 1/2 \leq x \leq 1 \end{cases},
\]

for \(0 < \alpha_j \leq \alpha < 1\). In the first part of this section we prove that for such a fixed sequence of maps \(\mathcal{T}^\infty\), a polynomial large deviations bound holds for the centered sums.
Theorem 4.1 (Sequential LD). Let $0 < \alpha < 1$ and $\varphi \in C^1([0,1])$. Then the centered sums satisfy the following large deviations upper bound: given $\delta > 0$, for any $\epsilon > 0$

$$
m \left\{ x : \sum_{j=1}^{n} \left[ \varphi(T^j)(x) - m(\varphi(T^j)) \right] > ne \right\} \leq C_{\alpha,\delta,\|\varphi\|_{C^1}} (ne^2)^{1-\frac{1}{\alpha}+\delta}
$$

where $C = C_{\alpha,\delta,\|\varphi\|_{C^1}}$ is a constant depending on $\alpha$, $\delta$ and the $C^1$ norm of $\varphi$, but not on the sequence $T^\infty$.

The same estimate (by the same proof) holds for the measure $\tilde{m}$.

Remark 4.2. In [MN08] these bounds are shown to be basically optimal since in the case of a single map $T_\alpha$ being iterated there exists an open and dense set of $C^1$ observables $\varphi$ such that for any $\delta > 0$, $\mu \left\{ x : \sum_{j=1}^{n} [\varphi(T^j)(x) - m(\varphi(T^j))] > ne \right\} \geq C_{\alpha,\delta,\|\varphi\|_{C^1}} (ne^2)^{1-\frac{1}{\alpha}+\delta}$ infinitely often for the absolutely continuous invariant measure $\mu$.

Proof of Theorem 4.1 Fix $n$ and for $i \in \{1, \ldots, n\}$, define the sequence of $\sigma-$algebras $\mathcal{F}_{i,n} = \mathcal{F}_i = \mathcal{T}^{-(n-i)}(B)$. Note that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ hence $\{\mathcal{F}_i\}$, $i = 1, \ldots, n$ is an increasing sequence of $\sigma-$algebras. Take $X_i = \varphi_{n-i} \circ T^{n-i}$, so that $X_i$ is $\mathcal{F}_i$ measurable. Recall $\psi_j = \varphi_j + H_j - H_{j+1} \circ T_{j+1}$ for all $j \geq 0$. We define $Y_i = \psi_{n-i} \circ T^{n-i}$, $h_i = H_{n-i} \circ T^{n-i}$ for $i \in \{1, \ldots, n\}$. Hence $Y_i = X_i + h_i - h_{i-1}$.

Note also that $\mathcal{G}_i := \sigma(X_1, \ldots, X_i) \subset \sigma(\mathcal{F}_1, \ldots, \mathcal{F}_i) = \mathcal{F}_i$, as $\sigma(X_i) \subset \mathcal{F}_i$ for all $i$. Since $\mathbb{E}(\psi_i \circ T^i|T^{-i-1}B) = 0$, $\mathbb{E}(Y_i|\mathcal{F}_i) = 0$ for all $j \geq i$. Hence $\mathbb{E}(Y_i|\mathcal{G}_i) = \mathbb{E}(\mathbb{E}(Y_i|\mathcal{F}_j)|\mathcal{G}_j) = 0$ for $j \geq i$.

Define $b_{i,n}$ as in Rio’s inequality, with $\mathcal{G}_i$, $X_i$ as described above so that

$$
b_{i,n} = \max_{i \leq u \leq n} \left\| X_i \sum_{k=i}^{n} \mathbb{E}(X_k|\mathcal{G}_i) \right\|_{L^p(m)}.
$$

Here all the expectations are taken with respect to $m$.

Recalling the expression we have for the martingale difference, we can write the sum inside the $p$-norm as

$$
\sum_{k=i}^{u} \mathbb{E}(X_k|\mathcal{G}_i) = \sum_{k=i}^{u} \left[ \mathbb{E}(Y_k|\mathcal{G}_i) - \mathbb{E}(h_k|\mathcal{G}_i) + \mathbb{E}(h_{k-1}|\mathcal{G}_i) \right]
$$

$$
= \left[ \sum_{k=i}^{u} \mathbb{E}(Y_k|\mathcal{G}_i) \right] + \mathbb{E}(h_{i-1}|\mathcal{G}_i) - \mathbb{E}(h_u|\mathcal{G}_i).
$$

If $k > i$, then $\mathbb{E}(Y_k|\mathcal{G}_i) = 0$. This reduces the expression above to

$$
\mathbb{E}(Y_i|\mathcal{G}_i) + \mathbb{E}(h_{i-1}|\mathcal{G}_i) - \mathbb{E}(h_u|\mathcal{G}_i).
$$

We note that $\|E[f|G]\|_p \leq \|f\|_p$ for any $f \in L^p(m)$, $p \geq 1$. Since $\|X_i\|_{\infty}$ is uniformly bounded we may bound $b_{i,n}$ by $\max_{i \leq u \leq n} \|X_i\|_{\infty} (\|Y_i\|_p + \|h_{i-1}\|_p + \|h_u\|_p)$.

Fix $1 \leq p < \frac{1}{\alpha} - 1$. Since $h_i = H_{n-i} \circ T^{n-i}$ are uniformly bounded in $L^p$ (see Lemma 3.7) and $Y_i = X_i + h_i - h_{i-1}$ we may bound $\max_{i \leq u \leq n} \|X_i\|_{\infty} (\|Y_i\|_p + \|h_{i-1}\|_p + \|h_u\|_p)$ by a constant $C_{\alpha,\|\varphi\|_{C^1}}$ independent of $n$. Thus $b_{i,n}$ is uniformly bounded in $i$ and $n$.

Therefore $(4p \sum_{i=1}^{n} b_{i,n}) \leq C_{\alpha,\varphi,p,n^p}$. By Rio’s inequality $E|X_1 + X_2 + \cdots + X_n|^{2p} \leq C_{\alpha,\varphi,p,n^p}$. Thus, by Markov’s inequality,

$$
m(|X_1 + \cdots + X_n|^{2p} > n^{2p} \epsilon^{2p}) \leq C_{\alpha,\varphi,p,n^p} (n^{2p-2p} \epsilon^{-2p}) = C_{\alpha,\varphi,p} \epsilon^{-2p}
$$

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for $p < \frac{1}{\alpha} - 1$. \qed

4.2. Random dynamical systems. Now we prove large deviations estimates for the randomized systems. First we recall some notation. The annealed transfer operator $P: L^1(m) \to L^1(m)$ is defined by averaging over all the transformations:

$$P = \sum_{\beta \in \Omega} p_\beta P_\beta = \int_\Sigma P_\omega \, d\nu(\omega).$$

This operator is dual to the annealed Koopman operator $U: L^\infty(m) \to L^\infty(m)$ and defined by

$$(U \varphi)(x) = \sum_{\beta \in \Omega} p_\beta \varphi(T_\beta x) = \int_\Sigma \varphi(T_\omega x) \, d\nu(\omega) = \int_\Sigma \tilde{\varphi}(F(\omega, x)) \, d\nu(\omega)$$

where $\tilde{\varphi}(\omega, x) := \varphi(x)$. The annealed operators satisfy the duality relationship

$$\int_X (U \varphi) \cdot \psi \, dm = \int_X \varphi \cdot P \psi \, dm$$

for all observables $\varphi \in L^\infty(m)$ and $\psi \in L^1(m)$.

Remark 4.3. It is easy to see that the averaged transfer operator $P$ has no worse rate of decay in $L^1$ than the slowest of the maps (so better than $n^{-\frac{1}{2}\alpha+1}(\log n)^{\frac{1}{2}}$, by Proposition 3.4). By taking a limit point of $\frac{1}{n} \sum_{k=1}^n P_k(1)$, there is an invariant vector $h$ for $P$ in the cone $C_2$, see [LSV99]. The measure $\mu = hm$ is stationary for the RDS; by Proposition 3.8, $h \geq D_\alpha > 0$.

Moreover, Bahsoun and Bose [BB16a, BB16b] have shown that there exists a unique absolutely continuous (with respect to the Lebesgue measure) stationary measure $\mu$. So $\nu \otimes \mu$ is ergodic.

Using the same idea as in the proof of Theorem 4.1, we can obtain an annealed result for the random dynamical system. Note that $P_\mu$, the transfer operator with respect to the stationary measure $\mu$, satisfies $P_\mu 1 = 1$ and so $\|P_\mu \varphi\|_{\infty} \leq P_\mu(\|\varphi\|_{\infty}) = \|\varphi\|_{\infty} P_\mu 1 = \|\varphi\|_{\infty}$. An easy calculation shows that $P_\mu(\varphi) = \frac{1}{h} P(h \varphi)$ where $h \in C_2$ is the density of the invariant measure $\mu$ and hence $h \geq D_\alpha m(h)$ is bounded below. As before this observation allows us to bootstrap in some sense the $L^1(\mu)$ decay rate to $L^p(\mu)$ for $p < 1 - \frac{1}{\alpha}$, a technique used in [MN08, Mel09].

Theorem 4.4 (Annealed LD). Let $\varphi \in C^1([0, 1])$ with $\mu(\varphi) = 0$ and let $0 < \alpha < 1$. Then, for each $\delta > 0$ the Birkhoff averages have annealed large deviations with respect to the measure $\nu \otimes \mu$ with rate $$(\nu \otimes \mu)\{((\omega, x) : \left| \sum_{j=1}^n \varphi \circ T^j_\omega (x) \right| \geq n\epsilon \} \leq C_{\alpha, \delta, \|\varphi\|_{C^1}} (n\epsilon^2)^{1-\frac{1}{\alpha}+\delta}.$$ Note that the Birkhoff sums above are not centered for a given realization $\omega$, only on average over $\Sigma$.

Proof. To prove this result we will use the construction used to prove the annealed CLT in [ANV15]: let $\Sigma_X := X^{\mathbb{N}_0}$, endowed with the $\sigma$-algebra $\mathcal{G}$ generated by the cylinders, and the left shift operator $\tau: \Sigma_X \to \Sigma_X$. 


Denote by $\pi$ the projection from $\Sigma_X$ onto the 0-th coordinate, that is, $\pi(x) = x_0$ for $x = (x_0, x_1, \ldots)$. We can lift any observable $\varphi: X \to \mathbb{R}$ to an observable on $\Sigma_X$ by setting $\varphi_\pi := \varphi \circ \pi: \Sigma_X \to \mathbb{R}$.

Following [ANV15, §4], one can introduce a $\tau$-invariant probability measure $\mu_c$ on $\Sigma_X$ such that $E_\mu(\varphi) = E_{\mu_c}(\varphi_\pi)$, and the law of $S_n(\varphi)$ on $\Sigma \times X$ under $\nu \otimes \mu$ is the same as the law of the $n$-th Birkhoff sum of $\varphi_\pi$ on $\Sigma_X$ under $\mu_c$ and $\tau$; thus it suffices to establish large deviations for the latter.

Define now

$$H_n := \sum_{k=1}^{n} P^k_\mu(\varphi) : X \to \mathbb{R}$$

From the relation $P^1_\mu(.) = \frac{1}{n} P(. h)$, we have that $\|P^n_\mu(\varphi)\|_{L^1(\mu)} \leq C_{\alpha, \varphi} n^{1-\frac{1}{p}}$ because $\mu(\varphi) = 0$. We calculate $E_\mu[|P^n_\mu(\varphi)|^p] = E_\mu[|P^n_\mu(\varphi)|^{p-1}] \leq \|P^n_\mu(\varphi)|_{L^1(\mu)}^p \|$. Hence $\|P^n_\mu(\varphi)|_{L^p(\mu)} \leq C_\varphi n^{(1-1/\alpha)/p}$ and thus $\|H_n\|_{L^p(\mu)}$ is uniformly bounded if $p < 1 - \frac{1}{\alpha}$.

We lift $\varphi$ and $H_n$ to $\Sigma_X$ and denote them by $\varphi_\pi$ and $H_n, n$, respectively, and define

$$\chi_n := \varphi_\pi + H_n, n - H_n, n \circ \tau : \Sigma_X \to \mathbb{R}$$

We now apply Rio’s inequality as in Theorem 4.4 with $p < 1 - \frac{1}{\alpha}$. For $i = 1, \ldots, n$ we take the sequence $\{X_i = \varphi_\pi \circ \tau^{-i}\}$, $\{Y_i = \chi_n - i \circ \tau^{-i}\}$ and $G_i = \tau^{-i} G_i$. We have $E_{\mu_c}[Y_i | G_k] = 0$ for $k > i$ and so

$$\max_{i \leq u \leq n} \left\| \frac{1}{n} \sum_{k=i}^{n} E_{\mu_c}(X_k | G_i) \right\|_{L^p(\mu_c)}$$

is uniformly bounded in $n$ and $n$. Thus $(4p \sum_{i=1}^{n} b_i n)^p \leq C_{\alpha, \varphi, p} n^p$ and therefore, for $p < 1 - \frac{1}{\alpha}$,

$$\mu_c(|X_1 + \ldots + X_n|^{2p} > n^{2p} \epsilon^{2p}) \leq C_{\alpha, \varphi, p} n^{p - 2p} \epsilon^{-2p} = C_{\alpha, \varphi, p} n^{-p} \epsilon^{-2p}$$

which implies the result.

Using similar ideas, it is possible to obtain an annealed central limit theorem. This has been established already by Young Tower techniques in [BB16a, Theorem 3.2]. We include the statement of the annealed central limit and an alternative proof for completeness and to give an expression for the annealed variance.

**Proposition 4.5 (Annealed CLT).** If $\alpha < \frac{1}{2}$ and $\varphi \in C^1$ with $\mu(\varphi) = 0$ then a central limit theorem holds for $S_n \varphi$ on $\Sigma \times X$ with respect to the measure $\nu \otimes \mu$, that is, $\frac{1}{\sqrt{n}} S_n \varphi$ converges in distribution to $N(0, \sigma^2)$, with variance $\sigma^2$ given by

$$\sigma^2 = -\mu(\varphi^2) + 2 \sum_{k=0}^{\infty} \mu(\varphi U^k \varphi)$$

**Proof.** We will use the results of [ANV15, Section 4] and [Liv96 Theorem 1.1] (see Theorem 6.3 in the Appendix). We proceed as in Theorem 4.4, using the averaged operators $U$ and $P$. As in [ANV15, Section 4], to $U$ corresponds a transition probability on $X$ given by $U(x, A) = \sum_{\beta} \rho_{\beta} : T_{\beta} x \in A$. The stationary measure $\mu$ is invariant under $U$. Extend $\mu$ to the unique probability measure $\mu_c$ on $\Sigma_X := X^{\mathbb{N}_0} = \{ x = (x_0, x_1, x_2, \ldots, x_n, \ldots) \}$ endowed with the $\sigma$-algebra $\mathcal{G}$ given by cylinder sets, such corresponding to $\mu$ such that $\{ x_n \}_{n \geq 0}$ is a Markov chain on $(\Sigma_X, \mathcal{G}, \mu_c)$ (where $x_n$ is the $n$-th coordinate of $x$) induced by the random
Choose Proof of Theorem 4.6. for almost each realization \( \omega \) that for almost each realization the large deviation estimates hold even without centering.

Observe now that \( \{ \}

By Theorem 4.4, we also have the annealed estimate for the non-centered sums:

\[ m \]

That is, the contribution of the means (with respect to the measure \( \mu \))

The point of the above Theorem, compared to the sequential Theorem 4.1, is

because

for some \( \sigma \). Theorem 1.1. The stated formula for \( \sigma^2 \) is also given in [Liv96, Theorem 1.1].

We will use the annealed and sequential results to obtain quenched large deviations for random systems of intermittent maps. We denote the Birkhoff sums by \( S_n(x) \) to stress the dependance on the realization \( \omega \).

**Theorem 4.6 (Quenched LD).** Suppose \( \varphi \in C^1 \) and \( \mu(\varphi) = 0 \). Fix \( 0 < \alpha < 1/2 \). Then, for every \( \epsilon > 0 \), \( \delta > 0 \), for \( \nu \)-almost every realization \( \omega \), the Birkhoff averages have large deviations with polynomial rate, even without centering:

\[ \mu\{ x : S_{n, \omega} \varphi > n \epsilon \} \leq C_\omega C_{\alpha, \varphi}(\epsilon^2 n)^{-\frac{1}{2} + 1 + \delta} \]

for some \( C_\omega, C_{\alpha, \varphi} > 0 \). The same decay holds with respect to the Lebesgue measure \( m \), because \( \frac{dm}{d\mu} = 1/h \) is bounded above.

Again, note that the Birkhoff sums \( S_{n, \omega} \varphi \) above are not centered with respect to the realization \( \omega \), only on average over \( \Sigma \).

**Remark 4.7.** The point of the above Theorem, compared to the sequential Theorem 4.1, is that for almost each realization the large deviation estimates hold even without centering. That is, the contribution of the means (with respect to the measure \( m \) on \( X \)) can be ignored for almost each realization \( \omega \).

It is likely that, maybe with additional hypotheses, this result holds for all \( \alpha < 1 \).

**Proof of Theorem 4.6.** Choose \( \delta > 0 \) and \( \epsilon > 0 \). By Theorem 4.1 for all \( \omega \in \Omega \),

\[ \mu\left\{ x : \left| \frac{1}{n} S_{n, \omega} \varphi(x) - \frac{1}{n} \sum_{j=1}^{n} m(\varphi \circ T_{\omega}^j) \right| \geq \epsilon \right\} \leq C_{\alpha, \varphi, \delta}(\epsilon^2 n)^{1 - \frac{1}{\alpha} + \delta} \]

with \( C_{\alpha, \varphi, \delta} \) independent of \( \omega \). Integrating over \( \Sigma \) with respect to \( \nu \) we obtain

\[ \nu \otimes \mu\left\{ (\omega, x) : \left| \frac{1}{n} S_{n, \omega} \varphi(x) - \frac{1}{n} \sum_{j=1}^{n} m(\varphi \circ T_{\omega}^j x) \right| \geq \epsilon \right\} \leq C_{\alpha, \varphi, \delta}(\epsilon^2 n)^{1 - \frac{1}{\alpha} + \delta} \]

By Theorem 4.4 we also have the annealed estimate for the non-centered sums:

\[ \nu \otimes \mu\left\{ (\omega, x) : \left| \frac{1}{n} S_{n, \omega} \varphi(x) \right| \geq \epsilon \right\} \leq C'_{\alpha, \varphi}(\epsilon^2 n)^{1 - \frac{1}{\alpha} + \delta} \]

Observe now that

\[ \left\{ (\omega, x) : \left| \frac{1}{n} \sum_{j=1}^{n} m(\varphi \circ T_{\omega}^j) \right| > 2\epsilon \right\} \]
\[ \subset \left\{ (\omega, x) : \left| \frac{1}{n} S_{n, \omega} \varphi(x) \right| < \epsilon, \left| \frac{1}{n} S_{n, \omega} \varphi(x) - \frac{1}{n} \sum_{j=1}^{n} m(\varphi \circ T_{j}^{\omega}) \right| \geq \epsilon \right\} \]

\[ \bigcup \left\{ (\omega, x) : \left| \frac{1}{n} S_{n, \omega} \varphi(x) \right| > \epsilon \right\} . \]

Thus

\[ \nu \otimes m \left\{ (\omega, x) : \left| \frac{1}{n} \sum_{j=1}^{n} m(\varphi \circ T_{j}^{\omega}) \right| > 2\epsilon \right\} \leq K_{\alpha, \varphi, \delta} (\epsilon^2 n)^{1 - \frac{1}{\alpha} + \delta} \]

and, as there is no dependence on \( x \in X \), this means

\[ \nu \left\{ \omega : \left| \frac{1}{n} \sum_{j=1}^{n} m(\varphi \circ T_{j}^{\omega}) \right| > 2\epsilon \right\} \leq K_{\alpha, \varphi, \delta} (\epsilon^2 n)^{1 - \frac{1}{\alpha} + \delta} . \]

Take now \( \delta > 0 \) so that \( \frac{1}{\alpha} - 1 - \delta > 1 \). Then a straightforward application of Borel-Cantelli gives the desired result, since \( \nu \)-a.e. \( \omega \) is in at most finitely many of these sets. \( \square \)

We remark that the methods used to prove these results in the uniformly expanding case are not applicable here, as they rely on the quasi-compactness of the transfer operator. In the uniformly expanding case, which has exponential large deviations for Hölder observables, it is possible to obtain a rate function.

### 5. The Role of Centering in the Quenched CLT for RDS

In this section we discuss two results: Proposition 5.1, that the quenched variance is the same for almost all realizations \( \omega \in \Sigma \), and Theorem 5.3, that generically one must center the observations in order to obtain a CLT (as opposed to LD Theorem 4.6, where centering did not affect the quenched LD). Note that these hinge on the rate of growth of the mean of the Birkhoff sums; we see that it is \( o(n) \) but not \( o(\sqrt{n}) \). We use the recent paper by Hella and Stenlund [HS20] to extend and clarify results of [NTV18].

In [NTV18, Theorem 3.1] a self-norming quenched CLT is obtained for \( \nu \)-a.e. realization \( \omega \) of the random dynamical system of Theorem 4.4. More precisely, recalling the definition of the centered observables \( [\varphi]_k(\omega, x) = \varphi(x) - m(\varphi \circ T_{k}^{\omega}) \) and \( \sigma_n^2(\omega) := \int [\sum_{k=1}^{n} [\varphi]_k(\omega, \cdot) \circ T_{k}^{\omega}]^2 \, dx \) it is shown that \( \frac{1}{\sigma_n(\omega)} \sum_{k=1}^{n} [\varphi]_k(\omega, \cdot) \circ T_{k}^{\omega} \rightarrow N(0, 1) \) provided \( \sigma_n^2 \approx n^\beta \), with \( \alpha < \frac{1}{2} \) and \( \beta > \frac{1}{2(1-2\alpha)} \). Various scenarios under which \( \sigma_n^2(\omega) > n^\beta \) are given in [NTV18]. See also [HL19].

If the maps \( T_{\omega} \) preserved the same invariant measure then it suffices to consider observables with mean zero, since the mean would be the same along each realization. In the setting of [ALS09] this is the case, namely all realizations preserve Haar measure, and the authors address the issue of whether the variance \( \sigma_n^2(\omega) \) can be taken to be the “same” for almost every quenched realization in the setting of random toral automorphisms. They show that for almost every quenched realization the variance in the quenched CLT may be taken as a uniform constant. The technique they use is adapted from random walks in random environments and consists in analyzing a random dynamical system on a product space.

A natural question is whether in our setup of random intermittent maps, after centering, \( \sigma_n(\omega) \) can be taken to be “uniform” over \( \nu \)-a.e. realization. Recent results of Hella and Stenlund [HS20] give conditions under which \( \frac{1}{n} \sigma_n^2(\omega) \rightarrow \sigma^2 \) for \( \nu \)-a.e. \( \omega \), as well as information
about rates of convergence. Note that this is also true in the context of uniformly expanding
maps considered by [AA16] using the same method used in [HS20].
A related question is whether we need to center at all. For example, if \( \mu(\varphi) = 0 \), where \( \mu \)
is the stationary measure on \( X \), then for \( \nu \)-a.e \( \omega \)
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} [\varphi(T^j_\omega x) - m(\varphi(T^j_\omega))] \to 0 \quad \text{for} \quad \mu\text{-a.e. } x
\]
by the ergodicity of \( \nu \otimes \mu \), but also
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} \varphi(T^j_\omega) \to 0 \quad \text{for} \quad \mu\text{-a.e. } x,
\]
see the proof of Theorem 4.6. So for the strong law of large numbers centering is not necessary. Using ideas of [AA16] we consider the related question of whether centering is necessary to obtain a quenched CLT with almost surely constant variance. We show the answer to this is positive as well: to obtain an almost surely constant variance in the quenched CLT we need to center.

5.1. Non-random quenched variance. For Proposition 5.1, we verify that our system satisfies the conditions SA1, SA2, SA3 and SA4 of [HS20]; then, by [HS20, Theorem 4.1], the quenched variance is almost surely the same, equal to the annealed variance.

**Proposition 5.1.** Let \( \alpha < \frac{1}{2} \), \( \varphi \in C^1 \) and define the annealed variance
\[
\sigma^2 := \lim_{n \to \infty} \frac{1}{n} \|S_n\|_{L^2}^2 = \lim_{n \to \infty} \frac{1}{n} \|S_n - \int_{\Sigma \times X} S_n d \nu \otimes m\|_{L^2(\nu \otimes m)}^2
\]
\[
= \sum_{k=0}^{\infty} (2 - \delta_{0k}) \lim_{i \to \infty} \int_{\Sigma} [m(\varphi_i \varphi_{i+k}) - m(\varphi_i)m(\varphi_{i+k})]d\nu
\]
If \( \sigma^2 > 0 \) then for \( \nu \)-a.e. \( \omega \)
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [\varphi(T^j_\omega \cdot)]^m \to^d N(0, \sigma^2)
\]
in distribution with respect to \( m \).

**Remark 5.2.** Proposition 4.6 shows that the annealed CLT holds for \( \alpha < \frac{1}{2} \) and under the usual genericity conditions the annealed variance satisfies \( \sigma^2 > 0 \). Thus Proposition 5.1 extends [NTV18, Theorem 5.3] from the parameter range \( \alpha < \frac{1}{9} \) to \( \alpha < \frac{1}{2} \). Note that [HL19], proved the CLT for \( \alpha < \frac{1}{3} \).

**Proof of Proposition 5.1.** We will verify conditions SA1, SA2, SA3 and SA4 of [HS20, Theorem 4.1] in our setting, with \( \eta(k) = C k^{-\frac{1}{2}+1} (\log k)^{\frac{1}{n}} \) in the notation of [HS20]; we again will ignore the log-correction.

**SA1:** If \( j > i \) then
\[
\left| \int \varphi \circ T^i_\omega(x) \varphi \cdot T^j_\omega(x) dm - \int \varphi \circ T^i_\omega(x) dm \int \varphi \circ T^j_\omega(x) dm \right|
\]
\[
= \left| \int \varphi \circ T^{i+1}_\omega T^{j-i}_\omega(x) \varphi(x) P^i_\omega 1 dm - \int \varphi P^i_\omega 1 dm \int \varphi(x) P^j_\omega 1 dm \right| \leq C(j-i)^{-\frac{1}{2}+1}
\]
by the same argument as in the proof of [NTV18 Proposition 1.3].

SA2: Our underlying shift $\sigma : \Sigma \to \Sigma$ is Bernoulli hence $\alpha$-mixing.

SA3: We need to check [HS20], equations (4), (5) that

$$\left| \int \varphi(T_{\omega_k} T_{\omega_{k-1}} \cdots T_{\omega_1} x) dm - \int \varphi(T_{\omega_k} T_{\omega_{k-1}} \cdots T_{\omega_{r+1}} x) dm \right| \leq C\eta(k-r).$$

This follows since

$$\left| \int \varphi(T_{\omega_k} T_{\omega_{k-1}} \cdots T_{\omega_1} x) dm - \int \varphi(T_{\omega_k} T_{\omega_{k-1}} \cdots T_{\omega_{r+1}} x) dm \right|$$

$$= \left| \int \varphi(x) P_{\omega_k} P_{\omega_{k-1}} \cdots P_{\omega_1} 1 dm - \int \varphi(x) P_{\omega_k} P_{\omega_{k-1}} \cdots P_{\omega_{r+1}} 1 dm \right|$$

$$\leq \|\varphi\|_{\infty} \|P_{\omega_k} P_{\omega_{k-1}} \cdots P_{\omega_{r+1}} [1 - P_{\omega_r} \cdots P_{\omega_1}]\|_{L^1}$$

Since $1$ and $P_{\omega_r} \cdots P_{\omega_1}$ both lie in the cone and have the same $m$-mean, we have

$$\|P_{\omega_k} P_{\omega_{k-1}} \cdots P_{\omega_{r+1}} [1 - P_{\omega_r} \cdots P_{\omega_1}]\|_{L^1} \leq C(k-r)^{-\frac{1}{2} + 1} \log(k-r)$$

by [NTV18 Theorem 1.2].

SA4: $(\sigma, \Sigma, \nu)$ is stationary so SA4 is automatic.

5.2. Centering is generically needed in the CLT. Now we address the question of the necessity of centering in the quenched central limit theorem. We show that if $\int \varphi d\mu_{\beta_j} \neq \int \varphi d\mu_{\beta_i}$ for two maps $T_{\beta_i}, T_{\beta_j}$, where $\mu_{\beta_i}$ is the invariant measure of $T_{\beta_i}$, then centering is needed: although

$$\lim_{n \to -\infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} [\varphi(T_{\omega_j}) - m(\varphi(T_{\omega_j}))] \to^d N(0, \sigma^2)$$

for $\nu$-a.e. $\omega$, it is not the case that

$$\lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \varphi(T_{\omega_j}) \to^d N(0, \sigma^2)$$

for $\nu$-a.e. $\omega$.

Our proof has the same outline as that of [AA16], adapted to our setting of polynomial decay of correlations. First we suppose that the maps $T_{\beta_i}$ do not preserve the same measure. After reindexing we can suppose that $T_{\beta_1}$ and $T_{\beta_2}$ have different invariant measures and that $\int \varphi d\mu_{\beta_1} \neq \int \varphi d\mu_{\beta_2}$, a condition satisfied by an open and dense set of observables. Recall that the RDS has the stationary measure $d\mu = hdm$, $h \geq D_\alpha > 0$ and we have assumed $\mu(\varphi) = 0$, $\varphi \in C^1$.

Here are the steps:

- construct a product random dynamical system on $X \times X$ and prove that it satisfies an annealed CLT for $\tilde{\varphi}(x, y) = \varphi(x) - \varphi(y)$ with distribution $N(0, \tilde{\sigma}^2)$;
- observe that almost every uncentered quenched CLT has the same variance only if $2\sigma^2 = \tilde{\sigma}^2$, where the original RDS with stationary measure $d\mu = hdm$ satisfies an annealed CLT for $\varphi$ with distribution $N(0, \sigma^2)$;
- observe that the conclusions of [AA16] Theorem 9 hold in our setting and $\tilde{\sigma}^2 = 2\sigma^2$ if and only if $\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n-1} \int_X \varphi(T_k x) hdm)^2 d\nu = 0$;
- use ideas of [AA16] to show the limit above is zero only if a certain function $G$ on $\Sigma$ is a Hölder coboundary, which in turn implies $\int \varphi d\mu_{\beta_1} = \int \varphi d\mu_{\beta_2}$, a contradiction.
Let $\varphi : X \to \mathbb{R}$ be $C^1$, with $\int_X \varphi d\mu = 0$, and define $S_n(\varphi) = \sum_{k=0}^{n-1} \varphi(T^k_\omega x)$ on $\Sigma \times X$. Recall the standard expression (e.g. see [AA16]) for the annealed variance,

$$
\sigma^2 = \lim_{n \to \infty} \frac{1}{n} \int \int_X [S_n(\varphi)]^2 \, d\mu \, d\nu.
$$

We also consider the product random dynamical system $(\Sigma := \Sigma \times X \times X, \nu := \nu \otimes \mu \otimes \mu, \bar{T})$ defined on $X^2$ by $\bar{T}(x, y) = (T_\omega x, T_\omega y)$. For an observable $\varphi$, define $\bar{\varphi} : X^2 \to \mathbb{R}$ by $\bar{\varphi}(x, y) = \varphi(x) - \varphi(y)$, and its Birkhoff sums $S_n(\bar{\varphi})$. In Theorem 6.1 and Corollary 6.2 of the Appendix we show $\frac{1}{\sqrt{n}} \sum_{j=1}^n \bar{\varphi} \circ \bar{T}^j \to_d N(0, \bar{\sigma}^2)$ with respect to $\nu \otimes \mu \otimes \mu$ for some $\bar{\sigma}^2 \geq 0$.

The following lemma from [ANV15] is general and does not depend upon the underlying dynamics. It is a consequence of Levy’s continuity theorem (Theorem 6.5 in [Kar93]).

**Lemma ([ANV15] Lemma 7.2]).** Assume that $\sigma^2 > 0$ and $\bar{\sigma}^2 > 0$ are such that

1. $S_n(\varphi) / \sqrt{n}$ converges in distribution to $N(0, \sigma^2)$ under the probability $\nu \otimes \mu$,
2. $S_n(\bar{\varphi}) / \sqrt{n}$ converges in distribution to $N(0, \bar{\sigma}^2)$ under the probability $\nu \otimes \mu \otimes \mu$,
3. $S_n(\bar{\varphi}) / \sqrt{n}$ converges in distribution to $N(0, \sigma^2)$ under the probability $\mu$, for $\nu$ almost every $\omega$.

Then $2\sigma^2 = \bar{\sigma}^2$.

Suppose two of the maps $T_{\beta_1}$ and $T_{\beta_2}$ have different invariant measures. It is possible to find a $C^1$ $\varphi$ such that $\int \varphi d\mu_{\beta_1} \neq \int \varphi d\mu_{\beta_2}$. In fact, $\int \varphi d\mu_{\beta_1} \neq \int \varphi d\mu_{\beta_2}$ for a $C^2$ open and dense set of $\varphi$.

**Theorem 5.3.** Let $\varphi \in C^1$ with $\mu(\varphi) = 0$ and suppose that $\varphi d\mu_{\beta_1} \neq \varphi d\mu_{\beta_2}$. Then it is not the case that

$$
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{j=1}^n \varphi(T_\omega^j) \to N(0, \bar{\sigma}^2)
$$

for almost every $\omega \in \Sigma$. Hence, the Birkhoff sums need to be centered along each realization.

**Proof.** We follow the counterexample method of [AA16] Section 4.3. We show that in the uncentered case $2\sigma^2 \neq \bar{\sigma}^2$. To do this we use [AA16] Theorem 9 which holds in our setting, namely $\bar{\sigma}^2 = 2\sigma^2$ if and only if

$$
(5.1) \quad \lim_{n \to \infty} \int_{\Sigma} \left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n-1} \int_X \varphi P_{\omega_k} \ldots P_{\omega_n}(h) dm \right)^2 \, d\nu = 0
$$

(as in [AA16] Section 4.3) we change the time direction and replace $(\omega_1, \omega_2, \ldots, \omega_n)$ by $(\omega_n, \omega_2, \ldots, \omega_1)$; this does not affect integrals with respect to $\nu$ over finitely many symbols).

Note that the sequence $P_{\omega_1}P_{\omega_2} \ldots P_{\omega_n} h$ is Cauchy in $L^1$, as $\alpha < \frac{1}{2}$ and

$$
\|P_{\omega_1}P_{\omega_2} \ldots P_{\omega_n}(h) - P_{\omega_1}P_{\omega_2} \ldots P_{\omega_n}P_{\omega_{n+1}}(h)\|_1 \leq Cn^{-\frac{1}{n}+1}
$$

by Proposition [3.4]. Thus $P_{\omega_1}P_{\omega_2} \ldots P_{\omega_n} h \to h_\omega$ in $L^1$ for some $h_\omega \in C_2$. This limit defines $h_\omega$, in terms of $\bar{\omega} := (\ldots, \omega_n, \omega_2, \ldots, \omega_1)$, i.e. $\omega$ reversed in time. We define $G(\omega) := \int_X \varphi h_\omega dm$. Note also that $\|P_{\omega_1}P_{\omega_2} \ldots P_{\omega_n} h - h_\omega\|_1 \leq Cn^{-1-\delta}$ for some $\delta > 0$, uniformly for $\omega \in \Sigma$. 


Hence
\[
\int_{\Sigma} \left( \sum_{k=1}^{n-1} \frac{1}{\sqrt{n}} \int_X \varphi P_{\omega_k} \cdots P_{\omega_n} h dm \right)^2 d\nu \\
= \int_{\Sigma} \left( \sum_{k=1}^{n-1} \frac{1}{\sqrt{n}} \left( \int_X \varphi h_{\tau^k \omega} dm + O \left( \sum_{k=1}^{n-1} \frac{1}{(n-k)^{1+\delta}} \right) \right) \right)^2 d\nu
\]
which gives, using (5.1), that
\[
(5.2) \quad \lim_{n \to \infty} \int_{\Sigma} \left( \frac{1}{\sqrt{n}} \left( \sum_{k=1}^{n-1} G(\tau^k \omega) \right) \right)^2 d\nu = 0.
\]

We put a metric on \( \Sigma \) by defining \( d(\omega, \omega') = s(\omega, \omega')^{-1-\frac{\epsilon}{2}} \) where \( s(\omega, \omega') = \inf\{n : \omega_n \neq \omega'_n\} \). With this metric \( \Sigma \) is a compact and complete metric space. Note that \( \|h_\omega - h_{\omega'}\|_{L^1} \leq Cs(\omega, \omega')^{-\frac{\epsilon}{2}} \) hence \( G(\omega) \) is Hölder with respect to our metric.

As in the Abdulkader-Aïmno counterexample, (5.2) implies that \( G = H - H \circ \tau \) for a Hölder function \( H \) on the Bernoulli shift \( (\tau, \Sigma, \nu) \): by [Liv96, Theorem 1.1] (see Theorem 6.3 in the Appendix) \( G \) is a measurable coboundary, and therefore a Hölder coboundary, by the standard Livšic regularity theorem (see for instance [VO16, Section 12.2]). Now consider the points \( \beta_1 := (\beta_1, \beta_2, \ldots) \) and \( \beta_2 := (\beta_1, \beta_2, \ldots) \) in \( \Sigma \); they are fixed points for \( \tau \), and correspond to choosing only the map \( T_{\beta_1} \), respectively only the map \( T_{\beta_2} \). This implies \( G(\beta_1) = G(\beta_2) = 0 \) which in turn implies \( \int \varphi dm_{\beta_1} = \int \varphi dm_{\beta_2} \), a contradiction. \( \square \)

6. Appendix

We will show that the system \( \tilde{F}(\omega, x, y) = (\tau \omega, T_{\omega_1} x, T_{\omega_1} y) \) with respect to the measure \( \nu \otimes \mu^2 \) on \( \Sigma \times [0, 1]^2 \) (recall that \( \nu := \mathbb{P} \otimes \mathbb{N} \) and \( \mu \) is a stationary measure of the RDS) has summable decay of correlations in \( L^2 \) for \( \alpha < \frac{1}{2} \), and as a corollary it satisfies the CLT.

**Theorem 6.1.** Suppose that for \( \omega \in \Sigma \), \( h = \frac{dm}{dm} \in C_2 \) and each \( \varphi \in C^1 \) with \( m(\varphi h) = 0 \)
\[
\|P_{\omega_n} \cdots P_{\omega_1} (\varphi h)\|_{L^1(m)} \leq C \rho(n)(\|\varphi\|_{C^1} + m(h))
\]
(that is, the setting of Proposition 3.4).

Then there is a constant \( \tilde{C} \), independent of \( \omega \), such that for each \( \psi \in C^1(X \times X) \) and \( \varphi \in L^\infty(X \times X) \) with \( (\mu \otimes \mu)(\psi) = 0 \), one has
\[
\left| \int \varphi(T^n \omega x, T^n \omega y) \psi(x, y) d\mu(x) d\mu(y) \right| \leq \tilde{C} \rho(n) \|\varphi\|_{L^\infty} (\|\psi\|_{C^1} + 1)
\]

**Proof.** Since \( X \times X \) is compact, \( \psi \) is uniformly \( C^1 \) in both variables in the sense that \( \psi(x_0, y) \) is uniformly \( C^1 \) for each \( x_0 \) and similarly for \( \psi(x, y_0) \). We want to estimate
\[
I := \int \varphi(T^n \omega x, T^n \omega y) \psi(x, y) d\mu(x) d\mu(y).
\]

Define
\[
\overline{\psi}(x) := \int \psi(x, y) d\mu(y), \quad h_x(y) := \psi(x, y) - \overline{\psi}(x).
\]

Then \( \overline{\psi}, h_x \in C^1(X) \), with \( C^1 \)-norms bounded by \( 2\|\psi\|_{C^1} \), uniformly with respect to \( x \).
We can write $I$ as

$$I = \int \varphi(T_\omega^n x, T_\omega^n y) \left[ \psi(x, y) - \bar{\psi}(x, y) \right] d\mu(x)d\mu(y)$$

$$= I_1 + \int \varphi(T_\omega^n x, T_\omega^n y)\bar{\psi}(x, y)d\mu(x)d\mu(y).$$

Define now $g_{\omega,x}(y) := \varphi(T_\omega^n x, y)$. Then (note that $\int h_x(y)h(y)dm(y) = 0$)

$$|I_1| = \left| \int \left( \int g_{\omega,x}(T_\omega^n y)h_x(y)dm(y) \right) d\mu(x) \right| = \left| \int \left( \int \varphi_{\omega,x}(y)P^n_\omega(h_x(y)h(y))dm(y) \right) d\mu(x) \right|$$

$$\leq \| \varphi \|_{L^\infty} \sup_x \| P^n_\omega(h_x(y)h(y)) \|_{L^1(m(y))}$$

$$\leq C'\| \varphi \|_{L^\infty} (\| \psi \|_{C^1} + m(h))\rho(n).$$

These imply that $|I| \leq 2C'\| \varphi \|_{L^\infty} (\| \psi \|_{C^1} + m(h))\rho(n).$ \hfill \Box

**Corollary 6.2.** Under the assumptions of Theorem 6.1, for $\psi \in C^1(X \times X)$ with $(\mu \otimes \mu)(\psi) = 0$, \( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \psi \circ \tilde{F}^k(\omega, x, y) \) satisfies a CLT with respect to $\nu \otimes \mu \otimes \mu$, that is

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \psi \circ \tilde{F}^k(\omega, x, y) \longrightarrow^d N(0, \tilde{\sigma}^2)$$

in distribution for some $\tilde{\sigma}^2 \geq 0$.

**Proof.** Let $Q$ be the adjoint of $\tilde{F}(\omega, x, y) = (\sigma_\omega T_{\omega_1} x, T_{\omega_1} y)$ with respect to the invariant measure $\nu \otimes \mu \otimes \mu$ on $\Sigma \times X^2$ so that

$$\int \varphi \circ \tilde{F}(\omega, x, y)\psi(\omega, x, y)d\mu(x)d\mu(y)d\nu(\omega) = \int \varphi(\omega, x, y)(Q\psi)(\omega, x, y)d\mu(x)d\mu(y)d\nu(\omega).$$

for $\varphi \in L^\infty(\Sigma \times X \times X)$. Iterating we have

$$\int \varphi \circ \tilde{F}^n(\omega, x, y)\psi(\omega, x, y)d\mu(x)d\mu(y)d\nu(\omega) = \int \varphi(\omega, x, y)(Q^n\psi)(\omega, x, y)d\mu(x)d\mu(y)d\nu(\omega).$$

Taking $\varphi = \text{sign}(Q^n\psi)$, we see from Theorem 6.1 that $\| Q^n\psi \|_{L^1} \leq C'\rho(n)$.

The proof now follows, as in Proposition 4.5 from [Liv96, Theorem 1.1] (see Theorem 6.3 in the Appendix). \hfill \Box
**Proof of Lemma 3.5.** Let \( f_1 = (\varphi + \lambda x + A)h + B \) and \( f_2 = (A + \lambda x)h + B \).

First we show that \( f_1 \in C_2 \). It is clear that \( f_1 \in C^0(0,1] \cap L^1(m) \). Choose \( \lambda < 0 \) such that \( |\lambda| > \|\varphi'\|_{L^\infty} \) and \( A > 0 \) large enough so that
\[ \varphi + \lambda x + A > 0. \]

This ensures that \( f_1 \geq 0 \) for any value of \( B \geq 0 \). Note now that
\[ (\varphi + \lambda x + A)' = \varphi' + \lambda \leq 0 \]
so \( \varphi + \lambda x + A \) is decreasing. Since both \( \varphi + \lambda x + A \) and \( h \) are positive and decreasing, we obtain that \( f_1 \) is decreasing as well. We show now that \( x^{a+1} f_2 \) is increasing. Since \( h \in C_2 \), \( h \) is non-increasing so \( h' \) exists \( m \)-a.e. and \( h' \leq 0 \) \( m \)-a.e. Then \( (x^{a+1}h)' \) exists \( m \)-a.e. as well, and we can compute this derivative as
\[ (x^{a+1}h)' = (\alpha + 1)x^\alpha h + x^{a+1}h' \geq 0 \]
because it is increasing.

We compute now the derivative of \( x^{a+1} f_2 \):
\[ (x^{a+1}[(\varphi + \lambda x + A)h + B])' = (\alpha + 1)x^\alpha \varphi h + x^{a+1}\varphi'h + x^{a+1}\varphi'h' + (\alpha + 2)x^{a+1}h\lambda + \lambda x^{a+2}h' + (\alpha + 1)Ax^\alpha h + Ax^{a+1}h' + (\alpha + 1)x^\alpha B. \]

We group terms conveniently: note that
\[ (\alpha + 1)x^\alpha \varphi h + (\alpha + 1)Ax^\alpha h + x^{a+1}\varphi'h + Ax^{a+1}h' = (\varphi + A)[(\alpha + 1)x^\alpha h + h'x^{a+1}] \geq 0 \]
m-a.e., since the term in the square brackets corresponds to \( (x^{a+1}h)' \geq 0 \). The term \( \lambda x^{a+2}h' \) is non-negative \( m \)-a.e. since \( \lambda, h' \leq 0 \). Since \( 0 \leq h(x)x^\alpha \leq am(h) \), we have \( 0 \leq -x^{a+1}h' \leq (\alpha + 1)x^\alpha h \leq (\alpha + 1)am(h) \) and then the terms \( (\alpha + 2)\lambda x^{a+1}h + x^{a+1}h\varphi' \) are bounded. Thus, we can take \( B > 0 \) big enough so that
\[ (\alpha + 1)x^\alpha B \geq (\alpha + 2)\lambda x^{a+1}h + x^{a+1}h\varphi'. \]
With this, we have that \( (x^{a+1}h)' \geq 0 \) and so \( x^{a+1}h \) is increasing.

Finally, we check that \( f_1(x)x^\alpha \leq am(f_1) \). Using that \( h(x)x^\alpha \leq am(h) \),
\[ [(\varphi + \lambda x + A)h + B]x^\alpha \leq (\varphi + \lambda x + A)hx^\alpha + B \leq \sup(\varphi + \lambda x + A)am(h) + B. \]
On the other hand, \( am((\varphi + \lambda x + A)h + B) \geq a \inf(\varphi + \lambda x + A)m(h) + aB \), so it suffices to have
\[ \sup(\varphi + \lambda x + A)am(h) + B \leq a \inf(\varphi + \lambda x + A)m(h) + aB \]
\[ \iff B \geq \frac{a}{a-1} [\sup(\varphi + \lambda x + A) - \inf(\varphi + \lambda x + A)]m(h). \]

Thus, we see that \( f_1 \in C_2 \). The proof that \( f_2 \in C_2 \) is the same, take \( \varphi(x) \equiv 0 \). \( \square \)

**Theorem 6.3** (special case of [Liv96, Theorem 1.1]). Assume \( T : Y \to Y \) preserves the probability measure \( \eta \) on the \( \sigma \)-algebra \( B \). Denote by \( P \) its transfer operator.

If \( \varphi \in L^\infty(\eta) \) with \( \eta(\varphi) = 0 \) and \( \sum_k \|P^k \varphi\|_{L^1(\eta)} < \infty \) then a central limit theorem holds for \( S_n \varphi := \sum_{k=1}^n \varphi \circ T^k \) with respect to the measure \( \eta \), that is, \( \frac{1}{\sqrt{n}} S_n \varphi \) converges in distribution to \( \mathcal{N}(0, \sigma^2) \). The variance is given by
\[ \sigma^2 = -\eta(\varphi^2) + 2 \sum_{k=0}^{\infty} \eta(\varphi \cdot \varphi \circ T^k). \]
In addition, $\sigma^2 = 0$ iff $\varphi \circ T$ is a measurable coboundary, that is $\varphi \circ T = g - g \circ T$ for a measurable g.

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