COHOMOLOGY OF DYNAMICAL SYSTEMS AND RIGIDITY OF PARTIALLY HYPERBOLIC ACTIONS OF HIGHER RANK LATTICES

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§1. Introduction

Let $SL(n,\mathbb{Z})$ be the group of $n \times n$ integer matrices of determinant one, and let π be the standard action by linear automorphisms of $SL(n,\mathbb{Z})$ on the torus $\mathbb{T}^n$. Let d be a positive integer and consider the action ρ_0 of $SL(n,\mathbb{Z})$ on $\mathbb{T}^{n+d} = \mathbb{T}^n \times \mathbb{T}^d$ given by $\rho_0(A)(x,y) = (\pi(A)x, y)$ (where $x \in \mathbb{T}^n$, $y \in \mathbb{T}^d$ and $A \in SL(n,\mathbb{Z})$). Then ρ_0 is a (non-irreducible and non-transitive) partially hyperbolic action.

This paper emerges from an attempt to understand the differentiable actions close to ρ_0. After Zimmer initiated the program to classify volume-preserving actions of higher rank lattices on compact manifolds (see [Z1], [Z2]), Hurder, Katok, Lewis, Qian, Spatzier, Zimmer and others tried to understand local, global and infinitesimal rigidity of these actions (see [H1], [H2], [K-L1], [K-L2], [K-L-H-Z], [K-L-Z], [Q1], [Q2], [Z3]). In the great majority of these papers, an essential use was made of the presence of an Anosov element. Here we show for the first time the deformation rigidity of non-irreducible and non-transitive partially hyperbolic actions (for terminology, see §7). One example is ρ_0, for $n \geq 3$ and any integer d; others are given in §7. Precise statements are Theorems 7.1 and 7.8.

As turns out from this study, there is a close connection between the rigidity of the natural actions of lattices on tori and the cohomology of hyperbolic dynamical systems. The first nontrivial results about this cohomology were obtained by Livsic and Sinai in [L1], [L2] and [L-S]. We recall some definitions and Livsic's main results.

Let G be a discrete group acting on a closed Riemannian manifold M, and let Γ be some topological group with unit 1_Γ. A cocycle β is a continuous function $\beta : G \times M \to \Gamma$ such that

$$\beta(g_1 g_2, x) = \beta(g_1, g_2 x)\beta(g_2, x),$$

for all $g_1, g_2 \in G$, $x \in M$. Two cocycles β_1 and β_2 are called cohomologous if there exists a continuous map $P : M \to \Gamma$ such that

$$\beta_1(g,x) = P(gx)\beta_2(g,x)P(x)^{-1},$$

for all $g \in G$, $x \in M$.

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We say that a cocycle β satisfies the closing conditions if the following holds:

$gx = x$ for some $x \in M$ and $g \in G$ implies that $\beta(g, x) = 1_\Gamma$.

In the case $G = \mathbb{Z}$ the cocycle is determined by the function $\overline{\beta}(x) := \beta(1, x)$.

We recall also the definition of an Anosov diffeomorphism. Note that this definition is equivalent to the standard one (see [K]). The Riemannian metric which appears is called the Lyapunov metric.

Definition. *Let M be a compact manifold and $T : M \to M$ a C^1-diffeomorphism. The diffeomorphism T is called Anosov if there is a continuous splitting of the tangent bundle into DT-invariant subbundles $TM = E^s \oplus E^u$ and a Riemannian metric such that:*

$$\|DTv^s\| \le \lambda \|v^s\|, \ v^s \in E^s,$$
$$\|DT^{-1}v^u\| \le \lambda \|v^u\|, \ v^u \in E^u,$$

for some $0 < \lambda < 1$.

We denote by $W^s(x)$ (resp. $W^u(x)$) the stable (resp. unstable) manifold of the point $x \in M$.

Theorem 1.1. *(Livsic* [L2]*) If the group G is $\mathbb{Z}$ and its action on the Riemannian manifold M is given by a topologically transitive C^1 Anosov diffeomorphism, then a Hölder function $\overline{\beta} : M \to \mathbb{R}$ determines a null-cohomological cocycle $\beta(g, x)$ if and only if β satisfies the closing conditions. The corresponding function P is also Hölder.*

A topological group Γ is called "good" if it admits right-invariant and left-invariant metrics ρ_R and ρ_L, which are equivalent in a neighborhood of the identity, and such that Γ metrized by these metrics is a complete metric space. Moreover, for any points $(x_1, x_2) \in \Gamma \times \Gamma$

$$\frac{\rho_L(x_1, x_2)}{\rho_L(gx_1g^{-1}, gx_2g^{-1})} \to 1 \text{ and } \frac{\rho_R(x_1, x_2)}{\rho_R(gx_1g^{-1}, gx_2g^{-1})} \to 1$$

as $g \to 1_\Gamma$, this convergence being uniform for (x_1, x_2) in a bounded set*. The main example of a good group is a finite dimensional Lie group.

Theorem 1.2. *(Livsic* [L2]*) If the group G is $\mathbb{Z}$ and its action on the Riemannian manifold M is given by a topologically transitive C^1 Anosov diffeomorphism, then for each Hölder class α and in each good group Γ there exists a neighborhood of the identity U such that an α-Hölder function $\overline{\beta} : M \to U$ defines a null-cohomological cocycles $\beta(g, x)$ if and only if β satisfies the closing conditions. The corresponding function P is also α-Hölder.*

Livsic proved similar results for Anosov flows and was able to extend Theorem 1.1 to cocycles with values in finite dimensional nilpotent Lie groups. After Livsic's

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* This is the wording in the translation of [L2]. However, it seems that one uses uniform convergence with respect to all $(x_1, x_2) \in \Gamma \times \Gamma$.

work, the main results obtained were about the regularity of P produced by Theorem 1.1, assuming some regularity for β (see [L1], [C-E-G], [G-K], [L-M-M], [Lla]). De la Llave-Marco-Moriyon showed in [L-M-M] that P is C^∞ if β is C^∞, and de la Llave showed in [Lla] that P is analytic if β is analytic. A somehow different direction is taken by A. Katok, S. Katok and Spatzier in [K-S] and [K-K], where real-valued cocycles over certain $\mathbb{Z}^k$ and $\mathbb{R}^k$ Anosov actions ($k \geq 2$) are shown to be cohomologous to constant cocycles (in these cases, the periodic conditions are automatically satisfied!).

Our first theorem, Theorem 3.1, which is maybe the most important result of the paper, extends Livsic's Theorem 1.2 to cocycles with values in the group of C^K-diffeomorphisms of a compact closed manifold with stably trivial tangent bundle ($4 \leq K < \infty$). The statement and the proof are in §3. The strategy of the proof is similar to the one used by Livsic. The main difference is that on the group of C^K-diffeomorphisms the natural metric is neither left nor right invariant. Nevertheless, in a small neighborhood of the identity one can show that this metric is "quasi-invariant", and this is enough for our purposes. These properties of the metric are presented in §2.

In order to show deformation rigidity for ρ_0, one needs a Livsic type result for a cocycle over an $SL(n,\mathbb{Z})$-action, instead of a $\mathbb{Z}$-action. We obtain much more. Using a generalization of the Anosov Closing Lemma to a generic class of Anosov group actions (see Theorem 4.1), we show that if a cocycle over an Anosov group action satisfies the closing conditions and if the restriction of the cocycle to the action of a (special) Anosov element is null-cohomological, then the whole cocycle is null-cohomological. The Anosov group action is not necessarily the action of a large group. Results like this seems to be the missing link between the theory of $\mathbb{Z}$-actions and the theory of actions of "big groups". This result and various combinations of it with Livsic's theorems and with Theorem 3.1 appear in §5. In particular, we are able to generalize the Livsic-Sinai criterion about the existence of a smooth invariant measure for an Anosov $\mathbb{Z}$-action, to more general Anosov group actions.

In §6 we prove a version of the Hirsch-Pugh-Shub structural stability theorem (see [H-P-S], Theorem 7.1), namely we show that for some partially hyperbolic automorphism of the torus, the leaf-wise conjugation between the automorphism and a small $C^{K+\mathrm{Lip}}$-perturbation is Hölder, differentiable of order K along the leaves and these derivatives are Hölder. This result is necessary in order to produce from a perturbation of ρ_0 a cocycle which is Hölder in the d_K-metric. Note that our approach differs from that in [H-P-S].

The rigidity of ρ_0 and of other partially hyperbolic higher rank lattice actions is proved in §7, Theorems 7.1 and 7.8. The proofs involve the Livsic type theorem for cocycles with values in diffeomorphisms, our Noncommutative Closing Lemma, and the version of the Hirsch-Pugh-Shub theorem.

In the paper [N-T] we prove the regularity of the solution of Livsic's equation with values in diffeomorphism groups and with values in a matrix Lie group.

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§2. Metrics on Diffeomorphism Groups

The purpose of this section is to define some metrics on the space of C^K or $C^{K+\mathrm{Lip}}$ functions between "flat spaces" and then to study their invariance properties with respect to composition. These metrics are almost invariant in a small neighborhood of the identity. We use this "quasi-invariance" property in two places. First, it is used in §3 to generalize Livsic's Theorem 1.2 to cocycles with values in diffeomorphism groups. Second, it is used in §6 to prove a version of the Hirsch-Pugh-Shub Theorem (Theorem 7.1 in [H-P-S]).

Definition 2.1. *A flat space is either a finite dimensional linear space $\mathbb{R}^d$ (with the Euclidean metric), or a flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ with the metric induced from $\mathbb{R}^d$.*

Note that if X is a flat space and $x \in X$, then we have a canonical identification of the tangent space at x, T_xX, with $\mathbb{R}^{\dim X}$.

Definition 2.2. *Let U be an open connected set in a flat space. For any path γ in U, denote by $l(\gamma)$ the length of γ, and for any x, y in U, define their distance in U by*

$$\mathbf{d}_U(x,y) := \inf_{\gamma} l(\gamma),$$

where the infimum is taken over all paths γ in U connecting x and y. Let X and Y be flat spaces, $U \subset X$ and $V \subset Y$ open connected sets. For any integer $K \geq 0$ denote by $C^K(U,V)$ the set of C^K-functions from U into V. Take $a \in C^K(U,V)$ and for $1 \leq k \leq K$ denote by $D^k_x a$ the k-th derivative of a at $x \in U$, which is a symmetric k-linear map $D^k_x a : (\mathbb{R}^{\dim U})^k \to \mathbb{R}^{\dim V}$. Denote by $\|D^k_x a\|$ the operatorial norm of $D^k_x a$. Denote by $C^{K+\mathrm{Lip}}(U,V)$ the set of K-differentiable functions from U into V with the K-th derivative Lipschitz, i.e. $\phi \in C^K(U,V)$ for which there exists $C > 0$ such that

$$\|D^K_{x'}\phi - D^K_{x''}\phi\| \leq C\ \mathbf{d}_U(x',x''), \text{ for all } x', x'' \in U.$$

The smallest constant C with this property is called the Lipschitz norm of $D^K\phi$ and is denoted by $\|D^K\phi\|_{\mathrm{Lip}}$.

For $a, b \in C^K(U,V)$ and any $1 \leq k \leq K$, define:

$$\begin{aligned}
d_0(a) &:= \sup_{x\in U} \mathbf{d}_V(a(x),0), \text{ if } 0 \in V,\\
d_0(a,b) &:= \sup_{x\in U} \mathbf{d}_V(a(x),b(x)),\\
d_{(k)}(a) &:= \sup_{x\in U} \|D^k_x a\|,\\
d'_K(a) &:= \sum_{k=1}^{K} \frac{d_{(k)}(a)}{k!},\\
d''_K(a) &:= \sum_{k=2}^{K} \frac{d_{(k)}(a)}{k!},
\end{aligned}$$

$$
\begin{aligned}
d'_{K+\mathrm{Lip}}(a) &:= d'_K(a) + \frac{\|D^K a\|_{\mathrm{Lip}}}{(K+1)!},\\
d''_{K+\mathrm{Lip}}(a) &:= d''_K(a) + \frac{\|D^K a\|_{\mathrm{Lip}}}{(K+1)!},\\
d_K(a) &:= d_0(a) + d'_K(a),\ \ if\ \ 0 \in V,\\
d_K(a,b) &:= d_0(a,b) + d'_K(a-b),\\
d_{K+\mathrm{Lip}}(a,b) &:= d_0(a,b) + d'_{K+\mathrm{Lip}}(a-b).
\end{aligned}
$$

Lemma 2.3. *If $a \in C^{K+1}(U,V)$, then for any $0 \le k \le K$*

$$\|D^k a\|_{\mathrm{Lip}} \le d_{(k+1)}(a).$$

In particular, $d'_{K+\mathrm{Lip}}(a) \le d'_{K+1}(a)$.

Proof. Let $\eta_1, \eta_2, \cdots, \eta_k$ be vectors in $\mathbb{R}^{\dim U}$. Let x', x'' be two points in U, and let $\gamma : [0,1] \to U$ be a path joining x' and x''. Then:

$$
\begin{aligned}
&\|D^k_{x'} a(\eta_1, \eta_2, \cdots, \eta_k) - D^k_{x''} a(\eta_1, \eta_2, \cdots, \eta_k)\| =\\
&= \|\int_0^1 \frac{d}{dt}\big[D^k_{\gamma(t)} a(\eta_1, \eta_2, \cdots, \eta_k)\big]dt\| =\\
&= \|\int_0^1 \big[D^{k+1}_{\gamma(t)} a(\gamma'(t), \eta_1, \eta_2, \cdots, \eta_k)\big]dt\| \le\\
&\le \sup_{t\in[0,1]} \|D^{k+1}_{\gamma(t)} a\| \big(\int_0^1 \|\gamma'(t)\|dt\big)\|\eta_1\|\|\eta_2\|\cdots\|\eta_k\| \le\\
&\le l(\gamma) d_{(k+1)}(a)\|\eta_1\|\|\eta_2\|\cdots\|\eta_k\|. \quad \square
\end{aligned}
$$

Notation. If $r_1, r_2, \cdots, r_n$ is a set of real numbers, denote

$$r_1 \vee r_2 \vee \cdots \vee r_n := \max_{1\le i\le n} r_i.$$

This notation will be used throughout the rest of the paper.

Proposition 2.4. *Let $K \ge 1$ be an integer. Let X, Y and Z be flat spaces and let $U \subset X$, $V \subset Y$ and $W \subset Z$ be open sets.*

(i) *If $a \in C^K(U,U)$, then:*

$$d'_K(a) \le 1 + d_K(a, Id), \tag{D1}$$
$$d''_K(a) = d''_K(a, Id). \tag{D2}$$

(ii) *If $a, b \in C^K(V,W)$ and $u \in C^K(U,V)$, then:*

$$d'_K(au) \le d'_K(a) \cdot [\max\{1, d'_K(u)\}]^K, \tag{R1}$$
$$d'_K(au, bu) \le d'_K(a,b) \cdot [\max\{1, d'_K(u)\}]^K, \tag{R2}$$
$$d_K(au, bu) \le d_K(a,b) \cdot [\max\{1, d'_K(u)\}]^K. \tag{R3}$$

(iii) *If $a, b \in C^K(U,V)$, $u \in C^{K+\mathrm{Lip}}(V,W)$ and $v \in C^{K+1}(V,W)$, then:*

$$\begin{aligned} d_K(ua,ub) &\le d_K(a,b)(K+1)d'_{K+\mathrm{Lip}}(u)\cdot \\ &\qquad \cdot [1 \vee (\max\{d'_K(a), d'_K(b)\} + d''_K(a) + d''_K(b))]^K, \qquad \text{(L1)} \\ d_K(va,vb) &\le d_K(a,b)(K+1)d'_{K+1}(v)\cdot \\ &\qquad \cdot [1 \vee (\max\{d'_K(a), d'_K(b)\} + d''_K(a) + d''_K(b))]^K. \qquad \text{(L2)} \end{aligned}$$

In particular, if $a, b \in C^K(V,V)$, then:

$$d_K(ua,ub) \le d_K(a,b)(K+1)d'_{K+\mathrm{Lip}}(u)[1 + 2d'_K(a,Id) + 2d'_K(b,Id)]^K, \quad \text{(L3)}$$
$$d_K(va,vb) \le d_K(a,b)(K+1)d'_{K+1}(v)[1 + 2d'_K(a,Id) + 2d'_K(b,Id)]^K. \quad \text{(L4)}$$

(iv) *If $a \in C^{K+\mathrm{Lip}}(V,W)$ and $b \in C^{K+\mathrm{Lip}}(U,V)$, then*

$$\begin{aligned} d'_{K+\mathrm{Lip}}(ab) &\le d'_{K+\mathrm{Lip}}(a)[\max\{1, d'_K(b)\}]^K\cdot \\ &\qquad \cdot [d'_{K+\mathrm{Lip}}(b) + K^2 d''_K(b)], \qquad \text{(Lip1)} \end{aligned}$$

therefore, if $0 \in W$,

$$\begin{aligned} d_{K+\mathrm{Lip}}(ab) &\le d_{K+\mathrm{Lip}}(a)[\max\{1, d'_K(b)\}]^K\cdot \\ &\qquad \cdot \max\{1, d'_{K+\mathrm{Lip}}(b) + K^2 d''_K(b)\}. \qquad \text{(Lip2)} \end{aligned}$$

In particular, if $b \in C^{K+\mathrm{Lip}}(V,V)$ and $0 \in W$, then

$$\begin{aligned} d_{K+\mathrm{Lip}}(ab) &\le d_{K+\mathrm{Lip}}(a)[1 + d_{K+\mathrm{Lip}}(b,Id)]^K\cdot \\ &\qquad [1 + (K^2+1)d_{K+\mathrm{Lip}}(b,Id)]. \qquad \text{(Lip3)} \end{aligned}$$

Remark. One can obtain in (iii) an estimate which is sharp for $u = Id$. Since we do not need it, we do not describe it here.

Proof. Part (i) follows from the definitions. We prove (ii), (iii) and (iv).

We introduce first a formula for the derivative of a composite function.

Assume $a \in C^K(V,W)$, $u \in C^K(U,V)$, $\ell \le K$ and $\xi, \xi_1, \dots, \xi_\ell \in \mathbb{R}^{\dim U}$. Using Taylor's formula, one has

$$\frac{1}{\ell!} D^\ell_x(au)(\xi^{(\ell)}) = \\ = \sum_{k=1}^{\ell} \sum_{\substack{i_1,\dots,i_k \ge 1 \\ i_1+\dots+i_k=\ell}} \frac{1}{k!} D^k_{u(x)} a \left(\frac{D^{i_1}_x u(\xi^{(i_1)})}{i_1!}, \frac{D^{i_2}_x u(\xi^{(i_2)})}{i_2!}, \dots, \frac{D^{i_k}_x u(\xi^{(i_k)})}{i_k!} \right),$$

where $\xi^{(k)}$ stands for the k-tuple $(\xi, \xi, \dots, \xi)$.

Therefore, $D_x^\ell(au)$ is given by the symmetrization of the ℓ-linear map

$$(\xi_1,\dots,\xi_\ell)\mapsto \ell!\cdot\sum_{k=1}^{\ell}\sum_{\substack{i_1,\dots,i_k\geq 1\\ i_1+\dots+i_k=\ell}}\frac{1}{k!}D^k_{u(x)}a\left(\frac{1}{i_1!}D_x^{i_1}u(\xi_1,\dots,\xi_{i_1})\,,\right.$$
$$\left.\frac{1}{i_2!}D_x^{i_2}u(\xi_{i_1+1},\dots,\xi_{i_1+i_2}),\dots,\frac{1}{i_k!}D_x^{i_k}u(\xi_{i_1+\dots+i_{k-1}+1},\dots,\xi_\ell)\right). \quad (2.1)$$

Note that as far as the norms computed below are concerned, this symmetrization has no effect.

(ii) Take $a\in C^K(V,W)$ and $u\in C^K(U,V)$. From (2.1) it follows that

$$d_{(\ell)}(au)=\sup_{x\in U}\|D_x^\ell(au)\|\leq$$
$$\leq \ell!\sum_{k=1}^{\ell} d_{(k)}(a)\cdot\frac{1}{k!}\left(\sum_{\substack{i_1,\dots,i_k\geq 1\\ i_1+\dots+i_k=\ell}}\prod_{j=1}^{k}\frac{1}{i_j!}\cdot\|D^{i_j}u\|\right).$$

Hence $$\frac{1}{\ell!}\cdot d_{(\ell)}(au)\leq\sum_{k=1}^{\ell}\frac{1}{k!}d_{(k)}(a)\cdot\sum_{\substack{i_1,\dots,i_k\geq 1\\ i_1+\dots+i_k=\ell}}\prod_{j=1}^{k}\frac{\|D^{i_j}u\|}{i_j!}. \quad (2.2)$$

In order to simplify the above expressions, we introduce the following notations:

$$X_i:=\|D^iu\|\cdot\frac{1}{i!}\ \ ,\deg X_i=i,\ X:=X_1+X_2+\dots+X_K$$

and for a polynomial F in some "graded" variables, by $F|_{\deg\leq k}$ we mean the sum of the monomials of total degree less or equal to k.

Using (2.2) one has

$$d'_K(au)\leq\sum_{k=1}^{K}\frac{1}{k!}d_{(k)}(a)\cdot(X^k|_{\deg\leq K})\leq$$
$$\leq d'_K(a)\cdot[\max\{1,d'_K(u)\}]^K,$$

hence (R1). The formulas (R2) and (R3) follow immediately from (R1).

(iii) Take $a,b\in C^K(U,V)$ and $u\in C^{K+\mathrm{Lip}}(V,W)$. We use now (2.1) for ua and ub:

$$d_{(\ell)}(ua,ub)=\sup_{x\in U}\|D_x^\ell(ua)-D_x^\ell(ub)\|\leq$$
$$\leq\ell!\cdot\sum_{k=1}^{\ell}\sum_{\substack{i_1,\dots,i_k\geq 1\\ i_1+\dots+i_k=\ell}}\frac{1}{k!}\cdot\frac{1}{i_1!}\cdots\frac{1}{i_k!}\cdot$$
$$\cdot\|D^k_{a(x)}u(D_x^{i_1}a,D_x^{i_2}a,\dots,D_x^{i_k}a)-D^k_{b(x)}u(D_x^{i_1}b,D_x^{i_2}b,\dots,D_x^{i_k}b)\|.$$

In order to estimate the norm of the differences, first add and subtract for each term $D^k_{a(x)}u(D^{i_1}_x a, \cdots)$ an intermediate term $D^k_{b(x)}u(D^{i_1}_x a, \cdots)$. Then apply the triangle inequality. It remains to estimate terms of the form

$$\|D^k_{a(x)}u(D^{i_1}_x a, D^{i_2}_x a, \ldots, D^{i_k}_x a) - D^k_{b(x)}u(D^{i_1}_x a, D^{i_2}_x a, \ldots, D^{i_k}_x a)\|, \tag{2.3}$$

and

$$\|D^k_{b(x)}u(D^{i_1}_x a, D^{i_2}_x a, \cdots, D^{i_k}_x a) - D^k_{b(x)}u(D^{i_1}_x b, D^{i_2}_x b, \cdots, D^{i_k}_x b)\|. \tag{2.4}$$

The quantity in (2.3) can be estimated using

$$\|D^k_{a(x)}u - D^k_{b(x)}u\| \le \|D^k u\|_{\mathrm{Lip}} d_0(a, b). \tag{2.5}$$

For (2.4), introduce again intermediate terms, use the linearity of the derivatives and the triangle inequality to obtain

$$\begin{gathered}\|D^k_{b(x)}u(D^{i_1}_x a, D^{i_2}_x a, \ldots, D^{i_k}_x a) - D^k_{b(x)}u(D^{i_1}_x b, D^{i_2}_x b, \ldots, D^{i_k}_x b)\| \le \\ \le \|D^k u\| \left[d_{(i_1)}(a,b) \cdot \|D^{i_1}a\| \cdots \|D^{i_k}a\| \ + \ \|D^{i_1}b\| \cdot d_{(i_2)}(a,b) \cdot \|D^{i_3}b\| \cdots \|D^{i_k}b\| + \right. \\ \left. + \ \cdots \ + \ \|D^{i_1}b\| \|D^{i_2}b\| \cdots \|D^{i_{k-3}}b\| \cdot d_{(i_k)}(a,b)\right].\end{gathered}$$

Therefore

$$\begin{gathered}d_{(\ell)}(ua, ub) \le \ell! \sum_{k=1}^{\ell} \left\{ \sum_{\substack{i_1, \ldots, i_k \ge 1 \\ i_1 + \cdots + i_k = \ell}} \frac{1}{k!} \left[\|D^k u\|_{\mathrm{Lip}} \cdot d_0(a,b) \cdot \right.\right. \\ \cdot \prod_{j=1}^{k} \frac{1}{i_j!} \|D^{i_j} a\| + \|D^k u\| \cdot \sum_{j=1}^{k} \left(\frac{d_{(i_j)}(a,b)}{i_j!} \cdot \prod_{\alpha=1}^{j-1} \frac{\|D^{i_\alpha} b\|}{i_\alpha!} \cdot \right. \\ \left.\left.\left. \cdot \prod_{\beta=j+1}^{k} \frac{\|D^{i_\beta} a\|}{i_\beta!} \right) \right] \right\}.\end{gathered} \tag{2.6}$$

In order to simplify the above formula, we introduce the following notations:

$$\begin{aligned}\delta_i &: = \frac{d_{(i)}(a,b)}{i!}, \\ \alpha_i &: = \max\left\{ \frac{\|D^i a\|}{i!}, \ \frac{\|D^i b\|}{i!} \right\}, \\ X_i &: = \frac{\|D^i u\|}{i!}, \\ X_{K+1} &: = \frac{\|D^K u\|_{\mathrm{Lip}}}{(K+1)!}, \\ \alpha &: = \sum_{i=1}^{K} \alpha_i, \qquad \delta := \sum_{i=1}^{K} \delta_i, \\ \deg \ \alpha_i &= \deg \ \delta_i = i, 1 \le i \le K.\end{aligned}$$

It is clear that:

$$\alpha_1 \leq \max\{d'_K(a), d'_K(b)\},$$
$$\alpha_i \leq \frac{d_{(i)}(a)}{i!} + \frac{d_{(i)}(b)}{i!},\ 2 \leq i \leq K.$$

Therefore

$$\alpha \leq \max\{d'_K(a), d'_K(b)\} + d''_K(a) + d''_K(b). \tag{2.7}$$

By (2.6) one has

$$\begin{aligned}\frac{1}{\ell!}\, d_{(\ell)}(ua, ub) \leq \delta_0 \cdot \sum_{k=1}^{\ell} (k+1) X_{k+1} \cdot (\alpha_1 + \cdots + \alpha_\ell)^k|_{\deg = \ell} + \\ + \sum_{k=1}^{\ell} X_k(\delta_1 + \cdots + \delta_\ell + \alpha_1 + \cdots + \alpha_\ell)^k|_{\deg=\ell, \#(\delta)=1} \quad ,\end{aligned} \tag{2.8}$$

where the subscript "deg $= \ell$, $\#(\delta) = 1$" means that we take the monomials of total degree ℓ which have exactly one δ_i factor.

Using (2.8) and the inequality $d_0(ua, ub) \leq \|Du\| d_0(a, b) = X_1 \cdot \delta_0$, one has

$$\begin{aligned} d_K(ua, ub) = d_0(ua, ub) + d'_K(ua, ub) \leq \\ \leq X_1 \cdot \delta_0 + \delta_0 \cdot \sum_{\ell=1}^{K} \sum_{k=1}^{\ell} (k+1) \cdot X_{k+1} \cdot \alpha^k|_{\deg=\ell} + \\ + \sum_{\ell=1}^{K} \sum_{k=1}^{\ell} X_k \cdot (\delta + \alpha)^k|_{\deg=\ell, \#(\delta)=1}. \end{aligned} \tag{2.9}$$

But

$$(\delta + \alpha)^k|_{\deg=\ell, \#(\delta)=1} = \sum_{i \geq 1} k\delta_i(\alpha^{k-1}|_{\deg=(\ell-i)}),$$

hence, if we denote by B the second double sum which appears in the RHS of (2.9), one has:

$$\begin{aligned} B &= \sum_{\ell=1}^{K} \sum_{k=1}^{\ell} \sum_{i \geq 1} k\delta_i X_k(\alpha^{k-1}|_{\deg=(\ell-i)}) = \\ &= \sum_{i \geq 1} \delta_i \sum_{k=1}^{K} X_k \sum_{\ell=k}^{K} k(\alpha^{k-1}|_{\deg=(\ell-i)}) = \\ &= \sum_{i \geq 1} \delta_i \sum_{k=1}^{K} kX_k(\alpha^{k-1}|_{k-i \leq \deg \leq K-i}) = \\ &= \sum_{i \geq 1} \delta_i \sum_{k=1}^{K} kX_k(\alpha^{k-1}|_{\deg \leq K-i}). \end{aligned}$$

Then inequality (2.9) becomes

$$\begin{gathered}
d_K(ua,ub)\le\delta_0\cdot\sum_{k=2}^{K+1}kX_k\alpha^{k-1}|_{\deg\le K}+\\
+\sum_{i\ge1}\delta_i\sum_{k=1}^{K}kX_k\alpha^{k-1}|_{\deg\le K-i}\quad\le\\
\le d_K(a,b)(K+1)d'_{K+\mathrm{Lip}}(u)[\max\{1,\alpha\}]^K
\end{gathered}$$

which together with (2.7) yields (L1). (L2) follows from (L1) using Lemma 2.3. In order to obtain (L3) and (L4), notice that for $a,b:V\to V$ one has:

$$\begin{gathered}
\max\{d'_K(a),d'_K(b)\}+d''_K(a)+d''_K(b)\le\\
\le 1+\max\{d'_K(a,Id),d'_K(b,Id)\}+d'_K(a,Id)+d'_K(b,Id)\le\\
\le 1+2d'_K(a,Id)+2d'_K(b,Id).
\end{gathered}$$

(iv) Take $a\in C^{K+\mathrm{Lip}}(V,W)$ and $b\in C^{K+\mathrm{Lip}}(U,V)$. We estimate $\|D^K_{x'}(ab)-D^K_{x''}(ab)\|$. Proceeding as in (iii) and making the appropriate changes in the formula (2.6) above, we have

$$\frac{1}{K!}\|D^K(ab)\|_{\mathrm{Lip}}\le\sum_{k=1}^{K}\left\{\sum_{\substack{i_1,\dots,i_k\ge1\\ i_1+\cdots+i_k=K}}\left[\frac{\|D^ka\|_{\mathrm{Lip}}}{k!}\cdot\|b\|_{\mathrm{Lip}}\cdot\right.\right.$$
$$\left.\left.\cdot\prod_{j=1}^{k}\frac{1}{i_j!}\|D^{i_j}b\|+\frac{\|D^ka\|}{k!}\Big(\sum_{j=1}^{k}\frac{\|D^{i_j}b\|_{\mathrm{Lip}}}{i_j!}\cdot\prod_{\substack{\alpha=1\\ \alpha\ne j}}^{k}\frac{\|D^{i_\alpha}b\|}{i_\alpha!}\Big)\right]\right\}.$$

We introduce now the following notations:

$$\begin{aligned}
X_i&:=\frac{d_{(i)}(b)}{i!},1\le i\le K,\\
X_{K+1}&:=\frac{\|D^Kb\|_{\mathrm{Lip}}}{(K+1)!},\\
X&:=X_1+X_2+\cdots+X_K,\\
X''&:=X_2+X_3+\cdots+X_K,\\
\alpha_i&:=\frac{d_{(i)}a}{i!},1\le i\le K,\\
\alpha_{K+1}&:=\frac{\|D^Ka\|_{\mathrm{Lip}}}{(K+1)!},\\
\deg X_i&=i,1\le i\le K+1.
\end{aligned}$$

From Lemma 2.3 it follows that

$$\frac{\|D^i a\|_{\mathrm{Lip}}}{i!} \leq \frac{d_{i+1}(a)}{i!} = (i+1)\alpha_{i+1},$$

and similarly for b.

Hence:

$$\begin{aligned}
\frac{1}{K!}\|D^K(ab)\|_{\mathrm{Lip}} \leq \ & \sum_{k=1}^{K}(k+1)\alpha_{k+1}X_1\Big(\sum_{\substack{i_1,\dots,i_k\geq 1\\ i_1+\dots+i_k=K}}\prod_{j=1}^{k}X_{i_j}\Big)+\\
&+\sum_{k=1}^{K}\alpha_k\sum_{\substack{i_1,\dots,i_k\geq 1\\ i_1+\dots+i_k=K}}\sum_{j=1}^{k}(i_j+1)X_{i_j+1}\prod_{\substack{\alpha=1\\ \alpha\neq j}}^{k}X_{i_\alpha} =
\end{aligned}$$

$$\begin{aligned}
&=\sum_{k=1}^{K}(k+1)\alpha_{k+1}X_1(X^k|_{\deg=K})+\sum_{k=1}^{K}\alpha_k\left[k(\sum_{i=2}^{K+1}iX_i)X^{k-1}|_{\deg=K+1}\right]=\\
&=\sum_{k=2}^{K+1}k\alpha_kX_1(X^{k-1}|_{\deg=K})+\sum_{k=1}^{K}k\alpha_k\left[(\sum_{i=2}^{K+1}iX_i)X^{k-1}|_{\deg=K+1}\right]=\\
&=\sum_{k=2}^{K}k\alpha_k\left[(\sum_{k=1}^{K+1}iX_i)X^{k-1}|_{\deg=K+1}\right]+(K+1)\alpha_{K+1}X_1(X^K|_{\deg=K})+\\
&\qquad+\alpha_1(\sum_{k=1}^{K+1}iX_i|_{\deg=K+1})=\\
&=\sum_{k=2}^{K}k\alpha_k\left[(\sum_{i=1}^{K}iX_i)X^{k-1}|_{\deg=K+1}\right]+(K+1)\alpha_{K+1}X_1(X^K|_{\deg=K})+\\
&\qquad+\alpha_1(K+1)X_{K+1}.
\end{aligned}$$

But, for $2\leq k\leq K$, one has:

$$\begin{aligned}
&(\sum_{i=1}^{K}iX_i)X^{k-1}|_{\deg=K+1}\leq KX^k|_{\deg=K+1}=K(X^k-X_1^k)|_{\deg=K+1}=\\
&=K(X-X_1)(X^{k-1}+X^{k-2}X_1+\dots+X_1^{k-1})|_{\deg=K+1}\leq\\
&\leq Kk(X''X^{k-1}|_{\deg=K+1}).
\end{aligned}$$

Also, $X^K|_{\deg=K}=X_1^K$. Therefore

$$\begin{aligned}
\frac{1}{K!}\|D^K(ab)\|_{\mathrm{Lip}}\leq \ &\sum_{k=2}^{K}\alpha_kk^2K(X''X^{k-1}|_{\deg=K+1})+\\
&+(K+1)(\alpha_{K+1}X_1^{K+1}+\alpha_1X_{K+1}).
\end{aligned}$$

From the proof of (ii), we know that

$$d'_K(ab) \leq \sum_{k=1}^{K} \alpha_k(X^k|_{\deg\leq K}).$$

Then

$$\begin{aligned} d'_{K+\mathrm{Lip}}(ab) = d'_K(ab) + \frac{\|D^K(ab)\|_{\mathrm{Lip}}}{(K+1)!} \leq \\ \leq \sum_{k=1}^{K} \alpha_k(X^k|_{\deg\leq K}) + \sum_{k=2}^{K} \alpha_k \frac{k^2K}{K+1}(X''X^{k-1}|_{\deg=K+1}) + \\ + \alpha_{K+1}X_1^{K+1} + \alpha_1 X_{K+1} \leq \\ \leq \sum_{k=2}^{K} \alpha_k \left[(X^k|_{\deg\leq K} + K^2(X''X^{k-1}|_{\deg=K+1}) \right] + \\ + \alpha_1(X + X_{K+1}) + \alpha_{K+1}X_1^{K+1} \leq \\ \leq \sum_{k=2}^{K} \alpha_k X^{k-1}(X + K^2X'') + \alpha_1(X + X_{K+1}) + \alpha_{K+1}X_1^{K+1} \leq \\ \leq (\sum_{k=1}^{K+1} \alpha_k)[\max\{1, X\}]^K(X + X_{K+1} + K^2X''), \qquad \text{q.e.d.} \quad \square \end{aligned}$$

Proposition 2.5. *Let U be an open connected subset of a flat space. Then $\mathrm{Diff}^K_{bdd}(U) := \{a \in \mathrm{Diff}^K(U) \mid d_K(a^{\pm1}, Id) < \infty\}$ is a group.*

Proof. We only have to show that $\mathrm{Diff}^K_{bdd}(U)$ is closed under composition. Since $d_K = d_0 + d'_K$, a diffeomorphism $a \in \mathrm{Diff}^K(U)$ is in $\mathrm{Diff}^K_{bdd}(U)$ if and only if

$$d'_K(a^{\pm1}) < \infty \text{ and } d_0(a^{\pm1}, Id) < \infty.$$

The first condition is clearly composition invariant, while for the second one, by the right-invariance of d_0, one has:

$$d_0(ab, Id) \leq d_0(ab, b) + d_0(b, Id) = d_0(a, Id) + d_0(b, Id). \quad \square$$

Proposition 2.6. *The group $\mathrm{Diff}^K_{bdd}(U)$ introduced in Proposition 2.5 is complete with respect to the metric*

$$\overline{d}_K(a, b) := \max\{d_K(a, b), d_{(1)}(a^{-1}, b^{-1})\}.$$

Proof. Assume $\{a_n\}_{n\geq0} \subset \mathrm{Diff}^K_{bdd}(U)$ is a Cauchy sequence in the metric $\overline{d}_K$. In particular, it is Cauchy in d_K, so it has a limit

$$a = \lim_{n\to\infty} a_n \in C^K(U)$$

with $d_K(a, Id) < \infty$.

It is enough to show that $a \in \mathrm{Diff}^1(U)$, because $d'_K(a) < \infty$ and $d_{(1)}(a^{-1}) < \infty$ implies $d'_K(a^{-1}) < \infty$ (see the proof of Proposition 2.9 (b)).

In order to obtain this, note that the family $\{a_n^{-1}\}_{n\geq 0}$ is uniformly equicontinuous (because the first derivatives are uniformly bounded) and pointwise bounded (because $d_0(a_n^{-1}, Id) = d_0(Id, a_n)$, hence $\sup_n\, d_0(a_n^{-1}, Id) < \infty$). Therefore, by Ascoli-Arzela's theorem (see [R], which works for our "vector valued" functions too), there is a subsequence $\{a_{n_k}^{-1}\}_{k\geq 0}$ which converges uniformly (i.e., in d_0) to some $b \in C^0(U)$. Clearly $d_0(b, Id) < \infty$ and since $\{a_{n_k}^{-1}\}_{k\geq 0}$ converges actually in d_1, we see that $b \in C^1(U)$ and $d_1(b, Id) < \infty$.

We have to show that $b = a^{-1}$.

But this follows from the continuity of the composition with respect to the the convergence we have here: if $f_n \to f$, $g_n \to g$ in d_0 and f is uniformly continuous, then $f_n g_n \to fg$ in d_0.

(Indeed, for $x \in U$

$$\begin{aligned}
&\mathbf{d}_U(f_n g_n(x), fg(x)) \leq \\
&\leq \mathbf{d}_U(f_n(g_n(x)), f(g_n(x))) + \mathbf{d}_U(f(g_n(x)), f(g(x))) \leq \\
&\leq d_0(f_n, f) + \mathbf{d}_U(f(g_n(x)), f(g(x))),
\end{aligned}$$

and now use the uniform convergence of $\{g_n\}$ and the uniform continuity of f.)

Therefore:

$$\begin{aligned}
ab &= \lim_{k\to\infty} a_{n_k} a_{n_k}^{-1} = Id_U, \\
ba &= \lim_{k\to\infty} a_{n_k}^{-1} a_{n_k} = Id_U,
\end{aligned}$$

and we are done. $\square$

The following definition describes manifolds on whose diffeomorphism groups one can still define a family of metrics with the desired properties.

Definition 2.7. *Let N be a connected C^∞ manifold without boundary. N is called a good manifold if there is an embedding of N into an Euclidean space $\mathbb{R}^d$ such that N has a tubular neighborhood U which is an open set in $\mathbb{R}^d$ and which is C^∞-diffeomorphic to $N \times D$, where D is an open bounded disk in $\mathbb{R}^{d-\dim N}$.*

Remark. Any good manifold has stably trivial tangent bundle and for closed manifolds the converse is also true. Examples of such manifolds are the n–torus, the n–sphere, or any closed compact hypersurface in $\mathbb{R}^d$.

Take now an integer $K \geq 1$ and let N be a good manifold. Then any diffeomorphism a in $\mathrm{Diff}^K(N)$ can be extended to a diffeomorphism $\widetilde{a}$ in $\mathrm{Diff}^K(U)$ by $\widetilde{a}(x, y) := (a(x), y)$, for $x \in N$ and $y \in D$ (here $U \cong N \times D$ is the open set given by the Definition 2.7).

By Propositions 2.4 and 2.6 we obtain the following:

Proposition 2.8. *For any diffeomorphisms a, b in $\mathrm{Diff}^K(N)$ define:*

$$\begin{aligned} d_0(a,b) :=& d_0(\widetilde{a},\widetilde{b}),\\ d_K(a,b) :=& d_K(\widetilde{a},\widetilde{b}),\\ d'_K(a,b) :=& d'_K(\widetilde{a},\widetilde{b}),\\ d'_K(a) :=& d'_K(\widetilde{a}),\\ \overline{d}_K(a,b) :=& \overline{d}_K(\widetilde{a},\widetilde{b}). \end{aligned}$$

Denote by $\mathrm{Diff}^K_{U,bdd}(N)$ the set of C^K-diffeomorphisms a of N such that $d_K(\widetilde{a}^{\pm 1}, Id_U) < \infty$.

Then, for any diffeomorphisms $a, b, u \in \mathrm{Diff}^K_{U,bdd}(N)$ and $v \in \mathrm{Diff}^{K+1}_{U,bdd}(N)$, the properties (D1), (D2), (R1), (R2), (R3) and (L4) hold.

Also, $\mathrm{Diff}^K_{U,bdd}(N)$ is a complete group with respect to the metric $\overline{d}_K$. □

Remark. We use the same notation for the metrics on $\mathrm{Diff}^K(N)$ and $\mathrm{Diff}^K(U)$, for N a good manifold and U an open set in a flat space, but this will produce no confusion.

The next proposition will be used in §6.

Proposition 2.9. *(a) Consider $a \in C^1(\mathbb{T}^d, \mathbb{T}^d)$. If $d_1(a, Id) < \frac{1}{2}$, then $a \in \mathrm{Diff}^1(\mathbb{T}^d)$.*

(b) Let $K \geq 1$ be an integer. Then there exists a constant $L_K > 0$ such that for any $a \in C^{K+\mathrm{Lip}}(\mathbb{T}^d)$ with $d_{K+\mathrm{Lip}}(a, Id) < \frac{1}{2}$, one has

$$d_{K+\mathrm{Lip}}(a^{-1}, Id) \leq L_K.$$

(c) Define

$$\begin{aligned} M'_K :=& (K^2+1)(1+L_K)^{K+1},\\ M_K :=& (1+K)(1+L_K)^{K+1}[1+3(1+L_K)^K]^K. \end{aligned}$$

Then, for any $a, b \in C^{K+\mathrm{Lip}}(\mathbb{T}^d)$ with $d_{K+\mathrm{Lip}}(a, Id) < \frac{1}{2}$ and $d_{K+\mathrm{Lip}}(b, Id) < \frac{1}{2}$, one has:

$$\begin{aligned} &(i)\ d_{K+\mathrm{Lip}}(a^{-1}, Id) \leq M'_K d_{K+\mathrm{Lip}}(a, Id),\\ &(ii)\ d_K(a^{-1}, b^{-1}) \leq M_K d_K(a,b). \end{aligned}$$

Proof. (a) If $d_0(a, Id) < \frac{1}{2}$, then a is homotopic to the identity map. Hence a is surjective (use the topological degree). If $d_{(1)}(a, Id) < \frac{1}{2}$, then for any $x, y \in \mathbb{T}^d$:

$$\begin{aligned} &\|a(x) - a(y)\| \geq \|x-y\| - \|(a(x)-x) - (a(y)-y\| \geq\\ \geq& \|x-y\| - \|x-y\| \sup_{z\in\mathbb{T}^d} \|D_z(a - Id)\| \geq \frac{1}{2}\|x-y\|, \end{aligned}$$

so a is injective. Since a is also a local diffeomorphism, $a \in \mathrm{Diff}^1(\mathbb{T}^d)$.

(b) By differentiating the equality $aa^{-1} = Id$, one can express $D_x^l a^{-1}$ as a "polynomial" in $D^s_{a^{-1}(x)} a$, $1 \le s \le l$, and $(D^1_{a^{-1}(x)} a)^{-1}$. For example:

$$\begin{gathered} D_x^1 a^{-1} = (D^1_{a^{-1}(x)} a)^{-1}, \\ D_x^2 a^{-1}(\xi,\eta) = -(D^1_{a^{-1}(x)} a)^{-1} D^2_{a^{-1}(x)} a((D^1_{a^{-1}(x)} a)^{-1}\xi, (D^1_{a^{-1}(x)} a)^{-1}\eta). \end{gathered}$$

(See formula (2.1)).

Use now that if $X \in \operatorname{End}(V)$ (V some normed vector space) is such that $\|I - X\| < 1$, then $\|I - X^{-1}\| \le \frac{\|I-X\|}{1-\|I-X\|}$ (by the power series expansion).

Hence, if $d_K(a, Id) < \frac{1}{2}$, then

$$\|D_x a^{-1} - Id\| = \|(D_{a^{-1}(x)} a)^{-1} - Id\| < 2\|D_{a^{-1}(x)} a - Id\| < 1$$

and $d_{(1)}(a^{-1}) < 2$.

Therefore, for $1 \le l \le K$, $d_{(l)}(a^{-1})$ is bounded by some constant which does not depend on a.

In order to estimate $\|D^K a^{-1}\|_{\mathrm{Lip}}$, notice that it will be bounded by some polynomial expression of $\|a^{-1}\|_{\mathrm{Lip}}, d_{(1)}(a^{-1}), \|(D^1 a)^{-1}\|_{\mathrm{Lip}}, d_{(s)}(a)$ and $\|D^s a\|_{\mathrm{Lip}}$ ($1 \le s \le K$). This proves our assertion.

(c) (i)

$$\begin{aligned} d_{K+\mathrm{Lip}}(a^{-1}, Id) &= d_{K+\mathrm{Lip}}((Id - a)a^{-1}) \le \\ &\overset{(\mathrm{Lip3})}{\le} d_{K+\mathrm{Lip}}(Id - a)[1 + d_{K+\mathrm{Lip}}(a^{-1}, Id)]^K \cdot \\ &\quad \cdot [1 + (K^2+1) d_{K+\mathrm{Lip}}(a^{-1}, Id)] \le \\ &\le (1 + L_K)^K [1 + (K^2+1) L_K] d_{K+\mathrm{Lip}}(a, Id) \le \\ &\le (1 + L_K)^{K+1} (K^2+1) d_{K+\mathrm{Lip}}(a, Id). \end{aligned}$$

(ii)

$$\begin{aligned} & d_K(a^{-1}, b^{-1}) = d_K(a^{-1} Id, a^{-1}(ab^{-1})) \le \\ &\overset{(\mathrm{L1})}{\le} d_K(Id, ab^{-1})(K+1) d'_{K+\mathrm{Lip}}(a^{-1})[1 + 2d'_K(ab^{-1}))]^K \le \\ &\overset{(\mathrm{R1})}{\le} d_K((bb^{-1}, ab^{-1})(K+1)(1+L_K) \cdot \Big(1 + 2d'_K(a)[1 + d_K(b^{-1}, Id)]^K\Big)^K \le \\ &\overset{(\mathrm{R3})}{\le} d_K(b, a)[1 + d_K(b^{-1}, Id)]^K (K+1)(L_K+1) \cdot \\ &\quad \cdot \Big\{1 + 2[1 + d_K(a, Id)][1 + d_K(b^{-1}, Id)]^K\Big\}^K \le \\ &\le (1 + L_K)^{K+1}(1+K)[1 + 3(1+L_K)^K]^K d_K(a, b). \quad \square \end{aligned}$$

§3. A Livsic's Type Theorem for Cocycles with Values in Diffeomorphism Groups

The goal of this section is to prove the following theorem:

Theorem 3.1. *Let M be a compact closed manifold, let $\mathbb{Z} \times M \to M$, $(g, x) \mapsto gx$ be an action given by a C^1 transitive Anosov diffeomorphism $T : M \to M$. Endow M with a Lyapunov metric $\mathbf{d}_M$ and for $k \geq 0$ let $\mathcal{G}^k$ be one of the groups:*

(a) $\mathrm{Diff}^k_{bdd}(U)$ endowed with the metric d_k described in Definition 2.2, where U is an open connected subset in $\mathbb{R}^d$;

(b) $\mathrm{Diff}^k_{U,bdd}(N)$ endowed with the metric d_k described in Proposition 2.8, where N is a good manifold.

Fix $0 < \alpha \leq 1$, $K \in \mathbb{N}$, $K \geq 4$.

Then there exists $\delta > 0$ with the following property: if $\overline{\beta}$ is any $\mathbb{Z}$-cocycle over the considered action

$$\overline{\beta} : \mathbb{Z} \times M \to \mathcal{G}^K$$

such that:

1. *$d_K(\overline{\beta}(1, x)^{\pm 1}, Id) < \delta$, for any $x \in M$;*
2. *$\overline{\beta}$ fulfills the closing conditions:*

$$x \in M, g \in \mathbb{Z} \text{ and } gx = x \text{ implies that } \overline{\beta}(g, x) = Id;$$

3. *$\overline{\beta}(1, \cdot) : M \to \mathcal{G}^K$ is α-Hölder,*

then there is an α-Hölder map $P : M \to \mathcal{G}^{K-2}$ which trivializes $\overline{\beta}$, i.e.

$$\overline{\beta}(g, x) = P(gx)P(x)^{-1}. \tag{T}$$

Remarks. 1. The proof uses the idea of Livsic (compare the proof of Theorem 1.2 in [L2] with Claim 1 in the proof of Theorem 3.1). Nevertheless, our situation is more complicated due to the absence of invariant metrics. We replace these by the "quasi-invariance" properties exhibited in Propositions 2.4 and 2.8. The proof of the theorem will be the same in the cases (a) and (b).

2. One can view a connected matrix Lie group $G \subset GL(n, \mathbb{R})$ as a subgroup of $\mathrm{Diff}^k_{bdd}(GL(n, \mathbb{R})^0)$, via its action by left translations. [$GL(n, \mathbb{R})^0$ is the connected component of the identity in the group of $n \times n$ invertible matrices.] Since $GL(n, \mathbb{R})^0$ is an open subset of $\mathbb{R}^{n^2}$, one can try to find a proof of Theorem 1.2 using directly Theorem 3.1 (a). The problem is that the natural metric on the Lie group might not be equivalent to the one we introduced in §2.

3. There is also a similar statement for cocycles over actions given by diffeomorphisms with an invariant locally maximal hyperbolic set.

Before beginning the proof, we record the following ingredients, also used by Livsic in the proof of Theorem 1.2.

Anosov's Closing Lemma. *([K-H]) Given $T : M \to M$ an Anosov diffeomorphism on a compact manifold, there are constants $\epsilon_{CL} > 0$, $K_{CL} > 0$ with the following property:*

> *for $0 < \delta < \epsilon_{CL}$, given a periodic δ-pseudo-orbit $(x_0, x_2, \ldots, x_{n-1})$, there is a periodic orbit of the same length $(x_*, Tx_*, \ldots, T^{n-1}x_*)$, $T^n x_* = x_*$ such that*
>
> $$\mathbf{d}_M(x_k, T^k x_*) \le K_{CL}\delta, \quad 0 \le k \le n-1.$$

Local Product Structure. *([K-H]) Given $T : M \to M$ an Anosov diffeomorphism on a compact manifold, there are $\epsilon_{PS} > 0$, $K_{PS} > 0$ such that:*

> *for any $x, y \in M$ with $\mathbf{d}_M(x, y) < \epsilon_{PS}$, the intersection $W^s(x) \cap W^u(y)$ contains exactly one element, whose distance to both x and y is at most $K_{PS} \cdot \mathbf{d}_M(x, y)$.*

Since in our case the metrics d_k are not as well behaved as those of Livsic, we isolate the main computational part in the following lemma:

Lemma 3.2. *Assume $0 < \mu < 1$ and $\delta > 0$ are such that $\mu(1 + 4\delta)^{2K} < 1$. Let $N \ge 1$ be an integer and, for any $1 \le k \le N$, let $a_k, b_k, c_k \in \mathcal{G}^K$ be such that*

(i) $d_{K-1}(a_k, b_k) \le C_0 \cdot \mu^{k-1} \cdot \omega_0$,
(ii) $d_{K-2}(b_k, c_k) \le C_0 \cdot \mu^{N-k} \cdot \omega_0$,
(iii) $d_{K-1}(a_k^{\pm 1}, Id) < \delta$, $d_K(b_k^{\pm 1}, Id) < \delta$, $d_{K-2}(c_k^{\pm 1}, Id) < \delta$.

Denote $a = a_1 a_2 \cdots a_N$, $b = b_1 b_2 \cdots b_N$, $c = c_1 c_2 \cdots c_N$. Then:

(a) $d_{K-1}(a, b) \le \omega_0 \cdot \frac{C_0 K}{1-\mu(1+4\delta)^{2K}} \max\{1, d'_{K-1}(a)\}^{K-1}$;
(b) $d_{K-2}(b, c) \le \omega_0 \cdot \frac{C_0 (K-1)}{1-\mu(1+4\delta)^{2K}} \cdot d'_{K-1}(b)$.

(Note that $d'_{K-1}(b) \le 1 + d_{K-1}(b, Id)$).

Proof. (a) $d_{K-1}(a, b) = d_{K-1}(a_1 a_2 \cdots a_N,\ b_1 b_2 \cdots b_N) \le \Delta_1 + \Delta_2 + \cdots + \Delta_N$, where

$$\Delta_k = d_{K-1}(b_1 \cdots b_{k-1} a_k a_{k+1} \ldots a_N, b_1 \ldots b_{k-1} b_k a_{k+1} \ldots a_N).$$

Replace in Δ_k $a_{k+1} \cdots a_N$ by $a_k^{-1} a_{k-1}^{-1} \cdots a_1^{-1} a$. Then apply (R3) successively for the final a-factors, and then (L4) once for the product of the leading b-factors to get:

$$\Delta_k \le d_{K-1}(b_1 \ldots b_{k-1} a_k, b_1 \ldots b_k) \cdot$$
$$\cdot \left[\prod_{j=1}^{k} \left(1 \vee d'_{K-1}(a_j^{-1})\right)^{K-1} \right] \left(1 \vee d'_{K-1}(a)\right)^{K-1} \le$$
$$\le d_{K-1}(a_k, b_k) \cdot K \cdot \left[\prod_{j=1}^{k} (1 \vee d'_{K-1}(a_j^{-1}))^{K-1} \right] \cdot (1 \vee d'_{K-1}(a))^{K-1} \cdot$$
$$\cdot d'_K(b_1 \cdots b_{K-1}) \cdot [1 + 2d'_{K-1}(a_k, Id) + 2d'_{K-1}(b_k, Id)]^{K-1}.$$

Applying successively (R1) for each factor, we get (for $k \geq 2$) that

$$d'_K(b_1 \cdots b_{k-1}) \leq d'_K(b_1) \cdot \prod_{\ell=2}^{k-1} [1 \vee d'_K(b_\ell)]^K.$$

By (D1) and (iii):

$$\begin{aligned} \Delta_k &\leq d_{K-1}(a_k, b_k) \cdot K \cdot [(1+\delta)^{K-1}]^k \cdot (1 \vee d'_{K-1}(a))^{K-1} \cdot \\ &\cdot (1+\delta)^{1+K(k-2)} \cdot (1+4\delta)^{K-1} \leq \\ &\leq d_{K-1}(a_k, b_k) \cdot K \cdot (1+4\delta)^{2kK} (1 \vee d'_{K-1}(a))^{K-1}. \end{aligned}$$

Together with (i), this yields (a).
(b) Write

$$d_{K-2}(b, c) = d_{K-2}(b_1 b_2 \cdots b_N, c_1 c_2 \cdots c_N) \leq \Delta_1 + \Delta_2 + \cdots + \Delta_N,$$

with

$$\Delta_k = d_{K-2} \left(b_1 \cdots b_{N-k} b_{N-k+1} c_{N-k+2} \cdots c_N, b_1 \cdots b_{N-k} c_{N-k+1} c_{N-k+2} \cdots c_N\right).$$

(Here we start interchanging factors at the right side of the products.)

Replace in Δ_k $b_1 \cdots b_{N-k}$ by $b\, b_N^{-1} \cdots b_{N-k+1}^{-1}$, apply (R3) successively for the last c-factors and (L4) once for the product of the leading b-factors to get

$$\begin{aligned} \Delta_k &\leq d_{K-2}(b_{N-k-1}, c_{N-k+1}) \cdot (K-1) \cdot \left[\prod_{j=N-k+2}^{N} (1 \vee d_{K-2}(c_j))^{K-2} \right] \cdot \\ &\cdot d'_{K-1}(b\, b_N^{-1} \cdots b_{N-k+1}^{-1}) \cdot [1 + d'_{K-2}(b_{N-k+1}, Id) + d'_{K-2}(c_{N-k+1}, Id)]^{K-2}. \end{aligned}$$

By successive use of (R1) we obtain that

$$d'_{K-1}(b\, b_N^{-1} \cdots b_{N-k+1}^{-1}) \leq d'_{K-1}(b) \cdot \prod_{j=N-k+1}^{N} (1 \vee d'_{K-1}(b_{N-k+1}))^{K-1},$$

hence, by (D1) and (iii), it follows that:

$$\begin{aligned} \Delta_k &\leq d_{K-2}(b_{N-k+1}, c_{N-k+1}) \cdot (K-1) \cdot (1+\delta)^{(K-2)(k-1)} \cdot d'_{K-1}(b) \cdot \\ &\cdot (1+\delta)^{(K-1)k} \cdot (1+4\delta)^{K-2} \leq \\ &\leq d_{K-2}(b_{N-k+1}, c_{N-k+1})(K-1)(1+4\delta)^{(2kK-3k)} \cdot d'_{K-1}(b) \leq \\ &\leq d_{K-2}(b_{N-k+1}, c_{N-k+1})(K-1)(1+4\delta)^{2kK} \cdot d'_{K-1}(b) \end{aligned}$$

which together with (ii) implies (b).

Proof of Theorem 3.1.

Let $TM = E^u \oplus E^s$ be the Anosov splitting of TM and $0 < \lambda < 1$ such that $\|DT|_{E^s}\| < \lambda$ and $\|DT^{-1}|_{E^u}\| < \lambda$. Take $\delta > 0$ such that $\lambda^\alpha(1+4\delta)^{2K} < 1$.

Let $x_0 \in M$ be a transitive point for T, i.e. one has $\overline{\mathcal{O}}_+ = M$, where $x_n := T^n x$, and $\mathcal{O}_+ = \{x_n\}_{n\geq 0}$.

Denote $\beta(\cdot) := \overline{\beta}(1,\cdot) : M \to \mathcal{G}^K$, where $\overline{\beta} : \mathbb{Z} \times M \to \mathcal{G}^K$ fulfills the conditions (1), (2) and (3) in the statement of Theorem 3.1.

The equation (T) has a unique solution on $\mathcal{O}_+$ up to a choice of $P(x_0)$. For $P(x_0) = Id$ we obtain the solution $P_0 : \mathcal{O}_+ \to \mathcal{G}^K,\ P_0(x_n) \overset{\text{not.}}{=} \varphi_n = \beta(T^{n-1}x_0)$ $\beta(T^{n-2}x_0)\cdots\beta(x_0)$.

We will show that P_0 can be extended to M to give the desired P.

Claim 1. There exist $1 \geq \epsilon > 0$ and $A \geq 1$ such that whenever

$$\begin{gathered}\mathbf{d}_M(x_n, x_m) < \epsilon \text{ for some } n > m, \text{ then}\\ d_{K-2}(\varphi_n\varphi_m^{-1}, Id) \leq A\,(\mathbf{d}_M(x_n, x_m))^\alpha.\end{gathered}$$

Proof. Set $\epsilon = \min\{1, \epsilon_{CL}, \frac{\epsilon_{PS}}{K_{CL}}\}$; A will be specified later.

Assume $n > m$ are such that $\mathbf{d}_M(x_n, x_m) \overset{\text{not.}}{=} \omega < \epsilon$. Denote $x = x_m,\ N = n-m$. Therefore $(x, Tx, \cdots, T^{N-1}x)$ is a periodic ω-pseudo-orbit, so there is an N-periodic point x_* such that

$$\mathbf{d}_M(T^k x, T^k x_*) < K_{CL}\,\omega < \epsilon_{PS}, \text{ for } 0 \leq k \leq N.$$

Let $\overline{x}$ be the unique element of $W^u(x_*) \cap W^s(x)$. Then one has the inequalities:

$$\begin{aligned}\mathbf{d}_M(T^k x, T^k\overline{x}) &\leq C\cdot\lambda^k\cdot\omega,\\ \mathbf{d}_M(T^k\overline{x}, T^k x_*) &\leq C\cdot\lambda^{N-k}\cdot\omega \quad,\quad 0 \leq k \leq N,\end{aligned}$$

where C depends only on T.

Denote:

$$\begin{aligned}a_k &= \beta(T^{N-k}x_*),\\ b_k &= \beta(T^{N-k}\overline{x}),\\ c_k &= \beta(T^{N-k}x),\ \ 1 \leq k \leq N,\\ a &= a_1a_2\cdots a_N,\\ b &= b_1b_2\cdots b_N,\\ c &= c_1c_2\cdots c_N.\end{aligned}$$

Then $\varphi_n\varphi_m^{-1} = c$, while $a = Id$ due to the closing conditions. We have to show that c is close to a.

Since β is α-Hölder,

$$\begin{aligned}d_K(a_k, b_k) &\leq C'\,\mu^k\cdot\omega^\alpha,\\ d_K(b_k, c_k) &\leq C'\,\mu^{N-k}\cdot\omega^\alpha,\end{aligned}$$

where $C' = C^\alpha\cdot\|\beta\|_{\text{Hölder}}$ and $\mu = \lambda^\alpha$.

Now we can apply Lemma 3.2 to obtain

$$d_{K-2}(\varphi_n\varphi_m^{-1}, Id) = d_{K-2}(c,a) \leq$$
$$\leq d_{K-1}(b,a) + d_{K-2}(c,b) \leq A\,\omega^\alpha = A \cdot \mathbf{d}_M(x_m,x_n)^\alpha,$$

where

$$A = \frac{C'K}{1-\mu(1+4\delta)^{2K}} +$$
$$+ \frac{C'(K-1)}{1-\mu(1+4\delta)^{2K}} \cdot \Big(1 + \frac{C'K}{1-\mu(1+4\delta)^{2K}} \cdot \epsilon^\alpha\Big) + 1$$

(Note: the last 1 appears only because we want $A \geq 1$.)

Claim 2. If $a \in \mathcal{G}^2$ and $d_2(a, Id) < \frac{1}{2}$, then

$$d_2'(a^{-1}, Id) \leq 8\, d_2(a, Id) < 4.$$

Proof. From the proof of Proposition 2.9 (b), it follows that $\|D_x a^{-1} - I\| < 2$ and

$$D_x^2 a^{-1} = (D_{a^{-1}(x)}a)^{-1} D^2_{a^{-1}(x)} a((D_{a^{-1}(x)}a)^{-1}, (D_{a^{-1}(x)}a)^{-1}).$$

Claim 3. One has:

(a) $M_1 := \sup_n d_{K-2}(\varphi_n, Id) < \infty$,

(b) $M_2 := \sup_n d_2'(\varphi_n^{-1}) < \infty$.

Proof. Define $\epsilon' = \max\{\epsilon, \frac{1}{(2A)^{1/\alpha}}\}$. Since M is compact, there is a finite set $I \subset \mathbb{N}$ such that the $\mathbf{d}_M$-balls of radius $\frac{1}{2}\epsilon'$ centered in $\{x_i\}_{i\in I}$ cover M. Call these balls U_i $(i \in I)$ respectively.
(a) Let $n \in \mathbb{N}$ be such that $x_n \in U_i,\ n \geq i$. Then:

$$d_{K-2}(\varphi_n, \varphi_i) = d_{K-2}((\varphi_n\varphi_i^{-1})\varphi_i, \varphi_i) \overset{(R3)}{\leq}$$
$$\leq d_{K-2}(\varphi_n\varphi_i^{-1}, Id) \cdot [1 + d_{K-2}(\varphi_i, Id)]^{K-2} \leq$$
$$\overset{\text{Claim 1}}{\leq} A(\frac{\epsilon'}{2})^\alpha [1 + d_{K-2}(\varphi_i, Id)]^{K-2}.$$

Note that the values of n which are not covered by the above argument are in $J = \{n \in \mathbb{N}; x_n \in U_i,\ n < i\}$, which is a finite set. Therefore:

$$\sup_{n\in\mathbb{N}} d_{K-2}(\varphi_n, Id) \leq \max_{i\in I}\{d_{K-2}(\varphi_i, Id) +$$
$$+ A(\frac{\epsilon'}{2})^\alpha [1 + d_{K-2}(\varphi_i, Id)]^{K-2}\} \vee \max_{n\in J}\{d_{K-2}(\varphi_n, Id)\} < \infty.$$

(b) Consider again $n \in \mathbb{N}$ such that $x_n \in U_i,\ n \geq i$. Then:

$$\begin{gathered} d_2'(\varphi_n^{-1}, \varphi_i^{-1}) \leq d_2(\varphi_i^{-1}(\varphi_n\varphi_i^{-1})^{-1}, \varphi_i^{-1}) \overset{(L4)}{\leq} \\ \leq 3d_2((\varphi_n\varphi_i^{-1})^{-1}, Id)d_3'(\varphi_i^{-1})[1 + 2d_2'((\varphi_n\varphi_i^{-1})^{-1}, Id) + 2d_2'(Id, Id)]^2 \leq \\ \leq d_3'(\varphi_i^{-1}))[1 + 3d_2((\varphi_n\varphi_i^{-1})^{-1}, Id)]^3. \end{gathered}$$

Using now Claim 1 and Claim 2, it follows that

$$d_2(\varphi_n\varphi_i^{-1}, Id) \leq A(\frac{\epsilon'}{2})^\alpha < \frac{1}{2}, \text{ for } 2 \leq K-2,$$

and

$$d_2((\varphi_n\varphi_i^{-1})^{-1}, Id) \leq 4,$$

and from here we proceed as at (a).

Claim 4. $P_0 : \mathcal{O}_+ \to \mathcal{G}^K$ is α-Hölder with respect to $\overline{d}_{K-2}$.

Proof. Let n, m be such that $\mathbf{d}_M(x_n, x_m) < \epsilon'$. Assume $n \geq m$. We consider separately the two metrics which define $\overline{d}_{K-2}$.

$$\begin{gathered} d_{K-2}(\varphi_n, \varphi_m) = d_{K-2}((\varphi_n\varphi_m^{-1})\varphi_m, \varphi_m) \leq \\ \overset{(R3)}{\leq} d_{K-2}(\varphi_n\varphi_m^{-1}, Id)[1 + d_{K-2}(\varphi_m, Id)]^{K-2} \leq (M_1)^{K-2}A\ (\mathbf{d}_M(x_m, x_n))^\alpha; \\ d_{(1)}(\varphi_n^{-1}, \varphi_m^{-1}) \leq d_1(\varphi_n^{-1}, \varphi_n^{-1}(\varphi_n, \varphi_m^{-1})) \leq \\ \overset{(L4)}{\leq} d_1(\varphi_n\varphi_m^{-1}, Id)\ d_2'(\varphi_n^{-1})[1 + 2d_1(\varphi_n\varphi_m^{-1}, Id)] \leq \\ \leq M_2(1 + 2A\epsilon)A(\mathbf{d}_M(x_m, x_n))^\alpha. \end{gathered}$$

Conclusion. Due to the α-Hölder continuity in the complete metric space $(\mathcal{G}^{K-2}, \overline{d}_{K-2})$, P_0 can be extended by continuity to an α-Hölder function $P : M \to \mathcal{G}^{K-2}$. Since P fulfills equation (T) on $\mathcal{O}_+$ and both sides of (T) are continuous functions into $\mathcal{G}^{K-2}$, P will trivialize $\overline{\beta}$ over M. $\square$

Remark. The correspondence "cocycle $\overline{\beta} \mapsto$ trivialization P" constructed in the Theorem 3.1 is continuous in the sense described below. (The notations are the same as in the Theorem 3.1.)

Notice first of all that a cocycle $\overline{\beta} : \mathbb{Z} \times M \to \mathcal{G}^K$ is completely determined by the function $\beta := \overline{\beta}(1, \cdot) : M \to \mathcal{G}^K$. From now on we will speak about such functions when we mean a $\mathbb{Z}$-cocycle. On the space of $\mathbb{Z}$-cocycles $\beta : M \to \mathcal{G}^K$ consider the topology given by the uniform convergence (note that for functions $\beta : M \to \mathcal{G}^K$ in a given Hölder class with uniformly bounded Hölder norm, this topology is the same as that of pointwise convergence).

Fix a degree of smoothness $K \in \mathbb{N}$, $K \geq 4$, and a Hölder class $0 < \alpha \leq 1$. Let $\delta > 0$ be the constant provided by Theorem 3.1.

Corollary (of the proof) 3.3.
Assume that β, β_ι are cocycles with values in $\mathcal{G}^K$, in the same Hölder class α, and $\|\beta\|_{\text{Hölder}}$, $\|\beta_\iota\|_{\text{Hölder}} \leq C_0$, where $C_0 > 0$ is a constant. If β_ι converges to β, and all these cocycles satisfy the conditions (1) and (2) of the theorem, then P_ι converges uniformly to P, as functions from M into $\mathcal{G}^{K-2}$ (where P, P_ι are the trivializations constructed for β, respectively β_ι).

Proof. Let $P_0, P_0^{(\iota)} : \mathcal{O}_+ \to \mathcal{G}^K$ be the trivializations constructed for β, respectively β_ι, on the orbit of the transitive point x_0, such that their value at x_0 is Id. Since all these are finite products of the values of the cocycles, $P_0 \to P_0^{(\iota)}$ pointwise on $\mathcal{O}_+$. But the functions $P_0, P_0^{(\iota)} : \mathcal{O}_+ \to \mathcal{G}^{K-2}$ are all α-Hölder, with uniformly bounded norms, (the estimate obtained in Claim 4, (a) depends "continuously on the cocycle"). Therefore the functions $P, P_\iota : M \to \mathcal{G}^{K-2}$ are uniformly equicontinuous, and then pointwise convergence on the dense set $\mathcal{O}_+$ implies uniform convergence on the compact set M. $\square$

§4. Noncommutative Closing Lemma

In this section we generalize the Anosov Closing Lemma. This will be used later to extend Livsic's theorems to Anosov group actions.

Theorem 4.1. *("Noncommutative Closing lemma") Let M be a smooth closed manifold (i.e. compact and without boundary). Let $A, B \in \mathrm{Diff}^1(M)$, A Anosov. Let $TM = E^u \oplus E^s$ be the splitting induced by DA, the differential of A. Endow M with an A-adapted metric $\mathbf{d}_M$. Assume that*

$$P_+(D\,B)|_{E^u} : E^u \to E^u \quad \text{and} \quad P_-(D\,B^{-1})|_{E^s} : E^s \to E^s$$

can be inverted fiberwise (hence the norm of the inverses is bounded uniformly on M), where $P_\pm : TM \to E^{u,s}$ is the projection along $E^{s,u}$. (Equivalently, that there is a constant $\alpha_B > 0$ such that

$$\measuredangle\,(E^s,\ D\,B(E^u)) > \alpha_B$$

and

$$\measuredangle\,(E^u,\ D\,B^{-1}(E^s)) > \alpha_B.)$$

Then there are $N_0 \in \mathbb{N}$, $\delta_0 > 0$, $K > 0$ such that for any $N \geqslant N_0$, $0 < \delta < \delta_0$, and for any sequence $(\overline{x}_0, \overline{x}_1, \cdots, \overline{x}_N)$ such that $\mathbf{d}_M(A\overline{x}_k, \overline{x}_{k+1}) < \delta$, for $0 \le k \le N-1$, and $\mathbf{d}_M(B\overline{x}_N, \overline{x}_0) < \delta$, there is a point $x \in M$ with the following properties:

(a) *$B\,A^N x = x$;*
(b) *the pseudo-orbit $(\overline{x}_0, \overline{x}_1, \cdots, \overline{x}_N)$ and the orbit of x are $K\delta$-close, i.e.*

$$\mathbf{d}_M(\overline{x}_k, A^k x) < K\delta,\ 0 \le k \le N.$$

Moreover, in a neighborhood of radius δ_0 of $\overline{x}_0$, there is only one point x that satisfies (a) and (b).

Notations. 1. A diffeomorphism B as above will be said to be in generic position with respect to A.

2. By abuse of notation, a sequence $(\overline{x}_0, \overline{x}_1, \cdots, \overline{x}_N)$ as in the theorem will be called a δ-pseudo-orbit.

Remarks. 1. The restriction on B is very weak (the required property is generic).

2. The theorem holds for a hyperbolic set too, provided the appropriate changes are done. Also, if the considered fiber bundle maps are not globally invertible, one can obtain a local result.

3. The proof of the theorem is an adaptation of the classic proof of Anosov's Closing Lemma as presented in [K-H], Theorem 6.4.15. Our contribution is to show that the analog of the operator denoted $(1 - L)$ there, is also invertible in our case, and the norm of the inverse is uniformly bounded with respect to N. For the reader's convenience, we present the complete proof.

4. Note that the proof is much simpler if A and B are linear automorphisms of the torus. This is the case we need to prove rigidity of ρ_0. The general case is used to generalize the Livsic-Sinai theorem to Anosov group actions (see §5).

Proof. We first introduce local coordinates on M.

Given $\delta > 0$, denote by $(TM)_\delta$ the bundle of balls of radius δ centered at the origin. By compactness, there are $\delta_1', \delta_1 > 0$ such that the map

$$\exp : (TM)_{\delta_1'} \to M$$

given in each fiber by

$$\phi_x = \exp_x : (T_xM)_{\delta_1'} \to M$$

has the following properties:

– for any x in M, the image of ϕ_x contains a ball of radius δ_1 centered at x;

– the dependence $x \mapsto \phi_x$ is continuous from M to the C^1-topology on the maps $\{\phi_x\}_{x\in M}$;

– $\|\phi_x^{\pm 1}\|_{Lip} \leq 2$ for each $x \in M$.

Recall that $D_0\phi_x = Id_{T_xM}$.

Since $\measuredangle(E^u, E^s) > \alpha_A > 0$, there is a constant $C_1 > 0$ with the property that for each $x \in M$ one can choose a linear isomorphism $\Lambda_x : T_xM \to \mathbb{R}^d$ such that:

– Λ_x takes the decomposition $T_xM = E_x^u \oplus E_x^s$ into the decomposition $\mathbb{R}^d = \mathbb{R}^{d_+} \oplus \mathbb{R}^{d_-}$;

– $\Lambda_x|_{E_x^{u,s}} : E_x^{u,s} \to \mathbb{R}^{d_\pm}$ are isometries, where $\mathbb{R}^d$ is endowed with the standard Euclidean structure;

– $\|\Lambda_x^{\pm 1}\| \leq C_1$.

Having done all these, we describe the procedure by which we reduce the maps along a pseudo-orbit to ones from $\mathbb{R}^d$ to $\mathbb{R}^d$.

Consider any $\frac{\delta_1}{2}$-pseudo-orbit $(\overline{x}_0, \cdots, \overline{x}_N)$. Describe (locally) the maps A and B by:

$$\mathcal{A}_k^0 = \Lambda_{\overline{x}_{k+1}}\phi_{\overline{x}_{k+1}}^{-1} A \phi_{\overline{x}_k}\Lambda_{\overline{x}_k}^{-1}, 0 \leq k \leq N-1;$$

$$\mathcal{B}_N^0 = \Lambda_{\overline{x}_0}\phi_{\overline{x}_0}^{-1} B \phi_{\overline{x}_N}\Lambda_{\overline{x}_N}^{-1}.$$

Using the decomposition $\mathbb{R}^d = \mathbb{R}^{d_+} \oplus \mathbb{R}^{d_-}$, write

$$D_0\mathcal{A}_k^0 = \begin{pmatrix} A_k^+ & * \\ * & A_k^- \end{pmatrix},$$

$$D_0\mathcal{B}_N^0 = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \overset{\text{not.}}{=} B_N,$$

$$(D_0\mathcal{B}_N^0)^{-1} = \begin{pmatrix} \overline{B}_{11} & \overline{B}_{12} \\ \overline{B}_{21} & \overline{B}_{22} \end{pmatrix} \overset{\text{not.}}{=} \overline{B}_N,$$

where we denoted by $*$ the entries which are not used in the sequel.

By the properties of the chosen coordinates and of the diffeomorphisms A and B, there are $\delta_2 > 0$, $0 < \lambda_1 < 1$ and $C_2, C_3 \geq 1$ which satisfy the following:

for any $0 < \epsilon$, there is $0 < \delta_3(\epsilon) < \frac{\delta_1}{2}$ such that if $(\overline{x}_0, \cdots, \overline{x}_N)$ is a $\delta_3(\epsilon)$-pseudo-orbit then:

– the maps $\mathcal{A}_k^0, 0 \leq k \leq N-1, \mathcal{B}_N^0$ are defined on a ball of radius δ_2 centered at the origin;

– on these balls

$$\begin{aligned} &\|\mathcal{A}_k^0 - \begin{pmatrix} A_k^+ & 0 \\ 0 & A_k^- \end{pmatrix}\|_{C^1} < \epsilon, \\ &\|\mathcal{B}_N^0 - \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\|_{C^1} < \epsilon; \end{aligned} \tag{4.1}$$

– the linear parts satisfy

$$\begin{aligned} \|A_k^-\| \le \lambda_1, \|(A_k^+)^{-1}\| \le \lambda_1, \\ \|(B_{11})^{-1}\| \le C_2, \|(\overline{B}_{22})^{-1}\| \le C_2, \\ \|B_N\| \le C_3, \|(B_N)^{-1}\| \le C_3. \end{aligned}$$

In order to make the following steps simpler, we will extend the maps $\mathcal{A}_k^0$ and $\mathcal{B}_N^0$ to the whole space $\mathbb{R}^d$. We use the following fact (see [K-H]):

> Let U, V be open bounded sets in $\mathbb{R}^d$ with $\overline{U} \subset V$. For any $\epsilon > 0$, there is $\delta_4(\epsilon)$ such that if $f \in C^1(V, V)$ and Λ is a linear map on $\mathbb{R}^d$ with $\|f - \Lambda\|_{C^1, V} < \delta_4(\epsilon)$, then there is a function $\widetilde{f} \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ such that $\|\widetilde{f} - \Lambda\|_{C^1, \mathbb{R}^d} < \epsilon$ and $\widetilde{f}|_U = f|_U$.

(Indeed, let $\chi : \mathbb{R}^d \to [0,1]$ be a C^1 function such that $\chi|_U \equiv 1$ and $\chi|_{\mathbb{R}^d \smallsetminus V} \equiv 0$, and set $\widetilde{f} = \chi \cdot f + (1-\chi)\Lambda$.)

Take U, V to be the balls of radii $\frac{\delta_2}{2}$, respectively δ_2, centered at the origin. Denote the extended maps by $\mathcal{A}_k$ and $\mathcal{B}_N$.

Using the extended maps, the problem becomes one about small perturbations of linear maps between Euclidean spaces of dimension d: if the ϵ used in (4.1) is less than $\delta_4(\omega)$, then

$$\begin{aligned} \mathcal{A}_k(x^+, x^-) &= (A_k^+ x^+ + \alpha_k^+(x^-, x^+), A_k^- x^- + \alpha_k^-(x^-, x^+)), \qquad 0 \le k \le N-1, \\ \mathcal{B}_N(x^+, x^-) &= (B_{11}x^+ + B_{12}x^- + \beta^+(x^-, x^+), B_{21}x^+ + B_{22}x^- + \beta_N{}^-(x^-, x^+)), \end{aligned}$$

where $A_k^\pm, B_{ij}$ are linear maps and $\alpha_k^\pm, \beta^\pm$ are such that $\|\alpha_k^\pm\|_{C^1} \le \omega$ for all k and $\|\beta_N^\pm\|_{C^1} \le \omega$.

Let

$$C_4 := \frac{1}{1-\lambda_1}(2C_3+1)^3 \max\{\lambda_1^{-1}, C_3\}.$$

After these preliminaries, we are ready to start the proof itself.

The constants mentioned in the theorem are determined by:

– $\delta_0 = \min\{\dfrac{\delta_2}{16C_1C_4}, \delta_3(\delta_4(\frac{1}{2}C_4^{-1}))\}$;

– $K = 8C_1^2$;

– N_0 satisfies

$$\begin{aligned} \lambda_1^{N_0} C_2 < \frac{1}{2}, \\ \lambda_1^{N_0} C_3(1 + 2C_2C_3) < \frac{1}{2}. \end{aligned} \tag{4.2}$$

Consider a δ-pseudo-orbit $(\overline{x}_0, \overline{x}_1, \cdots, \overline{x}_N)$, with $\delta < \delta_0$ and $N \geq N_0$. The equation we want to solve ($BA^N x = x$) becomes

$$F(\widetilde{x}) = \widetilde{x}, \tag{4.3}$$

where $\widetilde{x} = \text{column}(x_0, x_1, \cdots, x_N) \in (\mathbb{R}^d)^{N+1}$, and

$$F(x_0, x_1, \cdots, x_N) := (\mathcal{A}_0 x_1, \mathcal{A}_1 x_2, \cdots, \mathcal{A}_{N-1} x_N, \mathcal{B}_N x_0).$$

(Via our change of coordinates, x_{k+1} corresponds to $A^k x$ and x_0 to $A^N x$.)
Write $F = L_N + S_N$, where $L_N, S_N : (\mathbb{R}^d)^{N+1} \to (\mathbb{R}^d)^{N+1}$ are given by:

$$L_N(x_0, x_1, \cdots, x_N) = (A_0 x_1, \cdots, A_{N-1} x_N, B_N x_0),$$
$$S_N(x_0, x_1, \cdots, x_N) = (\alpha_0(x_1), \cdots, \alpha_{N-1}(x_N), \beta_N(x_0)).$$

Consider on $(\mathbb{R}^d)^{N+1}$ the norm

$$\|(x_0, x_1, \cdots, x_N)\| = \max_{0 \leq i \leq N} \|x_i\|.$$

Note that $\|S_N\|_{C_1} \leq \frac{1}{2} C_4^{-1}$ and

$$\|S(\tilde{0})\| = \|F(\tilde{0})\| = \|(\Lambda_{\overline{x}_1} \phi_{\overline{x}_1}^{-1} A\overline{x}_0, \Lambda_{\overline{x}_2} \phi_{\overline{x}_2}^{-1} A\overline{x}_1, \cdots, \Lambda_{\overline{x}_0} \phi_{\overline{x}_0}^{-1} B\overline{x}_N)\| \leq 2C_1\delta.$$

We will show that if $N \geq N_0$, then $1 - L_N$ is invertible and

$$\|(1 - L_N)^{-1}\| \leq C_4. \tag{4.4}$$

Then the equation (4.3) is equivalent to

$$\widetilde{x} = \mathcal{F}(\widetilde{x}),$$

where $\mathcal{F} = (I - L_N)^{-1} S_N$. Since $\mathcal{F}$ is a contraction, the equation above has a unique solution $\widetilde{x}_*$, and

$$\|\widetilde{x}_*\| \leq \frac{\|\mathcal{F}(\tilde{0})\|}{1 - \|\mathcal{F}\|_{\text{Lip}}} \leq \frac{C_4 \|S(\tilde{0})\|}{1 - \|\mathcal{F}\|_{\text{Lip}}} \leq 4C_1 C_4 \delta \leq \frac{\delta_2}{4}.$$

Therefore $\widetilde{x}_*$ corresponds to a solution of $BA^N x = x$ on the manifold, which will satisfy claim (b) as well.

The last claim of the theorem follows from the fact that the M-balls of radius δ_0 centered at the $\overline{x}_k$'s are mapped inside balls of radius $2C_1\delta_0 \leq 2C_1 \frac{\delta_2}{16 C_1 C_4} < \frac{\delta_2}{2}$ centered at the origin in the corresponding $\mathbb{R}^d$'s. Therefore, on this set, the action on the manifold is conjugated to the F-action on $(\mathbb{R}^d)^{N+1}$ and F has a unique fixed point.

It remains to prove (4.4).

Denoting $\widetilde{\eta}$=column$(\eta_0, \eta_1, \cdots, \eta_N)$ and $\widetilde{y}$=column$(y_0, y_1, \cdots, y_N)$ we have to solve in $\widetilde{y}$ the equation

$$(I - L_N)\widetilde{y} = \widetilde{\eta}$$

$$\iff \qquad \begin{cases} y_0 - A_0 y_1 = \eta_0 \\ y_1 - A_1 y_2 = \eta_1 \\ \qquad \cdots \\ y_{N-1} - A_{N-1} y_N = \eta_{N-1} \\ y_N - B_N y_0 = \eta_N. \end{cases} \tag{4.5}$$

Denote

$$\widetilde{A} := A_0 A_1 \cdots A_{N-1}$$

and

$$\begin{aligned} E :=& B_N A_0 \cdots A_{N-2} \eta_{N-1} + B_N A_0 \cdots A_{N-3} \eta_{N-2} + \cdots \\ & \cdots + B_N A_0 \eta_1 + B_N \eta_0 + \eta_N. \end{aligned} \tag{4.6}$$

Decompose both variables and maps with respect to $\mathbb{R}^d = \mathbb{R}^{d_+} \oplus \mathbb{R}^{d_-}$:

$$\widetilde{A} = \begin{pmatrix} \widetilde{A}_+ & 0 \\ 0 & \widetilde{A}_- \end{pmatrix},$$

$$y_k = \begin{pmatrix} y_k^+ \\ y_k^- \end{pmatrix}, \quad \eta_k = \begin{pmatrix} \eta_k^+ \\ \eta_k^- \end{pmatrix}, \quad 0 \le k \le N,$$

$$E = \begin{pmatrix} E^+ \\ E^- \end{pmatrix}.$$

Note that $\|\widetilde{A}_-\| \le \lambda_1^N$, $\|(\widetilde{A}_+)^{-1}\| \le \lambda_1^N$.

Consider the equations (4.5). One obtains, by eliminating the other variables,

$$y_N = B_N \widetilde{A} y_N + E. \tag{4.7}$$

One can separate (4.7) into two parts, according to the decomposition of $\mathbb{R}^d$:

$$\begin{cases} y_N^+ = B_{11} \widetilde{A}_+ y_N^+ + B_{12} \widetilde{A}_- y_N^- + E^+ & (4.8) \\ y_N^- = B_{21} \widetilde{A}_+ y_N^+ + B_{22} \widetilde{A}_- y_N^- + E^- & (4.9) \end{cases}$$

Rewrite (4.8) as

$$\begin{aligned} (I - B_{11}\widetilde{A}_+) y_N^+ &= B_{12} \widetilde{A}_- y_N^- + E^+ \\ \iff F_0 \widetilde{A}_+ y_N^+ &= B_{12} \widetilde{A}_- y_N^- + E^+, \end{aligned}$$

where $F_0 = (\widetilde{A}_+^{-1}(B_{11})^{-1} - I) B_{11}$.

Now $N \geq N_0$ implies that $\|\widetilde{A}_+^{-1}(B_{11})^{-1}\| \leq \frac{1}{2}$ (see (4.2)). Therefore F_0 is invertible,

$$\|F_0^{-1}\| \leq 2\|(B_{11})^{-1}\| \leq 2C_2 \tag{4.10}$$

and

$$y_N^+ = \widetilde{A}_+^{-1}(F_0^{-1}B_{12}\widetilde{A}_- y_N^- + F_0^{-1}E^+). \tag{4.11}$$

Substituting into (4.9), one has

$$y_N^- = F_1 y_N^- + B_{21}^{-1}F_0^{-1}E^+ + E^-, \tag{4.12}$$

where

$$F_1 = B_{22}\widetilde{A}_- + B_{21}F_0^{-1}B_{12}\widetilde{A}_-.$$

Compute

$$\begin{aligned}\|F_1\| &\leq \|B_{22}\| \cdot \|\widetilde{A}_-\| + \|B_{21}\| \cdot \|F_0^{-1}\| \cdot \|B_{12}\| \cdot \|\widetilde{A}_-\| \leq \\ &\overset{(4.10)}{\leq} \|B_N\|\lambda_1^N + \|B_N\| \cdot 2C_2 \cdot \|B_N\| \cdot \lambda_1^N.\end{aligned}$$

By (4.2), if $N \geq N_0$ then $\|F_1\| \leq \frac{1}{2}$. So $I - F_1$ is invertible, $\|(I - F_1)^{-1}\| \leq 2$, and the equation (4.12) has the (unique) solution

$$y_N^- = (I - F_1)^{-1}(B_{21}^{-1}F_0^{-1}E^+ + E^-)$$

with

$$\|y_N^-\| \leq 2(\|B_N\| \cdot 2 \cdot \|E^+\| + \|E^-\|) \leq (4C_3 + 2)\|E\| \tag{4.13}$$

In conclusion, for $N \geq N_0$ the system (4.5) has a unique solution, hence $1 - L_N$ is invertible.

It remains to estimate the norm of $(1 - L_N)^{-1}$. Denote $V^+ := (\mathbb{R}^{d_+})^{N+1}$ and $V^- := (\mathbb{R}^{d_-})^{N+1}$. In order to make the computation easier, we show separately that $\|(1 - L_N)^{-1}|_{V^\pm}\|$ are bounded by C_4, and hence so is $\|(1 - L_N)^{-1}\|$.

Consider $\widetilde{\eta} \in V^-$.

For $0 \leq k \leq N - 1$, by (4.5) and (4.11):

$$\begin{aligned} y_k^- &= A_k^- \cdots A_{N-1}^- y_N^- + A_k^- \cdots A_{N-2}^- \eta_{N-1}^- + \cdots + A_k^- \eta_{k+1}^- + \eta_k^-, \\ y_k^+ &= A_k^+ \cdots A_{N-1}^+ \widetilde{A}_+^{-1}(F_0^{-1}B_{12}\widetilde{A}_- y_N^- + F_0^{-1}E^+) \\ &= (A_{k-1}^+)^{-1} \cdots (A_0^+)^{-1}(F_0^{-1}B_{12}\widetilde{A}_- y_N^- + F_0^{-1}E^+). \end{aligned}$$

Hence

$$\begin{aligned} \|y_k^-\| &\leq \lambda_1^{N-k}\|y_N^-\| + (\lambda_1^{N-k-1}\|\widetilde{\eta}\| + \lambda_1^{N-k-2}\|\widetilde{\eta}\| + \cdots + \|\widetilde{\eta}\|) \leq \\ &\leq \lambda_1^{N-k}\|y_N^-\| + \frac{1}{1-\lambda_1}\|\widetilde{\eta}\|, \\ \|y_k^+\| &\leq \lambda_1^k(2C_3\lambda_1^N\|y_N^-\| + 2\|E\|). \end{aligned}$$

By (4.13), it is enough to estimate $\|E\|$. By (4.6), we see that for $\widetilde{\eta} \in V^-$

$$\|E\| \leq C_3 \frac{1}{1-\lambda_1} \|\widetilde{\eta}\|.$$

Hence

$$\|(1-L_N)^{-1}|_{V^-}\| \leq \frac{1}{1-\lambda_1}(2C_3+1)^3.$$

In order to estimate $\|(1-L_N)^{-1}|_{V^+}\|$, note that this case can be reduced to the one studied above. Indeed:

– $(1-L_N)^{-1}|_{V^+} = -L_N^{-1}(1-L_N^{-1})^{-1}|_{V^+}$ and $\|L_N^{-1}\| \leq \max\{\lambda_1^{-1}, C_3\}$;

– the matrix of $J(L_N^{-1})J^{-1}$ is of same form as that of L_N, where $J = (j_{kl})_{0 \leq k,l \leq N}$ is the isometry of $(\mathbb{R}^d)^{N+1}$ given by $j_{kl} = \delta_{k+l,N+1} Id_{\mathbb{R}^d}$;

– the "contracting space" of L_N^{-1} is V^+ (which is invariated by J^{-1}), and the corresponding entry of $\overline{B}$ is $\overline{B}_{22}$.

This completes the proof of Theorem 4.1. $\square$

§5. Livsic Type Results for Anosov Smooth Group Actions

In this section we extend Livsic's results about the cohomology of Anosov $\mathbb{Z}$-actions to more general Anosov smooth group actions.

Definition 5.1. *Let G be a discrete group and let $\alpha : G \times M \to M$ be a C^k-action ($k \geq 1$) of G by diffeomorphisms on a compact manifold M. We say that α is Anosov if there is an element $g \in G$ such that $\alpha(g)$ is Anosov. Let Γ be a topological group and $\beta : G \times M \to \Gamma$ a continuous cocycle. We say that β satisfies the isolated closing conditions if for any $g \in G$ and any $x \in M$ isolated fixed point of g, one has $\beta(g, x) = 1_\Gamma$.*

Remark. The theorem below shows that the notion of isolated closing conditions is natural. If we have an Anosov $\mathbb{Z}$-action, then the closing conditions are equivalent with the isolated closing conditions. As one will see in §7, the isolated closing conditions can be easily obtained for some natural examples.

The main result is the following theorem which is based essentially on the Noncommutative Closing Lemma, Theorem 4.1:

Theorem 5.2. *Let M be a smooth closed manifold. Let $A, B \in \mathrm{Diff}^1(M)$, A a topologically mixing Anosov and B in generic position with respect to A. Let $G = \langle A, B\rangle$ be the group generated by A and B in $\mathrm{Diff}^1(M)$. Let Γ be a topological group and $\beta : G \times M \to \Gamma$ a continuous cocycle which satisfies the isolated closing conditions. Denote by $\langle A\rangle$ the group generated by A in $\mathrm{Diff}^1(M)$ and assume that $\beta|_{\langle A\rangle \times M}$ is null cohomological, i.e. there is $P : M \to \Gamma$ continuous such that $\beta(A, x) = P(Ax)P(x)^{-1}$ for $x \in M$. Then β is null cohomological, being trivialized by P.*

Proof. For any $g \in G$, define $\overline{\beta}(g, x) = P(gx)^{-1}\beta(g, x)P(x)$. Then $\overline{\beta}(A, x) = 1$ and we have to show that $\overline{\beta}(B, x) = 1$ for any $x \in M$.

Pick x_0 close to Bx such that $A^n x_0$ is close to x for some large n. Then, by the continuity of B, $BA^n x_0$ is close to x_0. Apply now Theorem 4.1 (Noncommutative Closing Lemma) and find x_* such that $BA^n x_* = x_*$, x_* is close to x_0 and $A^n x_*$ is close to $A^n x_0$. Hence $A^n x_*$ is also close to x_0. But

$$\overline{\beta}(B, A^n x_*) = \overline{\beta}(BA^n, x_*) \cdot \overline{\beta}(A^n, x_*)^{-1} = \overline{\beta}(BA^n, x_*) = 1_\Gamma,$$

because $BA^n x_* = x_*$. Using now the fact that $A^n x_*$ approaches x when $A^n x_0$ approaches x, and the continuity of $\overline{\beta}$, we have $\overline{\beta}(B, x) = 1_\Gamma$. $\square$

It is clear that for many Anosov actions of "big groups" we can apply Theorem 5.2 several times for generators and obtain Livsic type results. This motivates the following definition:

Definition 5.3. *We say that a smooth action $G \times M \to M$ is generically Anosov, (GA), if G contains a set of generators $\{A, B_1, \dots, B_n\}$ such that A is transitive Anosov and each B_k is in generic position with respect to A ($1 \leq k \leq n$).*

Note that Theorem 5.2 has a straightforward analogue for cocycles over (GA)-actions.

The following theorems extend to (GA)-actions the theorems of Livsic and our Livsic type theorem for cocycles with values in diffeomorphisms. The element A in G which appears in Theorems 5.4, 5.5, 5.6 and 5.9 is the Anosov element specified in the definition of a (GA)-action.

Theorem 5.4. *(a) Let $\rho : G \times M \to M$ be a C^1 (GA)-action. Let $\beta : G \times M \to \mathbb{R}$ be a continuous cocycle over ρ which satisfies the isolated closing condition. Assume that $\beta(A, \cdot)$ is Hölder. Then there exists a Hölder function $P : M \to \mathbb{R}$ such that $\beta(g, x) = P(gx) - P(x)$.*

(b) Let $K \geq 1$. If $\rho(B)$ and $\beta(B, \cdot)$ are C^K for some Anosov $\rho(B)$, then P is C^{K-1}. In particular, if $\rho(B)$ and $\beta(B, \cdot)$ are C^∞, then P is C^∞. The same holds in the class C^ω.

Proof. The first part follows from Livsic's Theorem 1.1, and Theorem 5.2. The regularity is a result in [L-M-M] (Theorem 2.1), respectively [Lla]. □

Theorem 5.5. *(a) Let $\rho : G \times M \to M$ be a C^1 (GA)-action. Let Γ be a good group (see §1) and $\alpha \in (0, 1]$. Then there is a neighborhood U of 1_Γ with the property that for any continuous cocycle $\beta : G \times M \to \Gamma$ such that:*

1. *$\beta(A, \cdot)$ is α-Hölder;*
2. *$\beta(A, M) \subset U$;*
3. *β satisfies the isolated closing conditions,*

there exists an α-Hölder function $P : M \to \Gamma$ with $\beta(g, x) = P(gx)P(x)^{-1}$.

(b) Let $K \geq 1$ and Γ be a matrix Lie group. If $\rho(B)$ and $\beta(B, \cdot)$ are C^K for some Anosov $\rho(B)$, then P is C^{K-1}. In particular, if $\rho(B)$ and $\beta(B, \cdot)$ are C^∞, then P is C^∞.

Proof. The first part follows from Livsic's Theorem 1.2, and Theorems 5.2. The regularity follows from [N-T]. □

Theorem 5.6. *(a) Let $\rho : G \times M \to M$ be a C^1 (GA)-action. Fix $K \in \mathbb{N}$, $K \geq 4$ and let $\alpha \in (0, 1]$, $\mathcal{G}^K$, d_K be as in Theorem 3.1. Then there is $\delta > 0$ with the property that for any continuous cocycle $\beta : G \times M \to \mathcal{G}^K$ such that:*

1. *$\beta(A, \cdot)$ is α-Hölder;*
2. *$d_K(\beta(A, x)^{\pm 1}, Id) < \delta$, for any $x \in M$;*
3. *β satisfies the isolated closing conditions,*

there is an α-Hölder map $P : M \to \mathcal{G}^{K-2}$ which trivializes β.

(b) If, in addition, $\rho(B)$ and $\beta(B, \cdot)(\cdot)$ are C^{K-2} (the latter seen as a function $M \times U \to U$, respectively $M \times N \to N$) for some Anosov $\rho(B)$, then $P(\cdot)(\cdot)$ is C^{K-3}.

Proof. The first part follows from Theorems 3.1 and 5.2. The regularity follows from [N-T]. □

We give now an application of Theorem 5.4, which is a natural generalization of the Livsic-Sinai theorem (see [L-S]).

Theorem 5.7. *Let $\rho : G \times M \to M$ be a C^2 (GA)-action. Then there is a G-invariant measure on M which is equivalent to the Lebesgue measure if and only* *if for any x in M and g in G with $gx = x$, one has $|J_g(x)| = 1$. Here J_g is the Jacobian of g.*

Proof. Choose a Riemannian metric on M. Let m be the measure induced by it. Using the Riemannian metric we can introduce the C^1-cocycle

$$(g, x) \in G \times M \to \beta(g, x) = \log |J_g(x)| \in \mathbb{R}.$$

Assume that μ is a measure on M which is equivalent to m, $d\mu = \phi dm$, where $0 < \phi < \infty$. Note that μ is g-invariant for some $g \in G$ if and only if

$$\phi(x)dm(x) = \phi(gx)|J_g(x)|dm(x), \text{ a.e. with respect to } m. \tag{5.1}$$

Let $A \in G$ be the Anosov element given by the (GA)-property.

If μ is G-invariant, it is in particular A-invariant, hence by [L-S], ϕ is continuous. Then (5.1) implies the desired conditions for β.

Conversely, if the conditions are satisfied, β admits a continuous trivialization ψ, by Theorem 5.4. Then $e^{\psi}dm$ is a G-invariant measure. □

Remark. Using a de la Llave, Marco, Moriyon result (see [L-M-M]) it follows that if ρ is C^∞, then so is the invariant measure.

Take now $M = \mathbb{T}^n$ and let the group G be a subgroup of $SL(n, \mathbb{Z})$. Then G acts on $\mathbb{T}^n$ by diffeomorphisms. Many actions $G \times \mathbb{T}^n \to \mathbb{T}^n$ are (GA)-actions. Here we prove only a typical example, namely for $G = SL(n, \mathbb{Z})$.

Theorem 5.8. *The standard linear action $SL(n, \mathbb{Z}) \times \mathbb{T}^n \to \mathbb{T}^n$ is (GA).*

Proof. We define first an Anosov element A in $SL(n, \mathbb{Z})$.

The following matrices are diagonalizable Anosov elements:

$$A_1 = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \\ -1 & -2 & 2 \end{pmatrix}.$$

If n is even, take $A = \operatorname{diag}(A_1, \dots, A_1)$, and if n is odd take $A = \operatorname{diag}(A_1, \dots, A_1, A_2)$. Then A is diagonalizable and there exists B, which has the same block-diagonal form as A, such that $A = B \cdot \operatorname{diag}(\lambda_1, \dots, \lambda_n) \cdot B^{-1}$. Let $E^+ \oplus E^-$ be the splitting of $\mathbb{R}^n$ into expanding and contracting parts for $\operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Denote by $P_\pm$ the projection onto $E^\pm$. It is known that $SL(n, \mathbb{Z})$ is generated by $\{E_{ij} | 1 \le i, j \le n, \ i \ne j\}$, E_{ij} being the matrix with 1 on the diagonal and in the position (i, j) and 0 everywhere else. Therefore it is enough to prove that $P_+ \, B^{-1} E_{ij} B|_{E^+}$ and $P_- \, B^{-1} E_{ij} B|_{E^-}$ are invertible for any $i \ne j$.

But

$$B^{-1} E_{ij} B = I + \sum_{k,\ell=1}^{n} \overline{b}_{ki} b_{j\ell} e_{k\ell} \ ,$$

where $B = (b_{ij})$, $B^{-1} = (\overline{b}_{ij})$ and $\{e_{k\ell}\}_{k,\ell}$ are the matrix units of $\operatorname{Mat}(n, \mathbb{R})$. Because B and B^{-1} are block-diagonal, in the last sum nonzero terms can appear only in the block to which the position (i, j) belongs (see the $n \times n$ matrices as made of blocks, according to the pattern of A). Hence, unless (i, j) lies in one of the diagonal blocks, the matrix $B^{-1} E_{ij} B$ is triangular, and therefore $P_\pm B^{-1} E_{ij} B|_{E^\pm}$ are invertible. The few remaining cases can be checked numerically. □

Putting together Theorems 5.6 and 5.8, we obtain:

Theorem 5.9. *(a) Let* $\pi : SL(n,\mathbb{Z}) \times \mathbb{T}^n \to \mathbb{T}^n$ *be the standard linear action,* $K \geq 4$ *an integer and let* $\beta : SL(n,\mathbb{Z}) \times \mathbb{T}^n \to \mathrm{Diff}^K(\mathbb{T}^d)$ *be a continuous cocycle over* π*, close to the identity, which satisfies the isolated closing conditions. Assume that* $\beta(A,\cdot)$ *is Hölder. Then there is a Hölder function* $P : \mathbb{T}^n \to \mathrm{Diff}^{K-2}(\mathbb{T}^d)$ *which trivializes* β*.*

(b) If, in addition, there is $B \in SL(n,\mathbb{Z})$ *hyperbolic such that* $\beta(B,\cdot)(\cdot)$ *is* C^{K-2} *as a function* $\mathbb{T}^n \times \mathbb{T}^d \to \mathbb{T}^d$ *, then* P *is* C^{K-3}*.* $\square$

§6. A Version of the Hirsch-Pugh-Shub Structural Stability Theorem

In this section we prove a version of the Hirsch-Pugh-Shub structural stability theorem (Theorem 7.1 in [H-P-S]) about the leaf-wise conjugation between a partially hyperbolic diffeomorphism with a C^1 neutral foliation and a perturbation of it. This result is necessary in order to obtain rigidity for ρ_0. As one will see in §7, we use the conjugation to reduce the problem to one about cocycles with values in the diffeomorphism groups. The fact that such a cocycle is Hölder in the d_K-metric can be easily obtained once it is known that the conjugation in the Hirsch-Pugh-Shub Theorem is Hölder, differentiable of certain degree along the neutral leaves and with all these derivatives Hölder. We were not able to derive the fact that the derivatives are Hölder from [H-P-S]. We restrict our attention to the case of a partially hyperbolic automorphism of the torus and prove the following theorem:

Theorem 6.1. *Let $A \in SL(n, \mathbb{Z})$, $n \geq 2$, be a hyperbolic linear automorphism of $\mathbb{T}^n$, let $d \geq 1$ be an integer and let f be the linear automorphism of $\mathbb{T}^n \times \mathbb{T}^d$ given by $f(x,y) = (Ax, y)$ for $x \in \mathbb{T}^n$ and $y \in \mathbb{T}^d$. Let $K \geq 1$ be an integer and let $\widetilde{f}$ be a $C^{K+\mathrm{Lip}}$ diffeomorphism of $\mathbb{T}^n \times \mathbb{T}^d$, which is $C^{K+\mathrm{Lip}}$ close to f. Then there are functions $\overline{h} : \mathbb{T}^n \times \mathbb{T}^d \to \mathbb{T}^n$ and $\overline{q} : \mathbb{T}^n \times \mathbb{T}^d \to \mathbb{T}^d$ such that:*

— $\overline{h}(\cdot,\cdot)$, $\overline{q}(\cdot,\cdot)$ are uniformly $C^{K+\mathrm{Lip}}$ in the second variable and both functions, together with their first K derivatives in the second variable, are Hölder;

— the function $H : \mathbb{T}^n \times \mathbb{T}^d \to \mathbb{T}^n \times \mathbb{T}^d$ given by $H(x,y) = (x + \overline{h}(x,y), y)$ is a homeomorphism;

— the function $Q : \mathbb{T}^n \times \mathbb{T}^d \to \mathbb{T}^n \times \mathbb{T}^d$ given by $Q(x,y) = (x, y + \overline{q}(x,y))$ is a homeomorphism;

— the following equation holds: $HQf = \widetilde{f}H$. *(6.1)*

Remark. H is a leaf-wise conjugacy between the neutral foliation of f and the neutral foliation of $\widetilde{f}$.

Preliminaries. Write the perturbation $\widetilde{f}$ as

$$\widetilde{f}(x,y) = (Ax + \overline{\alpha}(x,y), \;\; y + \overline{\beta}(x,y)), \qquad (x,y) \in \mathbb{T}^n \times \mathbb{T}^d,$$

where $\overline{\alpha} : \mathbb{T}^n \times \mathbb{T}^d \to \mathbb{T}^n$ and $\overline{\beta} : \mathbb{T}^n \times \mathbb{T}^d \to \mathbb{T}^d$ are $C^{K+\mathrm{Lip}}$ functions which are $C^{K+\mathrm{Lip}}$ close to the constant zero function. Then, decomposing the equation (6.1) in components along $\mathbb{T}^n$ and $\mathbb{T}^d$, one obtains the equivalent system

$$\begin{cases} \overline{h}(Ax, \overline{q}(Ax,y) + y) = A\overline{h}(x,y) + \overline{\alpha}(x + \overline{h}(x,y), y) \\ \overline{q}(Ax,y) = \overline{\beta}(x + \overline{h}(x,y), y), \qquad (x,y) \in \mathbb{T}^n \times \mathbb{T}^d. \end{cases} \tag{6.2}$$

Substituting $\overline{q}$ in the first equation of (6.2) yields an equation only in $\overline{h}$:

$$\overline{h}(Ax, \overline{\beta}(x + \overline{h}(x,y), y) + y) = A\overline{h}(x,y) + \overline{\alpha}(x + \overline{h}(x,y), y). \tag{6.3}$$

We lift $\overline{h}$ to a function $h : \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{R}^n$ which is $\mathbb{Z}^n$-periodic in the first variable (because $\overline{h}$ is homotopic to the constant function zero). Since $\overline{\alpha}$ is close to zero, one can lift it to a function $\alpha : \mathbb{T}^n \times \mathbb{T}^d \to \mathbb{R}^n$. Note that $d_{K+\mathrm{Lip}}(\alpha) = d_{K+\mathrm{Lip}}(\overline{\alpha})$. Let p be the canonical projection $p : \mathbb{R}^n \to \mathbb{T}^n$. Then we obtain the equation

$$\begin{gathered} Ah(x,y) - h(Ax, \overline{\beta}(p(x+h(x,y)),y)+y) = -\alpha(p(x+h(x,y)),y), \\ (x,y) \in \mathbb{R}^n \times \mathbb{T}^n \end{gathered} \tag{6.4}$$

We can see equation (6.4) as a fixed point equation for the operator $\mathcal{F} : \chi \mapsto h$ given by:

$$\begin{aligned} A\,h(x,y) - h(Ax, \overline{\beta}(p(x+\chi(x,y)),y)+y) &= \\ = -\alpha(p(x+\chi(x,y)),y). \end{aligned}$$

One can give an explicit formula for h in terms of χ (see (6.6) below).

We will show that, on a suitable space of functions, $\mathcal{F}$ is a contraction. Therefore equation (6.4) has a solution.

Denote by $C^0_{\mathrm{per}}(\mathbb{R}^n \times \mathbb{T}^d, X)$ the space of continuous functions $\varphi : \mathbb{R}^n \times \mathbb{T}^d \to X$ which are $\mathbb{Z}^n$-periodic in the first variable. By $C^{K+\mathrm{Lip}}_{\mathrm{per}}$ we mean $C^{K+\mathrm{Lip}} \cap C^0_{\mathrm{per}}$.

We have to study the operator $\mathcal{F} : \chi \mapsto h$, which is determined by the functional equation

$$Ah - h \circ F_\chi = g_\chi, \tag{6.5}$$

where:

- A is a hyperbolic matrix in $SL(n,\mathbb{Z})$;
- h, $\chi \in C^0_{\mathrm{per}}\ (\mathbb{R}^n \times \mathbb{T}^d, \mathbb{R}^n)$ and are $C^{K+\mathrm{Lip}}$ in the second variable;
- $\alpha \in C^{K+\mathrm{Lip}}\ (\mathbb{T}^n \times \mathbb{T}^d, \mathbb{R}^n)$;
- $\overline{\beta} \in C^{K+\mathrm{Lip}}\ (\mathbb{T}^n \times \mathbb{T}^d, \mathbb{T}^d)$;
- $F_\chi(x,y) = (Ax, \varphi_x(y)) \quad (x \in \mathbb{R}^n \text{ and } y \in \mathbb{T}^d)$;
- $\varphi_x(y) = y + \overline{\beta} \circ S_\chi(x,y) \quad (\text{hence } \varphi_x \in C^{K+\mathrm{Lip}}(\mathbb{T}^d))$;
- $S_\chi(x,y) = (p(x+\chi(x,y)),y)$;
- $g_\chi = -\alpha \circ S_\chi$.

These notations will be valid throughout the rest of this paragraph.

Recall that M'_K and M_K where introduced in Proposition 2.9.

For $t > 0$, define the function

$$\Delta(t) = (1+t)^K[1+(1+K^2)t].$$

Notation. Denote $\delta_0 = \max\{d_{K+\mathrm{Lip}}(\overline{\alpha}),\, d_{K+\mathrm{Lip}}(\overline{\beta})\}$,

$$C_{\mathrm{Lip},\omega} = \sup_{v \in \mathbb{T}^d} \frac{\|v\|}{\|v\|^\omega}.$$

Lemma 6.2. *Let $r > 0$ and $\delta_0 > 0$ be constants. Let χ and $\overline{\beta}$ be such that:*

$$d_{K+\mathrm{Lip}}(\chi(x,\cdot)) \le r\ ,$$
$$d_{K+\mathrm{Lip}}(\overline{\beta}(x,\cdot)) \le \delta_0$$

for all $x \in \mathbb{R}^n$. Denote $\delta_1 = \delta_0\Delta(r), \delta_2 = M'_K\delta_1 > \delta_1$.

Then

$$d_{K+\mathrm{Lip}}(\varphi_x, Id) \le \delta_1,$$

for all $x \in \mathbb{R}^n$. Hence, by Proposition 2.9, if $\delta_1 < \frac{1}{2}$, then $\varphi_x \in \mathrm{Diff}^{K+\mathrm{Lip}}(\mathbb{T}^d)$ and

$$d_{K+\mathrm{Lip}}(\varphi_x^{-1}, Id) \le \delta_2.$$

Proof. We derive first some estimates for $S_\chi(x,y) = (p(x+\chi(x,y)), y)$.

$$\begin{aligned} d''_{K+\mathrm{Lip}}(S_\chi(x,\cdot)) &\le d''_{K+\mathrm{Lip}}(x+\chi(x,\cdot)) + d''_{K+\mathrm{Lip}}(Id) = d''_{K+\mathrm{Lip}}(\chi(x,\cdot)) \le r, \\ d'_{K+\mathrm{Lip}}(S_\chi(x,\cdot)) &\le d'_{K+\mathrm{Lip}}(x+\chi(x,\cdot)) + d'_{K+\mathrm{Lip}}(Id) = \\ &= d'_{K+\mathrm{Lip}}(\chi(x,\cdot)) + 1 \le r+1. \end{aligned}$$

Then

$$\begin{aligned} d_{K+\mathrm{Lip}}(\varphi_x, Id) = d_{K+\mathrm{Lip}}(\overline{\beta}\circ S_\chi(x,\cdot)) &\overset{(\mathrm{Lip2})}{\le} \\ \le d_{K+\mathrm{Lip}}(\beta)[1\vee\ d'_{K+\mathrm{Lip}}(S_\chi(x,\cdot))]^K &\cdot \\ \cdot[1\vee(d'_{K+\mathrm{Lip}}(S_\chi(x,\cdot)) + K^2\cdot d''_{K+\mathrm{Lip}}(S_\chi(x,\cdot)))] &\le \\ \le \delta_0(1+r)^K\cdot[1+r+K^2 r] = \delta_1. \quad \square \end{aligned}$$

Remark. In what follows, we always choose χ and β such that φ_x is invertible for all $x \in \mathbb{R}^m$. Then F_χ is invertible and

$$F_\chi^{-1}(x,y) = (A^{-1}x,\ \varphi^{-1}_{A^{-1}x}(y)).$$

We decompose $\mathbb{R}^n = E^+ \oplus E^-$ in expansive and contractive subspaces for A. With respect to this decomposition, A can be written as

$$A = \begin{pmatrix} A_+ & 0 \\ 0 & A_- \end{pmatrix}$$

where $\|(A_+)^{-1}\| < 1$, $\|A_-\| < 1$. Denote $\lambda = \max\{\|(A_+)^{-1}\|,\ \|A_-\|\}$.

If u is a function with values in $\mathbb{R}^n$, denote by either u_+, u_- or u^+, u^- the components of u with respect to the decomposition $\mathbb{R}^n = E^+ \oplus E^-$.

We introduce $\omega \in (0,1)$ and $0 < \lambda_1 < 1$ such that

$$\begin{aligned} \|A\|^\omega \|(A_+)^{-1}\| &< \lambda_1, \\ \|A^{-1}\|^\omega \|A_-\| &< \lambda_1. \end{aligned}$$

The equation (6.5) can be solved explicitly in C^0 as follows: $h = \mathcal{F}(\chi)$ has the components

$$\begin{cases} h_+ = \displaystyle\sum_{k=0}^{\infty} A_+^{-k-1}\cdot g_\chi^+ \circ F_\chi^k, \\ h_- = -\displaystyle\sum_{k=0}^{\infty} A_-^k g_\chi^- \circ F_\chi^{-k-1}. \end{cases} \tag{6.6}$$

Since $F_\chi \in C^0_{\rm per}(\mathbb{R}^n \times \mathbb{T}^d, \mathbb{R}^n \times \mathbb{T}^d)$ and g_χ are bounded, the series converge uniformly. Therefore $\mathcal{F} : C^0_{\rm per}(\mathbb{R}^n \times \mathbb{T}^d, \mathbb{R}^n) \to C^0_{\rm per}(\mathbb{R}^n \times \mathbb{T}^d, \mathbb{R}^n)$ is well defined. In order to obtain the desired result, we consider the restriction of $\mathcal{F}$ to a subspace, where it is a contraction.

For $C_0 > 0$ and $r > 0$ denote:

$$\begin{gathered}
B_{K+\mathrm{Lip}}(r) = \{\chi \in C^0_{\rm per}(\mathbb{R}^n \times \mathbb{T}^d, \mathbb{R}^n) \mid \chi(x, \cdot) \in C^{K+\mathrm{Lip}}(\mathbb{T}^d, \mathbb{R}^n) \text{ and} \\
d_{K+\mathrm{Lip}}(\chi(x,\cdot)) \le r, \quad \text{for all } x \in \mathbb{R}^n\}, \\
B_\omega(C_0) = \{\chi \in C^0_{\rm per}(\mathbb{R}^n \times \mathbb{T}^d, \mathbb{R}^n) \mid \chi(x, \cdot) \in C^K(\mathbb{T}^d, \mathbb{R}^n) \text{ for all } x \in \mathbb{R}^n, \\
\text{and } d_K(\chi(x',\cdot), \chi(x'',\cdot)) \le C_0 \|x' - x''\|^\omega, \quad \text{for all } x', x'' \in \mathbb{R}^n\}, \\
X_{r,C_0} = B_{K+\mathrm{Lip}}(r) \cap B_\omega(C_0).
\end{gathered}$$

On X_{r,C_0} we introduce the metric

$$d^Y_K(\chi, \chi') := \sup_{x \in \mathbb{R}^n} d_K(\chi(x,\cdot),\ \chi'(x,\cdot)) \qquad (\le 2r).$$

We remark that X_{r,C_0} is complete with respect to this metric.

Theorem 6.1 follows from Propositions 6.3, 6.4 and 6.5 below, using the contraction principle.

Notation. For $r > 0$ and $\delta_0 > 0$ introduced above, let

$$\begin{aligned}
\delta_1 &:= \delta_0 \Delta(r), \\
\delta_2 &:= M'_K \delta_1 > \delta_1.
\end{aligned}$$

Proposition 6.3. *If $r > 0$ and $\delta_0 > 0$ are such that $\delta_1 < \frac{1}{2}$ and $\lambda\Delta(\delta_2) < 1$, then*

$$\mathcal{F}(B_{K+\mathrm{Lip}}(r)) \subset B_{K+\mathrm{Lip}}\left(\delta_0 \frac{\sqrt{2}\Delta(r)\Delta(\delta_2)}{1 - \lambda\Delta(\delta_2)}\right).$$

Proposition 6.4. *If $r > 0$ and $\delta_0 > 0$ are such that $\delta_1 < \frac{1}{2}$ and $\lambda_1\Delta(\delta_2) < 1$, then*

$$\mathcal{F}(X_{r,C_0}) \subset B_\omega\left(\delta_0 \frac{\sqrt{2}\|A_-\|^{-1} M_K C_1 \Delta(r)(1 + 4\delta_2)^K (K+1)}{(1 - \lambda_1\Delta(\delta_2))^2}\right),$$

where $C_1 = (C_{\mathrm{Lip},\omega} + C_0)(K+1)(1+3r)^K$.

Proposition 6.5. *Assume that $r > 0$ and $\delta_0 > 0$ are such that $\delta_1 < \frac{1}{2}$ and $\lambda\Delta(\delta_2) < 1$. Let $\chi, \overline{\chi} \in B_{K+\mathrm{Lip}}(r)$, and denote $h = \mathcal{F}(\chi)$, $\overline{h} = \mathcal{F}(\overline{\chi})$. According to Proposition 6.3, $h, \overline{h} \in B_{K+\mathrm{Lip}}(r_1)$ for $r_1 = \delta_0 \frac{\sqrt{2}\Delta(r)\Delta(\delta_2)}{1-\lambda\Delta(\delta_2)}$.*

If $\lambda(1+\delta_2)^K < 1$, then

$$d^Y_K(h, \overline{h}) \le \delta_0 \frac{C_K(\delta_0, r, r_1)}{1 - \lambda(1+\delta_2)^K} \cdot d^Y_K(\chi, \overline{\chi}),$$

where $C_K(\delta_0, r, r_1)$ is a polynomial in δ_0, r, r_1, which has positive coefficients depending on $\|A_-\|$, $\|A_+^{-1}\|$, M_K and M'_K. (The precise expression of $C_K(\delta_0, r, r_1)$ can be determined from Claims 2 and 3 in the proof of Proposition 6.5).

Proof of Theorem 6.1.

Let $\epsilon_{XP} > 0$ be the expansiveness constant of the matrix A (i.e., for any $x', x'' \in \mathbb{T}^n, x' \neq x''$, there is an integer N such that $\mathbf{d}_{\mathbb{T}^n}(A^N x', A^N x'') > \epsilon_{XP}$).

Choose $r > 0$, $C_0 > 0$ and $\delta_0 > 0$ such that:

$$\begin{gathered} r < \frac{\epsilon_{XP}}{2}, \\ \delta_1 < \frac{1}{2}, \\ \lambda_1 \Delta(\delta_2) < 1, \\ \lambda(1+\delta_2)^K < 1, \\ \delta_0 \frac{\sqrt{2}\Delta(r)\Delta(\delta_2)}{1-\lambda\Delta(\delta_2)} < r, \\ \delta_0 \frac{\sqrt{2}\|A_-\|^{-1} M_K C_1 \Delta(r)(1+4\delta_2)^K (K+1)}{(1-\lambda_1\Delta(\delta_2))^2} < C_0, \\ \delta_0 \frac{C_K(\delta_0, r, r_1)}{1-\lambda(1+\delta_2)^K} < 1. \end{gathered}$$

(Recall that r_1, C_1 and C_K were introduced in Propositions 6.3–6.5, $\delta_1 = \delta_0 \Delta(r)$ and $\delta_2 = M'_K \delta_1$.)

Then, by Propositions 6.3 and 6.4, $X_{r,C_0} = B_{K+\mathrm{Lip}}(r) \cap B_\omega(C_0)$ is $\mathcal{F}$-invariant, and, by Proposition 6.5, $\mathcal{F}$ is a contraction on X_{r,C_0}, with respect to the metric d_K^Y. Therefore, by the contraction principle, $\mathcal{F}$ has a unique fixed point h in X_{r,C_0}.

This h, being in $C^0_{\mathrm{per}}(\mathbb{R}^n \times \mathbb{T}^d, \mathbb{R}^n)$, gives the desired function $\overline{h} \in C^0(\mathbb{T}^n \times \mathbb{T}^d, \mathbb{T}^n)$. Then we can construct $\overline{q}$ using the second formula in (6.2):

$$\overline{q}(x, y) = \overline{\beta}(A^{-1}x + \overline{h}(A^{-1}x, y), y).$$

We obtain a solution for the system (6.2), hence a solution of (6.1).

By a similar computation to that done in Claim 1 of Proposition 6.3, we obtain that $\overline{q} \in B_{K+\mathrm{Lip}}(\delta_0 \Delta(r))$. Adapting the computations of Claim 2 of Proposition 6.4, we obtain that $\overline{q} \in B_\omega(\delta_0 C_1 \|A\|^{-\omega})$.

It remains to show that H and Q are homeomorphisms. Clearly H and Q are continuous and close to the identity. Then surjectivity follows using the topological degree, and it remains to prove that they are one-to-one. We first prove that Q is one-to-one. Consider two points $(x', y'), (x'', y'') \in \mathbb{T}^n \times \mathbb{T}^d$ such that $Q(x', y') = Q(x'', y'')$. Then $x' = x''$ and $y' + \overline{q}(x', y') = y'' + \overline{q}(x', y'')$. Since the function $y \mapsto y + \overline{q}(x', y)$ is $C^{K+\mathrm{Lip}}$ and $C^{K+\mathrm{Lip}}$-close to the identity, it follows from Proposition 2.9 (a) that $y' = y''$. Hence Q is a homeomorphism.

Consider now $(x', y'), (x'', y'') \in \mathbb{T}^n \times \mathbb{T}^d$ such that $H(x', y') = H(x'', y'')$. Then clearly $y' = y''$. Assume by contradiction that $x' \neq x''$. Then there exists an integer

N such that $\mathbf{d}_{\mathbb{T}^n}(A^N x', A^N x'') > \epsilon_{XP}$. If $N \geq 0$, one has:

$$H(Qf)^N(x', y') \overset{(6.1)}{=} \widetilde{f}^N H(x', y') = \widetilde{f}^N H(x'', y'') \overset{(6.1)}{=} H(Qf)^N(x'', y'').$$

Hence, introducing $y_1, y_2 \in \mathbb{T}^d$ such that:

$$(Qf)^N(x', y') = (A^N x', y_1),$$
$$(Qf)^N(x'', y'') = (A^N x'', y_2)$$

we obtain

$$A^N x' + \overline{h}(a^N x', y_1) = A^N x'' + \overline{h}(a^N x'', y_2)$$

and therefore

$$\epsilon_{XP} < \mathbf{d}_{\mathbb{T}^n}(A^N x', A^N x'') \leq 2d_0(\overline{h}) < 2r,$$

which contradicts our initial choice of r.

If $N < 0$, then

$$H(f^{-1}Q^{-1})^{-N}(x', y') \overset{(6.1)}{=} \widetilde{f}^{-N} H(x', y') =$$
$$= \widetilde{f}^{-N} H(x'', y'') \overset{(6.1)}{=} H(f^{-1}Q^{-1})^{-N}(x'', y''),$$

and now proceed as in the previous case. $\square$

Notation. Denote, for $s \in \mathbb{Z}$, $F^s_\chi(x, y) = (A^s x, \phi^s_x(y))$.

Hence

$$\phi^s_x = \begin{cases} \varphi_{A^{s-1}x} \quad \varphi_{A^{s-2}x} \quad \cdots \quad \varphi_x & \text{if } s > 0 \\ Id_{\mathbb{T}^d} & \text{if } s = 0 \\ \varphi^{-1}_{A^s x} \quad \varphi^{-1}_{A^{s+1}x} \quad \cdots \quad \varphi^{-1}_{A^{-1}x} & \text{if } s < 0 \end{cases}$$

and the following recurrence relation holds:

$$\phi^s_x = \begin{cases} \phi^{s-1}_{Ax} \ \varphi_x & \text{if } s > 0 \\ \phi^{s+1}_{A^{-1}x} \ \varphi^{-1}_{A^{-1}x} & \text{if } s < 0. \end{cases}$$

Proof of Proposition 6.3.

Claim 1. $d_{K+\mathrm{Lip}}(g_\chi(x, \cdot)) \leq \delta_0 \Delta(r)$, for all $x \in \mathbb{R}^n$.

Proof. Indeed,

$$d_{K+\mathrm{Lip}}(g_\chi(x, \cdot)) = d_{K+\mathrm{Lip}}(\alpha \circ S_\chi(x, \cdot)) \leq$$
$$\overset{(\mathrm{Lip2})}{\leq} d_{K+\mathrm{Lip}}(\alpha)[1 \vee d'_{K+\mathrm{Lip}}(S_\chi(x, \cdot)]^K \cdot$$
$$\cdot [1 \vee (d'_{K+\mathrm{Lip}}(S_\chi(x, \cdot)) + K^2 d''_{K+\mathrm{Lip}}(S_\chi(x, \cdot)))] \leq$$
$$\leq \delta_0 \cdot (1 + r)^K [1 \vee (1 + r + K^2 r)] = \delta_0 \Delta(r),$$

where in the last inequality we use estimates from Lemma 6.2.

Claim 2. $d_{K+\mathrm{Lip}}(g_\chi \circ F_\chi^s(x,\cdot)) \le \delta_0 \Delta(r)\Delta(\delta_*)^{|s|}$, for any $x \in \mathbb{R}^n$ and any $s \in \mathbb{Z}$, where

$$\delta_* = \begin{cases} \delta_1, & \text{if } s > 0 \\ \delta_2 = M'_K \delta_1, & \text{if } s < 0. \end{cases}$$

Proof. Indeed, if $s > 0$, then

$$\begin{aligned} d_{K+\mathrm{Lip}}(g_\chi \circ F_\chi^s(x,\cdot)) &= d_{K+\mathrm{Lip}}(g_\chi(A^s x, \phi_x^s(\cdot))) = \\ &= d_{K+\mathrm{Lip}}(g_\chi(A^s x, \phi_{Ax}^{s-1}\phi_x(\cdot))) \le \\ &\overset{\text{(Lip3)}}{\le} d_{K+\mathrm{Lip}}(g_\chi(A^s x, \phi_{Ax}^{s-1}(\cdot)) \cdot [1 + d_{K+\mathrm{Lip}}(\varphi_x, Id)]^K \cdot \\ &\cdot [1 + (K^2+1) d_{K+\mathrm{Lip}}(\varphi_x, Id)] \le \\ &\le d_{K+\mathrm{Lip}}(g_\chi(A^s x, \phi_{Ax}^{s-1}(\cdot))) \cdot (1+\delta_1)^K [1 + (K^2+1)\delta_1] = \\ &= \Delta(\delta_1) d_{K+\mathrm{Lip}}(g_\chi \circ F_\chi^{s-1}(Ax,\cdot)). \end{aligned}$$

Iterate and then use Claim 1 to obtain the desired estimate.

The case $s < 0$ is dealt with similarly.

Claim 3. If $h = \mathcal{F}(\chi)$, then:

$$d^Y_{K+\mathrm{Lip}}(h_+) \le \delta_0 \cdot \frac{\Delta(r)\cdot \|A_+^{-1}\|}{1 - \|A_+^{-1}\|\Delta(\delta_1)},$$
$$d^Y_{K+\mathrm{Lip}}(h_-) \le \delta_0 \cdot \frac{\Delta(r)\Delta(\delta_2)}{1 - \|A_-\|\Delta(\delta_2)}.$$

Proof. Use formula (6.6), and the previous two claims. $\square$

Proof of Proposition 6.4.

Since $\lambda < \lambda_1$, the hypothesis implies that $\lambda\Delta(\delta_2) < 1$, hence the conclusion of Proposition 6.3 holds.

Consider $\chi \in X_{r,C_0}$.

Claim 1. For any $x', x'' \in \mathbb{R}^n$, one has

$$d_K(\varphi_{x'}, \varphi_{x''}) \le \delta_0 C_1 \cdot \|x' - x''\|^\omega,$$

where $C_1 = (C_{\mathrm{Lip},\omega} + C_0)(K+1)(1+3r)^K$ and (conform Proposition 2.9 (c), (ii))

$$d_K(\varphi_{x'}^{-1}, \varphi_{x''}^{-1}) \le \delta_0 C_1 \cdot M_K \cdot \|x' - x''\|^\omega.$$

Proof. Using the periodicity of $x \mapsto \varphi_x$, one has:

$$\begin{aligned} d_K(\varphi_{x'}, \varphi_{x''}) &= d_K(\beta \circ S_\chi(x'), \beta \circ S_\chi(x'')) \le \\ &\overset{\text{(L1)}}{\le} d_K(S_\chi(x',\cdot), S_\chi(x'',\cdot)) \cdot (K+1) \cdot d'_{K+\mathrm{Lip}}(\beta) \cdot \\ &\cdot \{1 \vee [d'_K(S_\chi(x',\cdot)) \vee d'_K(S_\chi(x'',\cdot)) + d''_K(S_\chi(x',\cdot)) + d''_K(S_\chi(x'',\cdot))]\}^K \le \\ &\le \|p(x'-x'')\| + d_K(p\chi(x',\cdot)\ ,\ p\chi(x'',\cdot)) \cdot \delta_0 (K+1)(1+3r)^K \le \\ &\le \|x'-x''\|^\omega \cdot (C_{\mathrm{Lip},\omega} + C_0)\delta_0(K+1)(1+3r)^K. \end{aligned} \tag{6.7}$$

Claim 2. For any x', $x'' \in \mathbb{R}^n$, one has

$$d_K(g_\chi(x',\cdot), g_\chi(x'',\cdot)) \le \delta_0 C_1 \|x' - x''\|^\omega.$$

Proof. $d_K(g_\chi(x',\cdot), g_\chi(x'',\cdot)) = d_K(\alpha \circ S_\chi(x')\ ,\ \alpha \circ S_\chi(x''))$, which is exactly (6.7) in the proof of Claim 1, written with α instead of β. We can use the same estimates.

Claim 3. For $s > 0$ and $x', x'' \in \mathbb{R}^n$, one has

$$\begin{gathered} d_K(g_\chi \circ F_\chi^s(x',\cdot), g_\chi \circ F_\chi^s(x'',\cdot)) \le \\ \le \delta_0 C_1 \Delta(r)(1+4\delta_1)^K (K+1)(s+1)\|A\|^{\omega s} \cdot \Delta(\delta_1)^{s-1}\|x' - x''\|^\omega. \end{gathered}$$

Proof.

$$\begin{gathered} d_K(g_\chi F_\chi^s(x',\cdot), g_\chi F_\chi^s(x'',\cdot)) = \\ = d_K(g_\chi(A^s x', \phi^s_{x'}(\cdot)), g_\chi(A^s x'', \phi^s_{x''}(\cdot))) \le \\ \le d_K(g_\chi(A^s x', \phi^{s-1}_{Ax'} \circ \varphi_{x'}(\cdot)), g_\chi(A^s x'', \phi^{s-1}_{Ax''} \circ \varphi_{x''}(\cdot))) \le \\ \le d_K(g_\chi(A^s x', \phi^{s-1}_{Ax'} \circ \varphi_{x'}(\cdot)), g_\chi(A^s x', \phi^{s-1}_{Ax'} \circ \varphi_{x''}(\cdot))) + \\ + d_K(g_\chi(A^s x', \phi^{s-1}_{Ax'} \circ \varphi_{x''}(\cdot)), g_\chi(A^s x'', \phi^{s-1}_{Ax''} \circ \varphi_{x''}(\cdot))) \le \\ \overset{\text{(L3),(R3)}}{\le} d_K(\varphi_{x'}, \varphi_{x''}) d'_{K+\mathrm{Lip}}(g_\chi(A^s x', \phi^{s-1}_{Ax'}(\cdot))) \cdot \\ \cdot (K+1)[1 + 2d'_K(\varphi_{x'}, Id) + 2d'_K(\varphi_{x''}, Id)]^K + \\ + d_K(g_\chi(A^s x', \phi^{s-1}_{Ax'}(\cdot)), g_\chi(A^s x'', \phi^{s-1}_{Ax''}(\cdot)))[1 + d_K(\varphi_{x''}, Id)]^K \le \\ \overset{(*)}{\le} \delta_0 C_1 \|x' - x''\|^\omega \cdot \delta_0 \cdot \Delta(r)\Delta(\delta_1)^{s-1}(1+4\delta_1)^K(K+1) + \\ + d_K(g_\chi F_\chi^{s-1}(Ax',\cdot), g_\chi F_\chi^{s-1}(Ax'',\cdot)) \cdot (1+\delta_1)^K. \end{gathered}$$

For the inequality $(*)$, note that $g_\chi(A^s x', \phi^{s-1}_{Ax'}(\cdot)) = g_\chi \circ F_\chi^{s-1}(Ax',\cdot)$, and use also Claim 2 from the proof of Proposition 6.3.

We iterate now the recurrence relation obtained above and obtain

$$\begin{gathered} d_K(g_\chi F_\chi^s(x',\cdot), g_\chi F_\chi^s(x'',\cdot)) \le \\ \le \sum_{\ell=0}^{s-1} \delta_0 C_1 \|A^\ell x' - A^\ell x''\|^\omega \cdot \delta_0 \Delta(r) \cdot \Delta(\delta_1)^{s-\ell-1} \cdot (K+1)(1+\delta_1)^{K\ell}(1+4\delta_1)^K + \\ + \delta_K(g_\chi(A^s x',\cdot), g_\chi(A^s x'',\cdot))(1+\delta_1)^{Ks} \le \\ \overset{\text{Claim 2}}{\le} \sum_{\ell=0}^{s-1} \|A^\ell x' - A^\ell x''\|^\omega \cdot \delta_0^2 \cdot C_1 \cdot \Delta(r)(K+1) \cdot (1+4\delta_1)^K \cdot \\ \cdot \{(1+\delta_1)^K[1 + (1+K^2)\delta_1]\}^{s-\ell-1}(1+\delta_1)^{K\ell} + \\ + \|A^s x' - A^s x''\|^\omega \cdot \delta_0 \cdot C_1 (1+\delta_1)^{Ks} \le \\ \overset{\Delta(r)>1, \delta_0 \le 1}{\le} \left[\sum_{\ell=0}^{s} \|A^\ell x' - A^\ell x''\|^\omega\right] \delta_0 C_1 \Delta(r)(1+4\delta_1)^K(K+1) \cdot \end{gathered}$$

$$\cdot(1+\delta_1)^{K(s-1)}[1+(1+K^2)\delta_1]^{s-1}\|x'-x''\|^\omega \leq$$

$$\overset{\|A\|\geq 1}{\leq} \delta_0 C_1 \Delta(r)(1+4\delta_1)^K(K+1)(s+1)\|A\|^{\omega s}\cdot\{(1+\delta_1)^K[1+(1+K^2)\delta_1]\}^{s-1}\cdot$$
$$\cdot\|x'-x''\|^\omega =$$
$$= \delta_0 C_1 \Delta(r)(1+4\delta_1)^K(K+1)(s+1)\|A\|^{\omega s}\cdot\Delta(\delta_1)^{s-1}\|x'-x''\|^\omega.$$

Claim 4. For $s<0$ and $x',x''\in\mathbb{R}^n$, one has

$$d_K(g_\chi F^s_\chi(x',\cdot)), g_\chi F^s_\chi(x'',\cdot)) \leq$$
$$\leq \delta_0\cdot M_K\cdot C_1\cdot\Delta(r)(1+4\delta_2)^K(K+1)(|s|+1)\|A^{-1}\|^{\omega|s|}\cdot\Delta(\delta_2)^{|s|}\cdot\|x'-x''\|^\omega.$$

Proof. As in Claim 3, one has the recurrence

$$d_K(g_\chi F^s_\chi(x',\cdot), g_\chi F^s_\chi(x'',\cdot)) \leq$$
$$\leq \delta_0 C_1 M_K\|A^{-1}x'-A^{-1}x''\|^\omega\delta_0\Delta(r)\Delta(\delta_2)^{-s-1}(K+1)(1+4\delta_2)^K+$$
$$+d_K(g_\chi F^{s+1}_\chi(A^{-1}x',\cdot), g_\chi F^{s+1}_\chi(A^{-1}x'',\cdot))\cdot(1+\delta_2)^K.$$

Hence

$$d_K(g_\chi F^s_\chi(x',\cdot), g_\chi F^s_\chi(x'',\cdot)) \leq$$
$$\leq \sum_{\ell=1}^{-s}\delta_0 C_1 M_K\|A^{-\ell}x'-A^{-\ell}x''\|\delta_0\Delta(r)\Delta(\delta_2)^{-s-\ell}(1+\delta_2)^{K\ell}\cdot$$
$$\cdot(K+1)(1+4\delta_2)^K + d_K(g_\chi(A^{-s}x',\cdot), g_\chi(A^{-s}x'',\cdot))(1+\delta_2)^{K(-s)} \leq$$
$$\leq \sum_{\ell=1}^{-s}\|A^{-\ell}x'-A^{-\ell}x''\|^\omega\cdot\delta_0^2\cdot C_1\cdot M_K(K+1)(1+4\delta_2)^K\cdot(1+\delta_2)^{K(-s)}\cdot$$
$$\cdot[1+(1+K^2)\delta_2]^{-s-\ell} + \|A^{-s}x'-A^{-s}x''\|^\omega\cdot\delta_0\cdot C_1\cdot(1+\delta_2)^{K(-s)} \leq$$
$$\leq \delta_0 M_K C_1\Delta(r)(1+4\delta_2)^K(K+1)(|s|+1)\|A^{-1}\|^{\omega(-s)}\Delta(\delta_2)^{-s}\|x'-x''\|^\omega.$$

Claim 5. Let $h=\mathcal{F}(\chi)$. Then, for any $x',x''\in\mathbb{R}^n$:

$$d_K(h_+(x',\cdot), h_+(x'',\cdot)) \leq$$
$$\leq \delta_0\cdot\frac{\|A_+^{-1}\|C_1\Delta(r)(1+4\delta_1)^K(K+1)}{(1-\|A_+^{-1}\|\cdot\|A\|^\omega\cdot\Delta(\delta_1))^2}\cdot\|x'-x''\|^\omega,$$
$$d_K(h_-(x',\cdot), h_-(x'',\cdot)) \leq$$
$$\leq \delta_0\cdot\frac{\|A_-\|^{-1}\cdot M_K\cdot C_1\cdot\Delta(r)(1+4\delta_2)^K(K+1)}{(1-\|A_-\|\,\|A^{-1}\|^\omega\Delta(\delta_2))^2}\cdot\|x'-x''\|^\omega.$$

Proof. We use relation (6.6):

$$d_K(h_+(x',\cdot),h_+(x'',\cdot)) \le \sum_{k=0}^{\infty} \|A_+^{-k-1}\| d_K(g_\chi F_\chi^K(x',\cdot), g_\chi F_\chi^K(x'',\cdot)) \overset{\text{Claim 2, Claim 3}}{\le}$$

$$\le \left(\sum_{k=0}^{\infty} \|A_+^{-}\|^{k+1} \delta_0 C_1 \Delta(r)(1+4\delta_1)^K (K+1)(k+1)\|A\|^{\omega k}\Delta(\delta_1)^k\right) \|x'-x''\|^{\omega} \le$$

$$\le \delta_0 \|A_+^{-1}\| C_1 \Delta(r)(1+4\delta_1)^K (K+1)\|x'-x''\|^{\omega} \sum_{k=0}^{\infty}(k+1)(\|A_+^{-1}\|\|A\|^{\omega}\Delta(\delta_1))^k =$$

$$= \delta_0 \cdot \frac{\|A_+^{-1}\| \cdot C_1\Delta(r)(1+4\delta_1)^K(K+1)}{(1-\|A_+^{-1}\|\cdot\|A\|^{\omega}\Delta(\delta_1))^2} \cdot \|x'-x''\|,$$

and

$$d_K(h_-(x',\cdot),\ h_-(x'',\cdot)) \le \sum_{k=0}^{\infty}\|A_-^k\| d_K(g_\chi F_\chi^{-k-1}(x',\cdot), g_\chi F_\chi^{-k-1}(x'',\cdot)) \le$$

$$\overset{\text{Claim 4}}{\le} \left(\sum_{k=0}^{\infty}\|A_-\|^k \cdot \delta_0 \cdot M_K \cdot C_1\Delta(r)\cdot(1+4\delta_2)^K(K+1)(k+2)\cdot \right.$$

$$\left. \cdot\|A^{-1}\|^{\omega(k+1)}\cdot\Delta(\delta_2)^{k+1}\right)\|x'-x''\|^{\omega} =$$

$$= \delta_0\|A_-\|^{-1}\cdot M_K\cdot C_1\cdot\Delta(r)(1+4\delta_2)^K(K+1)\cdot\|x'-x''\|^{\omega}\cdot$$

$$\cdot\sum_{k=0}^{\infty}(k+2)(\|A_-\|\cdot\|A^{-1}\|^{\omega}\cdot\Delta(\delta_2))^{k+1} <$$

$$< \delta_0\cdot\frac{\|A_-\|^{-1}M_K C_1\Delta(r)(1+4\delta_2)^K(K+1)}{(1-\|A_-\|\cdot\|A^{-1}\|^{\omega}\Delta(\delta_2))^2}\|x'-x''\|^{\omega}. \quad \square$$

Proof of Proposition 6.5.
Denote the "φ_x" corresponding to $\overline{\chi}$ by $\overline{\varphi}_x$.
Claim 1. One has:

$$d_{K+\mathrm{Lip}}(\varphi_x, Id) \le \delta_1,$$
$$d_{K+\mathrm{Lip}}(\overline{\varphi}_x, Id) \le \delta_1,$$
$$d_K(\varphi_x,\overline{\varphi}_x) \le \delta_0(K+1)(1+4r)^K\ d_K(\chi(x,\cdot),\ \overline{\chi}(x,\cdot)),$$

therefore, applying Proposition 2.9, it follows that

$$d_K(\varphi_x^{-1}, Id) \le M_K'\delta_1,\ \text{and similarly for } \overline{\varphi}_x^{\,-1},$$
$$d_K(\varphi_x^{-1},\overline{\varphi}_x^{\,-1}) \le M_K\delta_0(K+1)(1+4r)^K d_K(\chi(x,\cdot),\overline{\chi}(x,\cdot)).$$

Proof. The first two inequalities follow from Lemma 6.2. We prove the third inequality

$$\begin{aligned} d_K(\varphi_x,\overline{\varphi}_x) = d_K(\beta\circ S_\chi(x,\cdot)\ ,\ \beta\circ S_{\overline{\chi}}(x,\cdot)) \overset{(L3)}{\le} \\ \le d_K(S_\chi(x,\cdot),S_{\overline{\chi}}(x,\cdot))d'_{K+\mathrm{Lip}}(\beta)(K+1)\cdot \\ \cdot[1+2d'_K(S_\chi(x,\cdot),Id)+2d'_K(S_{\overline{\chi}}(x,\cdot),Id)]^K \le \\ \le d_K(\chi(x,\cdot),\overline{\chi}(x,\cdot))\delta_0(K+1)(1+4r)^K. \end{aligned}$$

Claim 2.

$$d^Y_K(h_+,\overline{h}_+) \le$$
$$\le \frac{\delta_0\|A_+^{-1}\|(1+4r)^K(K+1)[(K+1)r_1(1+4\delta_1)^K+1]}{1-\|A_+^{-1}\|(1+\delta_1)^K}\cdot d^Y_K(\chi,\overline{\chi}).$$

Proof. From equation (6.5) one has

$$h_+-\overline{h}_+ = A_+^{-1}(h_+F_\chi-\overline{h}_+F_{\overline{\chi}})+A_+^{-1}(g^+_\chi-g^+_{\overline{\chi}}),$$

therefore

$$d^Y_K(h_+,\overline{h}_+)\le\|A_+^{-1}\|[d^Y_K(h_+\,F_\chi,\overline{h}_+\,F_{\overline{\chi}})+d^Y_K(g_\chi,g_{\overline{\chi}})].$$

The desired estimates follow from the inequalities (6.8) – (6.11) below:

$$d^Y_K(h_+\,F_\chi,\overline{h}_+\,F_{\overline{\chi}})\le d^Y_K(h_+\,F_\chi,h_+\,F_{\overline{\chi}})+d^Y_K(h_+\,F_{\overline{\chi}},\overline{h}_+\,F_{\overline{\chi}}), \tag{6.8}$$

$$\begin{aligned} d_K(g_\chi(x,\cdot),\overline{g}_\chi(x,\cdot)) = d_K(\alpha\circ S_\chi(x,\cdot),\alpha\circ S_{\overline{\chi}}(x,\cdot)) \overset{(L3)}{\le} \\ \le d_K(S_\chi(x,\cdot),S_{\overline{\chi}}(x,\cdot))(K+1)d'_{K+\mathrm{Lip}}(\alpha)\cdot \\ \cdot[1+2d'_K(S_\chi(x,\cdot),Id)+2d'_K(S_{\overline{\chi}}(x,\cdot),Id)]^K\le \\ \le d_K(\chi(x,\cdot),\ \overline{\chi}(x,\cdot))(K+1)\ \delta_0(1+4r)^K, \end{aligned} \tag{6.9}$$

$$\begin{aligned} d_K(h_+\,F_\chi(x,\cdot),\ h_+\,F_{\overline{\chi}}(x,\cdot) = d_K(h_+(Ax,\varphi_x(\cdot))\ ,\ h_+(Ax,\overline{\varphi}_x(\cdot)))\le \\ \overset{(L3)}{\le} d_K(\varphi_x(\cdot),\overline{\varphi}_x(\cdot))\ d'_{K+\mathrm{Lip}}(h_+(Ax,\cdot))(K+1)\cdot \\ \cdot[1+2d'_K(\varphi_x,Id)+2d'_K(\overline{\varphi}_x,Id)]^K\le \\ \overset{\text{Claim 1}}{\le} d^Y_K(\chi,\overline{\chi})\ \delta_0\cdot(K+1)^2(1+4r)^K\cdot r_1(1+4\delta_1)^K, \end{aligned} \tag{6.10}$$

$$d'_K(F_{\overline{\chi}}(x,\cdot))\le 1+d'_K(\overline{\varphi}_x,Id)\le 1+\delta_1,$$

$$\begin{aligned} d_K(h_+F_{\overline{\chi}}(x,\cdot),\overline{h}_+F_{\overline{\chi}}(x,\cdot)) \overset{(R3)}{\le} d^Y_K(h_+,\overline{h}_+)[\max\{1,d'_K(F_{\overline{\chi}}(x,\cdot))\}]^K\le \\ \le d^Y_K(h_+,\overline{h}_+)\cdot(1+\delta_1)^K. \end{aligned} \tag{6.11}$$

Claim 3.

$$d^Y_K(h_-,\overline{h}_-)\le\frac{\delta_0 d^Y_K(\chi,\overline{\chi})}{1-\|A_-\|(1+M'_K\delta_1)^K}\cdot$$
$$\cdot\left\{\|A_-\|\cdot M_K\,\delta_0(K+1)^2r_1(1+4r)^K\,(1+4M'_K\delta_1)^K+C_\infty\right\},$$

where

$$C_\infty = (K+1)\cdot(1+4r)^K \cdot \big[M_K\delta_0\cdot(K+1)\,(1+4r)^K\cdot \\ \cdot\big\{1+r(K+1)[1+4M'_K\delta_1]^K\big\}\cdot[1+4(M'_K\delta_1+r(1+\delta_1)^K)]+ \\ +(1+M'_K\delta_1)^K\big]$$

Proof. From equality (6.5) one has

$$h_- - \overline{h}_- = A_-(h_-F_\chi^{-1} - \overline{h}_-F_{\overline{\chi}}^{-1}) + g_\chi^- F_\chi^{-1} - g_{\overline{\chi}}^- F_{\overline{\chi}}^{-1}.$$

Therefore

$$d_K^Y(h_-,\overline{h}_-) \le \|A_-\|\, d_K^Y(h_-\,F_\chi^{-1},\overline{h}_-\,F_{\overline{\chi}}^{-1}) + d_K^Y(g_\chi F_\chi^{-1},\ g_{\overline{\chi}}F_{\overline{\chi}}^{-1}).$$

Since

$$d_K^Y(h_-\,F_\chi^{-1},\overline{h}_-\,F_{\overline{\chi}}^{-1}) \le d_K^Y(h_-\,F_\chi^{-1},h_-\,F_{\overline{\chi}}^{-1}) + d_K^Y(h_-\,F_{\overline{\chi}}^{-1},\overline{h}_-\,F_{\overline{\chi}}^{-1}),$$

the statement follows if we have estimates in terms of $d_K^Y(\chi,\overline{\chi})$ and $d_K^Y(h_-,\overline{h}_-)$ for $d_K^Y(h_-\,F_\chi^{-1},h_-\,F_{\overline{\chi}}^{-1})$, $d_K^Y(h_-\,F_{\overline{\chi}}^{-1},\overline{h}_-\,F_{\overline{\chi}}^{-1})$ and $d_K^Y(g_\chi F_\chi^{-1},\ g_{\overline{\chi}}F_{\overline{\chi}}^{-1})$. These are given below in (6.12), (6.13) and (6.19).

Recall first that

$$F_\chi^{-1}(x,\cdot) = (A^{-1}x,\varphi^{-1}_{A^{-1}x}(\cdot)).$$

Then

$$d_K(h_-\,F_\chi^{-1}(x,\cdot),h_-\,F_{\overline{\chi}}^{-1}(x,\cdot)) \overset{\text{(L3)}}{\le} d_K^Y(F_\chi^{-1},F_{\overline{\chi}}^{-1})d'_{K+\mathrm{Lip}}(h_-(A^{-1}x,\cdot))\cdot \\ \cdot(K+1)[1+2d'_K(F_\chi^{-1}(x,\cdot),Id)+2d'_K(F_{\overline{\chi}}^{-1}(x,\cdot),Id)]^K.$$

Using now Claim 1, one has:

$$d_K(h_-\,F_\chi^{-1},h_-\,F_{\overline{\chi}}^{-1}) \le d_K^Y(\chi,\overline{\chi})M_K\delta_0(K+1)^2(1+4r)^K r_1\cdot \\ \cdot(1+4M'_K\delta_1)^K, \tag{6.12}$$

$$d_K^Y(h_-\,F_{\overline{\chi}}^{-1}(x,\cdot),\overline{h}_-\,F_{\overline{\chi}}^{-1}(x,\cdot)) \overset{\text{(R3)}}{\le} d_K^Y(h_-,\overline{h}_-)\cdot[\max\{1,d'_K(F_{\overline{\chi}}^{-1}(x,\cdot))\}]^K \le \\ \le d_K^Y(h_-,\overline{h}_-)(1+M'_K\delta_1)^K. \tag{6.13}$$

To estimate $d_K^Y(g_\chi F_\chi^{-1},g_{\overline{\chi}}F_{\overline{\chi}}^{-1})$, we divide it into two parts:

$$d_K^Y(g_\chi F_\chi^{-1},\ g_{\overline{\chi}}F_{\overline{\chi}}^{-1}) \le d_K^Y(g_\chi F_\chi^{-1},\ g_\chi F_{\overline{\chi}}^{-1}) + d_K^Y(g_\chi F_{\overline{\chi}}^{-1},\ g_{\overline{\chi}}F_{\overline{\chi}}^{-1}).$$

The two terms of the RHS are estimated by (6.17) and (6.18) below.

To obtain (6.17), we proceed as follows:

$$
\begin{aligned}
d_K(g_\chi F_\chi^{-1}(x,\cdot), g_\chi F_{\overline{\chi}}^{-1}(x,\cdot)) &= d_K(\alpha\, S_\chi F_\chi^{-1}(x,\cdot),\ \alpha S_\chi F_{\overline{\chi}}^{-1}(x,\cdot)) \le \\
&\overset{(L3)}{\le} d_K(S_\chi F_\chi^{-1}(x,\cdot),\ S_\chi F_{\overline{\chi}}^{-1}(x,\cdot))(K+1)d'_{K+\mathrm{Lip}}(\alpha)\cdot \\
&[1+2d'_K(S_\chi F_\chi^{-1}(x,\cdot), Id)+2d'_K(S_\chi F_{\overline{\chi}}^{-1}(x,\cdot), Id)]^K.
\end{aligned}
\tag{6.14}
$$

Now

$$
\begin{aligned}
d'_K(S_\chi F_\chi^{-1}(x,\cdot), Id) &\le d'_K(\varphi^{-1}_{A^{-1}x}(\cdot), Id) + d'_K(\chi(A^{-1}x, \varphi^{-1}_{A^{-1}x}(\cdot))) \le \\
&\overset{(R1)}{\le} M'_K\cdot\delta_1 + d'_K(\chi(A^{-1}x,\cdot))[\max\{1, d'_K(\varphi^{-1}_{A^{-1}x}(\cdot))\}]^K \le \\
&\le M'_K\delta_1 + r(1+M'_K\delta_1)^K,
\end{aligned}
\tag{6.15}
$$

and a similar estimate holds for $d_K(S_\chi F_{\overline{\chi}}^{-1}(x,\cdot), Id)$. Also,

$$
\begin{aligned}
&d_K(S_\chi F_\chi^{-1}(x,\cdot), S_\chi F_{\overline{\chi}}^{-1}(x,\cdot)) \le \\
&\le d_K(\varphi^{-1}_{A^{-1}x}(\cdot), \overline{\varphi}^{-1}_{A^{-1}x}(\cdot)) + d_K(\chi(A^{-1}x, \varphi^{-1}_{A^{-1}x}(\cdot)), \chi(A^{-1}x, \overline{\varphi}^{-1}_{A^{-1}x}(\cdot))) \le \\
&\overset{(L3)}{\le} d_K(\varphi^{-1}_{A^{-1}x}(\cdot), \overline{\varphi}^{-1}_{A^{-1}x}(\cdot)) + \\
&+ d_K(\varphi^{-1}_{A^{-1}x}(\cdot), \overline{\varphi}^{-1}_{A^{-1}x}(\cdot)) d'_{K+\mathrm{Lip}}(\chi(A^{-1}x,\cdot))(K+1)\cdot \\
&\cdot[1+2d'_K(\varphi^{-1}_{A^{-1}x}(\cdot), Id)+2d'_K(\overline{\varphi}^{-1}_{A^{-1}x}(\cdot), Id)]^K \le \\
&\overset{\text{Claim 1}}{\le} M_K\cdot\delta_0(K+1)(1+4r)^K d_K^Y(\chi,\overline{\chi})\{1+r(K+1)[1+4M'_K\delta_1]^K\}.
\end{aligned}
\tag{6.16}
$$

Therefore, by (6.14), (6.15) and (6.16),

$$
\begin{aligned}
&d_K^Y(g_\chi F_\chi^{-1}, g_\chi F_{\overline{\chi}}^{-1}) \le \\
&\le M_K\,\delta_0^2(K+1)^2(1+4r)^K\{1+r(K+1)[1+4M'_K\delta_1]^K\}\cdot \\
&\cdot[1+4(M'_K\delta_1+r(1+M'_K\delta_1)^K)]^K\cdot d_K^Y(\chi,\overline{\chi}).
\end{aligned}
\tag{6.17}
$$

For the second part:

$$
\begin{aligned}
d_K(g_\chi F_{\overline{\chi}}^{-1}(x,\cdot), g_{\overline{\chi}} F_{\overline{\chi}}^{-1}(x,\cdot)) &\overset{(R3)}{\le} d_K(g_\chi(A^{-1}x,\cdot), g_{\overline{\chi}}(A^{-1}x,\cdot))\cdot \\
\cdot[\max\{1, d'_K(\overline{\varphi}^{-1}_{A^{-1}x}(\cdot))\}]^K &\overset{(6.9)}{\le} d_K^Y(\chi,\overline{\chi})(K+1)\delta_0\cdot \\
&\cdot(1+4r)^K(1+M'_K\delta_1)^K.
\end{aligned}
\tag{6.18}
$$

From (6.17) and (6.18) it follows that

$$
\begin{aligned}
&d_K^Y(g_\chi F_\chi^{-1} g_{\overline{\chi}} F_{\overline{\chi}}^{-1}) \le \\
&\le d_K^Y(\chi,\overline{\chi})(K+1)\delta_0(1+4r)^K\big[M_K\delta_0(K+1)(1+4r)^K\cdot \\
&\cdot\{1+r(K+1)[1+4M'_K\delta_1]^K\}[1+4(M'_K\delta_1+r(1+M'_K\delta_1)^K)]^K + \\
&+(1+M'_K\delta_1)^K\big] = d_K^Y(\chi,\overline{\chi})\delta_0 C_\infty. \quad \square
\end{aligned}
\tag{6.19}
$$

In §7 we will need a parametric version of Theorem 6.1. This follows from the next two lemmas.

Lemma 6.6. *Let $(X, \mathbf{d})$ be a complete metric space, let $\mathcal{F}_\iota : X \to X$, be a family of uniformly contractive operators on X, $\|\mathcal{F}_\iota\|_{\mathrm{Lip}} \le \lambda < 1$. Then the application $\mathcal{F}_\iota \mapsto x_\iota$, where x_ι is the unique fixed point of $\mathcal{F}_\iota$, is continuous with respect to the metric $d(\mathcal{F}, \mathcal{G}) := \sup_{x \in X} \mathbf{d}(\mathcal{F}(x), \mathcal{G}(x))$.*

Lemma 6.7. *For δ_0, r and C_0 chosen as at the beginning of the proof of Theorem 6.1, the application*

$$\widetilde{f} \in \left\{ g \in \mathrm{Diff}^{K+\mathrm{Lip}}(\mathbb{T}^n \times \mathbb{T}^d) | d_{K+\mathrm{Lip}}(f, g) \le \delta_0 \right\} \mapsto \mathcal{F} \in C^0(X_{r,C_0}; X_{r,C_0})$$

given by (6.5) is continuous (the first space has the metric $d_{K+\mathrm{Lip}}$, and the second the metric described in Lemma 6.6), and the range is a family of uniformly contractive operators.

The first lemma is standard. The second lemma follows using similar estimates as in Proposition 6.5, of which we spare the reader.

§7. Deformation Rigidity of Certain Partially Hyperbolic Actions

In this section, as an application of the Livsic type theorem for cocycles with values in diffeomorphism groups, we show the deformation rigidity of the product action ρ_0. First we introduce some definitions.

Let Γ be a finitely-generated discrete group, M a compact manifold and $\varphi : \Gamma \times M \to M$ a C^∞-action. Let $k \geq 1$, $\ell \geq 0$ be integers. Fix a set of generators $\{\gamma_1, \dots, \gamma_p\}$ of Γ. We say that a C^∞-action $\widetilde{\varphi}$ is $C^{k+\mathrm{Lip}}$-close to φ if the C^∞-diffeomorphisms $\varphi(\gamma_i)$ and $\widetilde{\varphi}(\gamma_i)$ are $C^{k+\mathrm{Lip}}$-close for all i. A $C^{k+\mathrm{Lip}}$-deformation of the action φ is a $d_{k+\mathrm{Lip}}$-continuous path of C^∞-actions φ_t, $(0 \leq t \leq 1)$, with $\varphi_0 = \varphi$. An action φ is said to be $C^{k+\mathrm{Lip},\ell}$-deformation rigid if every $C^{k+\mathrm{Lip}}$-deformation of φ, with φ_t contained in a sufficiently small $C^{k+\mathrm{Lip}}$-neighborhood of φ, is C^ℓ-conjugate to φ by a continuous path of C^ℓ-diffeomorphisms.

The main result is the following:

Theorem 7.1. *Let π be the standard linear action of $SL(n,\mathbb{Z})$ on $\mathbb{T}^n$, $n \geq 3$. Let $d \geq 1$ be an integer and consider ρ_0 to be the action of $SL(n,\mathbb{Z})$ on $\mathbb{T}^{n+d} = \mathbb{T}^n \times \mathbb{T}^d$ given by $\rho_0(A)(x,y) = (Ax,y)$, for $x \in \mathbb{T}^n$, $y \in \mathbb{T}^d$ and $A \in SL(n,\mathbb{Z})$. Then, for any integer $K \geq 4$, ρ_0 is $C^{K+\mathrm{Lip},K-3}$-deformation rigid.*

The proof of this theorem will be done in the rest of this paragraph. It relies on our results about cocycles, and uses the techniques of [H1] and [K-L1].

An important fact used in the proof is Stowe's theorem (see [S1] and [S2]). We recall it here in the form used in [H1] and [K-L1].

Let Γ and M be as above. Let $\mathrm{Rep}(\Gamma, \mathrm{Diff}^1(M))$ be the set of all homomorphisms from Γ to $\mathrm{Diff}^1(M)$ with the compact–open topology. Assume α is in $\mathrm{Rep}(\Gamma, \mathrm{Diff}^1(M))$. Denote by $F(\alpha) = \{x \in M | \alpha(\gamma)x = x \text{ for any } \gamma \in \Gamma\}$ the fixed-point set of the action α and fix $x \in F(\alpha)$. Then α induces a finite-dimensional representation of Γ on T_xM, the tangent space to M at x. Denote by $H^1(\Gamma, T_xM)$ the group cohomology with coefficients in this representation.

Theorem 7.2. *(Stowe) If $F \subset F(\alpha)$ is a compact connected component of $F(\alpha)$ such that $H^1(\Gamma, T_xM) = 0$ for any $x \in F$, then F is a stable C^1-submanifold of M. More precisely, there is a tubular neighborhood U of F in M, a neighborhood V of α in* $\mathrm{Rep}(\Gamma, \mathrm{Diff}^1(M))$*, and for each $\beta \in V$ a closed embedding $q_\beta \in C^1(F,U)$, such that:*

(i) *$q_\beta(F) = U \cap F(\beta)$ and*
(ii) *$\beta \in V \mapsto q_\beta \in C^1(F,U)$ is continuous.* □

Using that $H^1(\Gamma, V) = 0$ for any finite-dimensional representation V of a subgroup Γ of finite index of $SL(n,\mathbb{Z})$ (see Margulis [M]) and Theorem 7.2, one has:

Proposition 7.3. *Consider $\{\rho_t\}_t$, a small C^1-deformation of ρ_0. Fix $0 \leq s \leq 1$ and suppose that $G \subset SL(n,\mathbb{Z})$ is a subgroup of finite index. Denote by $F_s \subset \mathbb{T}^n \times \mathbb{T}^d$ a connected component of the set of fixed points of $\rho_s(G)$ (hence $H^1(G, T_y(\mathbb{T}^n \times \mathbb{T}^d)) = 0$ for all $y \in F_s$).*

Then there exists $\epsilon > 0$, a tubular neighborhood U of F_s in $\mathbb{T}^n \times \mathbb{T}^d$ and embeddings $q_{t,s} \in C^1(F_s, U)$, close to the inclusion $F_s \hookrightarrow U$ and defined for $s - \epsilon \leq t \leq s + \epsilon$, *such that $F_t := q_{t,s}(F_s)$ is equal to $U \cap F(\rho_t)$ (and is connected). Moreover, the embeddings $q_{t,s}$ depend continuously on t.* $\square$

We return now to the proof of Theorem 7.1. First we recall a general situation when a cocycle appears naturally. Let G be a group and $\alpha : G \to \mathrm{Diff}^k(M \times N)$ an action of G by bundle morphisms on the trivial bundle $M \times N \to M$. Then the action is equivalent to a cocycle over the action in the base M, with values in the diffeomorphism group of the fiber N:

$$\alpha_g(x, y) = (\overline{\alpha}_g(x), \beta(g, x)(y)) \,, \; g \in G, \; x \in M, \; y \in N,$$

where $\beta : G \times M \to \mathrm{Diff}^k(N)$ is the cocycle and $\overline{\alpha} : G \times M \to M$ is the action induced in the base.

Therefore, once we can show that a conjugate of ρ_t (or its restriction) preserves the neutral foliation of ρ_0, we reduce the problem to one about cocycles.

In order to achieve this, consider the hyperbolic element $A \in SL(n, \mathbb{Z})$ given in Theorem 5.8. Assume that the perturbation $\{\rho_t\}_{0 \le t \le 1}$ is small enough in the $C^{K+\mathrm{Lip}}$-metric. Then we can apply the parametric version of the Theorem 6.1 (the version of the Hirsch-Pugh-Shub theorem; we use the Lemmas 6.6 and 6.7). We obtain two continuous paths, $\{H_t\}_t$ and $\{Q_t\}_t$, of Hölder homeomorphisms of $\mathbb{T}^n \times \mathbb{T}^d$, with the following property: H_t takes the leaves of the neutral foliation of $\rho_0(A)$ to those of $\rho_t(A)$, and

$$H_t^{-1} \circ \rho_t(A) \circ H_t = Q_t \circ \rho_0(A). \tag{7.1}$$

I.e., we obtain a path of $\mathbb{Z}$-cocycles $\beta_t : \mathbb{Z} \times \mathbb{T}^n \to \mathrm{Diff}^{K+\mathrm{Lip}}(\mathbb{T}^d)$ over the action of A on $\mathbb{T}^n$, given by

$$\beta_t(1, x)(y) = y + \overline{q}_t(Ax, y).$$

Because $\{\overline{q}_t\}_t$ are $C^{K+\mathrm{Lip}}$ along the neutral leaves and the first K derivatives are Hölder, we see immediately that $\beta_t(1, \cdot) : \mathbb{T}^n \to \mathrm{Diff}^{K+\mathrm{Lip}}(\mathbb{T}^d)$ are Hölder, if we consider on $\mathrm{Diff}^{K+\mathrm{Lip}}(\mathbb{T}^d)$ the d_K-metric.

The next step is to show that $H_t^{-1} \circ \rho_t(\gamma) \circ H_t$ preserves the neutral foliation of $\rho_0(A)$, for any $\gamma \in SL(n, \mathbb{Z})$. Here we follow a strategy similar to that used by Hurder to prove topological deformation rigidity in [H1]. We recall some of his definitions and results.

Consider the linear action $\pi : SL(n, \mathbb{Z}) \times \mathbb{T}^n \to \mathbb{T}^n$. A periodic point for π is a point in $\mathbb{T}^n$ which has a finite orbit under $\pi(SL(n, \mathbb{Z}))$. The set Λ of periodic points of π is the set of rational points in $\mathbb{T}^n$. A π-filtration of Λ is an increasing sequence of finite sets $\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_p \subset \cdots$ whose union is all of Λ, and such that $x_i \in \Lambda_i$ implies $\pi(\gamma)(x_i) \in \Lambda_i$ for all $\gamma \in SL(n, \mathbb{Z})$. From Lemma 3.2 in [H1] we know that Λ admits a natural π-filtration. Denote by G_p the π-stabilizer of Λ_p in $SL(n, \mathbb{Z})$, which is a normal subgroup of finite index. For $x \in \mathbb{T}^n$, denote by G_x the π-stabilizer of x in $SL(n, \mathbb{Z})$.

The following proposition is a generalization of Proposition 3.6 from [H1].

Proposition 7.4. *For $x \in \Lambda$, the set $F_t := H_t(\{x\} \times \mathbb{T}^d)$ is an isolated submanifold of fixed points of the action $\rho_t(G_x)$, for all $0 \le t \le 1$.*

Moreover, for any subgroup of finite index $G \subset SL(n, \mathbb{Z})$,

$$F(\rho_t(G)) \subset \bigcup_{x \in \Lambda'} H_t(\{x\} \times \mathbb{T}^d),$$

where Λ' is a finite subset of $\mathbb{T}^d$.

Proof. Since the π-orbit of x is finite, G_x has finite index in $SL(n, \mathbb{Z})$ and there is $m > 0$ such that $A^m \in G_x$. We apply Proposition 7.3 for $G = G_x$ and $s = 0$ to obtain:

- a value $t_1 > 0$;
- a continuous path $\{\widetilde{F}_t \mid 0 \le t \le t_1\}$ with $\widetilde{F}_0 = F_0$;
- that $\widetilde{F}_t$ is a connected compact submanifold of fixed points for $\rho_t(G_x)$, isolated in $F(\rho_t)$.

In particular, $\widetilde{F}_t$ is a connected submanifold of the fixed points of $\rho_t(A^m)$.

We claim that the path F_t coincides with $\widetilde{F}_t$, for $0 \le t \le t_1$.

Denote by $\mathcal{F}_t$ the neutral foliation of $\rho_t(A^m)$, $0 \le t \le 1$. Since $\mathcal{F}_t$ is invariant under $\rho_t(A^m)$, one obtains a continuous action on $\mathbb{T}^n \times \mathbb{T}^d/\sim_{\mathcal{F}_t}$, and, by (7.1), H_t induces a continuous conjugacy between the actions $\rho_0(A^m)/\sim_{\mathcal{F}_0}$ and $\rho_t(A^m)/\sim_{\mathcal{F}_t}$. But $\rho_0(A^m)/\sim_{\mathcal{F}_0}$ is Anosov, hence x is an isolated fixed point in the base. Therefore, due to the conjugacy of the actions on the quotient spaces, $H_t(\{x\} \times \mathbb{T}^d)$ is the unique neutral leaf close to F_0 which is invariated by $\rho_t(A^m)$. Hence $\widetilde{F}_t$ is included in $H_t(\{x\} \times \mathbb{T}^d)$. But $\widetilde{F}_t$ is the image of $\{x\} \times \mathbb{T}^d$ by a diffeomorphism and H_t is a homeomorphism, hence by a topological argument (e.g., the degree) it follows that $\widetilde{F}_t = H_t(\{x\} \times \mathbb{T}^d)$ for $0 \le t \le t_1$, which proves our claim.

Let $s > 0$ be the supremum of the values of $\epsilon \le 1$ with the property that F_t is an isolated compact connected submanifold of fixed points for $\rho_t(G_x)$ for all $0 \le t < \epsilon$. Then $s < 1$ leads to a contradiction by Proposition 7.3.

The last assertion follows from the fact that $F(\rho_t(G)) \subset F(\rho_t(A^m))$, where $m \in \mathbb{Z}$ is such that $A^m \in G$, and the latter set is included in the union of those neutral leaves of $\rho_t(A^m)$ which are over the fixed points of the action $\rho_t(A^m)/\sim_{\mathcal{F}_t}$. This action is conjugated by $H_t/\sim$ to the action $\pi(A^m)$ on $\mathbb{T}^n$, which has a finite number of fixed points. □

The previous proposition implies:

Proposition 7.5. *For each $x \in \Lambda$ and $\gamma \in SL(n, \mathbb{Z})$*

$$H_t^{-1} \circ \rho_t(\gamma) \circ H_t(\{x\} \times \mathbb{T}^d) = \rho_0(\gamma)(\{x\} \times \mathbb{T}^d) \text{ for all } 0 \le t \le 1. \tag{7.2}$$

Proof. Choose p such that $x \in \Lambda_p$. Denote, for $0 \le t \le 1$, $F_t := H_t(\{x\} \times \mathbb{T}^d)$. Then (7.2) is equivalent to

$$H_t \rho_0(\gamma)(F_0) = \rho_t(\gamma)(F_t), \qquad \text{for } 0 \le t \le 1.$$

We will show that

$$t \mapsto H_t \rho_0(\gamma)(F_0) \quad \text{and} \quad t \mapsto \rho_t(\gamma)(F_t)$$

are two continuous paths of connected components of $F(\rho_t(G_p))$, equal to neutral leaves of $\rho_t(A)$. Since such components are isolated (by Proposition 7.4) and the two paths are equal at $t = 0$, the paths have to coincide, thus proving (7.2).

It remains to prove the statement about the two paths.

For the first, we obtain

$$\rho_0(\gamma)(F_0) = \rho_0(\gamma)(\{x\} \times \mathbb{T}^d) = \{\gamma x\} \times \mathbb{T}^d.$$

But $\gamma x \in \Lambda_p$, hence Proposition 7.4 applied for γx gives the desired conclusion.

For the second path, notice that $F_t \subset F(\rho_t(G_p))$ (by Proposition 7.4) and $\gamma^{-1} g \gamma \in G_p$ for $g \in G_p$ (because G_p is normal in $SL(n, \mathbb{Z})$). Therefore

$$\rho_t(g)\rho_t(\gamma)(u, v) = \rho_t(\gamma)\rho_t(\gamma^{-1} g \gamma)(u, v) = \rho_t(\gamma)(u, v)$$

for any $(u, v) \in F_t$. Hence $\rho_t(\gamma)(F_t) \subset F(\rho_t(G_p))$, and, by Proposition 7.4, is a subset of a neutral leaf of $\rho_t(A)$. But both the leaf and this subset are homeomorphic to $\mathbb{T}^d$, which can happen only if they are equal (a proper subset of a torus has the top homology group trivial). $\square$

Corollary 7.6. *For each $\gamma \in SL(n, \mathbb{Z})$ and $0 \le t \le 1$, we have*

$$H_t^{-1} \circ \rho_t(\gamma) \circ H_t(\{x\} \times \mathbb{T}^d) = \rho_0(\gamma)(\{x\} \times \mathbb{T}^d)\ , \quad \textit{for all } x \in \mathbb{T}^d.$$

Proof. Because the set of periodic points Λ is dense in $\mathbb{T}^n$, we can use the continuity of both sides in (7.2). $\square$

It follows that $H_t^{-1} \circ \rho_t(\gamma) \circ H_t$ invariates the foliation $\{\{x\} \times \mathbb{T}^d\}_{x \in \mathbb{T}^n}$ for any $\gamma \in SL(n, \mathbb{Z})$, and the action induced on the base is exactly π. By our previous remark, we get a (continuous!) path of cocycles over π

$$\beta_t : SL(n, \mathbb{Z}) \times \mathbb{T}^n \to \mathrm{Diff}^K(\mathbb{T}^d)$$

such that

$$H_t^{-1} \circ \rho_t(\gamma) \circ H_t = (\gamma x, \beta_t(\gamma, x)(y))$$

for all $\gamma \in SL(n, \mathbb{Z})$, $x \in \mathbb{T}^n$, $y \in \mathbb{T}^d$. (We use the same notation for the cocycle over the $SL(n, \mathbb{Z})$-action because it is an extension of the previous cocycle over the action of A.) Moreover, from Proposition 7.4, the isolated closing conditions are satisfied:

> if $g \in SL(n, \mathbb{Z})$ and $x \in \mathbb{T}^n$ is an isolated fixed point of g, then $\beta_t(g, x) = Id_{\mathbb{T}^d}$, for all $t \in [0, 1]$.

Here we use the fact that if $x \in \mathbb{T}^n$ is an isolated fixed point for some $g \in SL(n,\mathbb{Z})$, then x is a rational point, hence is in Λ. We know that $\beta_t(A,\cdot)$ is Hölder in the d_K-metric on $\mathrm{Diff}^K(\mathbb{T}^d)$. Therefore, by Corollary 3.3, Theorem 5.8 and Theorem 5.2, there is a Hölder function $P_t : \mathbb{T}^n \to \mathrm{Diff}^{K-2}(\mathbb{T}^d)$ such that $\beta_t(\gamma,x) = P_t(\gamma x)P_t(x)^{-1}$ for any $\gamma \in SL(n,\mathbb{Z})$, $x \in \mathbb{T}^n$. Define now $\mathcal{P}_t : \mathbb{T}^n \times \mathbb{T}^d \to \mathbb{T}^n \times \mathbb{T}^d$ by $\mathcal{P}_t(x,y) := (x, P_t(x,y))$. Then

$$\mathcal{P}_t^{-1} \circ H_t^{-1} \circ \rho_1(\gamma) \circ H_t \circ \mathcal{P}_t(x,y) = (\gamma x, y),$$
$$\text{for all } x \in \mathbb{T}^n,\ y \in \mathbb{T}^d,\ \gamma \in SL(n,\mathbb{Z}),\ 0 \le t \le 1.$$

Hence there is a continuous path of homeomorphisms $\phi_t := \mathcal{P}_t^{-1} \circ H_t^{-1}$ conjugating ρ_0 and ρ_t. Recall that H_t is C^K-along the neutral foliation and $\mathcal{P}_t$ is C^{K-2} along the neutral foliation, so ϕ_t is C^{K-2} along the neutral foliation. The regularity of ϕ_t transverse to the neutral foliation can be obtained using the same method Katok and Lewis used to show regularity for conjugation between certain Anosov $\mathbb{Z}^k$-actions (see [K-L1]). Because for a partially hyperbolic element the distribution $E^{ss} \oplus E^{uu}$ is not, in general, integrable, we have to make a minor change in proof. The following lemma was also used in [K-L1].

Lemma 7.7. *(*[P-R]*) Let $\Gamma = SL(n,\mathbb{Z})$, $n \ge 2$. Then there is a subgroup $\mathcal{A} \subset \Gamma$ such that*

- (i) *the elements of $\mathcal{A}$ are simultaneously diagonalizable over $\mathbb{R}$;*
- (ii) *$\mathcal{A}$ is isomorphic to a free abelian group of rank $n-1$;*
- (iii) *if $v_1,\dots,v_n \in \mathbb{R}^n$ is a basis of simultaneous eigenvectors for the group $\mathcal{A}$, and $\lambda_i : \mathcal{A} \to \mathbb{R}^n$ is the character of $\mathcal{A}$ defined via $Av := \lambda_i(A)v_i$, $A \in \mathcal{A}$, then for each i, $1 \le i \le n$, there exists $A_i \in \mathcal{A}$ such that $\lambda_i(A_i) < 1$ and $\lambda_j(A_i) > 1$ for each $j \ne i$;*
- (iv) *there exist $C_i \in \mathcal{A}$, $1 \le i \le n-1$, such that $\lambda_j(C_i) < 1$, $j \le i$, and $\lambda_j(C_i) > 1$, $j > i$.* $\square$

We finish now the proof of Theorem 7.1.

Let $A_i, C_i \in SL(n,\mathbb{Z})$, $1 \le i \le n$, be as given by Lemma 7.7. For a small enough perturbation, we can apply Theorem 6.1 in [H-P-S] for the images of A_i and C_i through ρ_t, $0 \le t \le 1$.

Denote by $\mathcal{F}^s_{i,t}$, $\mathcal{G}_{i,t}$ the strong stable foliation of $\rho_t(A_i)$, respectively the weak stable foliation of $\rho_t(C_i)$. These foliations have uniform C^∞-leaves of dimension one, respectively uniform C^{K-2}-leaves of dimension $i+n$. Note that $\mathcal{G}_{i,t}$ and $\mathcal{F}^s_{i+1,t}$ are Hölder foliations whose tangent spaces span the tangent space of $\mathcal{G}_{i+1,t}$.

Fix $0 \le t \le 1$.

Since ϕ_t is a topological conjugacy, it carries $\mathcal{F}^s_{i,t}$ to $\mathcal{F}^s_{i,0}$ and $\mathcal{G}_{i,t}$ to $\mathcal{G}_{i,0}$. Then Lemma 4.10 from [K-L1] shows that $\phi_t, \phi_t^{-1} : \mathbb{T}^n \times \mathbb{T}^d \to \mathbb{T}^n \times \mathbb{T}^d$ are uniformly C^∞ along the leaves of $\mathcal{F}^s_{i,t}$ and $\mathcal{F}^s_{i,0}$, respectively.

We use now an induction argument. We describe it only for ϕ_t, as ϕ_t^{-1} is dealt with similarly.

We apply first Journé's Theorem (see [J]), on each leaf of $\mathcal{G}_{1,t}$, for $\mathcal{F}^s_{1,t}$ and the neutral foliation of $\rho_t(A_1)$. We get that ϕ_t is $C^{K-2-\alpha_1}$ along the leaves of $\mathcal{G}_{1,t}$, for some $0 < \alpha_1 < 1$. Assume now that ϕ_t is $C^{K-2-\alpha_i}$, $0 < \alpha_i < 1$, along the leaves of $\mathcal{G}_{i,t}$. Then we can apply again Journé's Theorem on each leave of $\mathcal{G}_{i+1,t}$, for the foliations $\mathcal{G}_{i,t}$ and $\mathcal{F}^s_{i+1,t}$, to obtain that ϕ_t is $C^{K-2-\alpha_{i+1}}$, $0 < \alpha_{i+1} < 1$, along the leaves of $\mathcal{G}_{i+1,t}$. At the end, we obtain that ϕ_t is $C^{K-2-\beta}$, $0 < \beta < 1$, on $\mathbb{T}^n \times \mathbb{T}^d$. $\square$

Note that the proof depends on only a few properties of the action. We summarize these in Theorem 7.8.

We say that Γ is a higher rank lattice if Γ is a discrete subgroup of a connected semisimple algebraic $\mathbb{R}$-group G, with the $\mathbb{R}$-split rank of each factor of G at least two, G having finite center, $G^0_{\mathbb{R}}$ having no compact factors and such that G/Γ has finite volume with respect to the Haar measure (see [Z4]). Note that from Margulis [M], $H^1(\Gamma, V) = 0$ for any finite-dimensional representation V of Γ.

Theorem 7.8. *(a) Let $\Gamma \subset SL(n, \mathbb{Z})$ be a higher rank lattice. Let π be the standard linear action of Γ on $\mathbb{T}^n$ and assume that π is generically Anosov. Let $d \geq 1$ be an integer and let ρ_0 be the action of Γ on $\mathbb{T}^{n+d} = \mathbb{T}^n \times \mathbb{T}^d$ given by $\rho_0(A)(x, y) = (Ax, y)$ (where $x \in \mathbb{T}^n$, $y \in \mathbb{T}^d$ and $A \in \Gamma$). Let $K \geq 4$ be any integer. Then ρ_0 is $C^{K+\mathrm{Lip},0}$-deformation rigid.*

(b) If, in addition, Γ contains an abelian subgroup $\mathcal{A}$ which satisfies properties (i)–(iv) of Lemma 7.7, then ρ_0 is $C^{K+\mathrm{Lip},K-3}$-deformation rigid. $\square$

References

[C-E-G] P. Collet, H. Epstein, G. Gallavoti, *Perturbations of geodesic flows on surfaces of constant negative curvature and their mixing properties*, Com. Math. Phys. **95** (1984), 61–112.

[G-K] V. Guillemin, D. Kazhdan, *On the cohomology of certain dynamical systems*, Topology **19** (1980), 291–299.

[H1] S. Hurder, *Rigidity of Anosov actions of higher rank lattices*, Annals of Math. **135** (1992), 361–410.

[H2] S. Hurder, *Infinitesimal rigidity of hyperbolic actions*, preprint (1993).

[H-K-L-Z] S. Hurder, A. Katok, J. Lewis, R. Zimmer, *Rigidity of Cartan actions of higher rank lattices*, preprint (1991).

[H-P-S] M. Hirsch, C. Pugh, M. Shub, *Invariant manifolds*, Lect. Notes in Math. 583, Springer Verlag, 1977.

[J] J. -L. Journé, *On a regularity problem occurring in connection with Anosov diffeomorphisms*, Com. Math. Phys. **106** (1986), 345-352.

[K] A. Katok, *Dynamical systems with hyperbolic structure*, Three Papers in Dynamical Systems, American Mathematical Society Translations Series 2, vol. 116, Providence, Rhode Island, 1981.

[K-H] A. Katok, B. Hasselblat, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.

[K-K] A. Katok, S. Katok, *Higher cohomology for abelian groups of toral automorphisms*, preprint (1993).

[K-L1] A. Katok, J. Lewis, *Local rigidity for certain groups of toral automorphisms*, Israel J. Math. **75(2-3)** (1992), 203-242.

[K-L2] A. Katok, J. Lewis, *Global rigidity results for lattice actions on tori and new examples of volume-preserving actions*, preprint (1991).

[K-L-Z] A. Katok, J. Lewis, R. Zimmer, *Cocycle superrigidity and rigidity for lattice actions on tori*, preprint (1992).

[K-S] A. Katok, R. Spatzier, *Differentiable rigidity of hyperbolic abelian actions*, preprint (1992).

[L1] A. Livsic, *Homology properties of U systems*, Math. Notes **10** (1971), 758–763.

[L2] A. Livsic, *Cohomology of dynamical systems*, Math. U.S.S.R., Izvestia **6** (1972), 1278–1301.

[L-S] A. Livsic, J. Sinai, *On invariant measures compatible with the smooth structures for transitive U systems*, Soviet Math. Doklady **13** (1972), no. 6, 1656–1659.

[Lla] R. de la Llave, *Analytic regularity of solutions of Livsic's cohomology equations and some applications to analytic conjugacy of hyperbolic dynamical systems*, preprint (1989).

[L-M-M] R. de la Llave, J. M. Marco, R. Moriyon, *Canonical perturbation theory of Anosov systems and regularity results for Livsic cohomology equation*, Annals of Math. **123** (1986), 537–612.

[M] G. A. Margulis, *Discrete subgroups of semisimple Lie groups*, Springer-Verlag, 1991.

[N-T] V. Niţică, A. Török, *Regularity results for the solution of Livsic cohomology equation with values in diffeomorphism groups* (to appear in Erg. Th. & Dyn. Syst.).

[P-R] G. Prasad, M. S. Raghunathan, *Cartan subgroups and lattices in semisimple groups*, Annals of Math. **96** (1972), 296–317.

[Q1] N. Qian, *Local rigidity of Anosov actions on tori*, preprint (1993).

[Q2] N. Qian, *Topological deformation rigidity of actions of higher rank lattices on tori*, preprint (1993).

[R] W. Rudin, *Principles of mathematical analysis*, 3rd ed., McGraw-Hill, 1976.

[S1] D. Stowe, *The stationary set of a group action*, Proc. A.M.S. **79** (1980), 139–146.

[S2] D. Stowe, *Stable orbits of differentiable group actions*, Trans. A.M.S. **277** (1983), 665–684.

[Z1] R. Zimmer, *Actions of semisimple groups and discrete groups*, Proc. Int. Congress Math., Berkeley (1986), 1247–1254.

[Z2] R. Zimmer, *Discrete groups in geometry and analysis*, Progress in Mathematics, Birkhäuser,, Boston, 1987.

[Z3] R. Zimmer, *Infinitesimal rigidity for smooth actions of discrete subgroups*, J. Diff. Geom. **31** (1990), 301–322.

[Z4] R. Zimmer, *Ergodic theory and semisimple groups*, Birkhauser, Boston, 1984.

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