On the Cohomology of Anosov Actions

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Abstract The goal of this survey is to present recent cohomological results in the theory of Anosov actions, as well as to show how they are related to other results in the field. Due to constraints of space, the presentation is biased toward results related to our work.

1 Cocycles

We start with an introduction to cocycles. The standard reference is the book by Zimmer [45]. The emphasis in [45] is on measurable cocycles, while here we are more concerned with Hölder and differentiable cocycles. Nevertheless, the basic definitions are the same.

Let G be a (discrete or continuous) group acting on a manifold M by $\alpha : G \times M \to M$, $(g, x) \mapsto \alpha_g(x) := gx$. Let Γ be a topological group with unit 1_Γ. A (continuous) *cocycle* β over the action α is a continuous function $\beta : G \times M \to \Gamma$ such that

$$\beta(g_1 g_2, x) = \beta(g_1, g_2 x)\beta(g_2, x)\,, \tag{1}$$

for all $g_1, g_2 \in G$, $x \in M$.

One important example of cocycle is the derivative cocycle. Let M be a manifold of dimension n, and let G act on M by diffeomorphisms. Denote by TM the tangent bundle of M and define $\beta(g, x) := (Dg)|_x : TM_x \to TM_{gx}$. Then the chain rule for differentiation is exactly the cocycle equation (1). If M has trivial tangent bundle (an example is the n-dimensional torus), then we can choose a smooth section in the principal bundle of n-frames of M and β becomes a cocycle with values in $GL(n, R)$. For an arbitrary manifold M, one can choose enough Borel sections of the tangent bundle, and hence measurably TM is trivial. Thus β can be identified with a measurable cocycle $G \times M \to GL(n, R)$.

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The natural equivalence relation for cocycles is the cohomology. Two cocycles β_1 and β_2 are called (continuously) *cohomologous* if there exists a continuous map $P : M \to \Gamma$ such that

$$\beta_1(g,x) = P(gx)\beta_2(g,x)P(x)^{-1}\,, \tag{2}$$

for all $g \in G$, $x \in M$. The map P is called a *transfer map*.

Let α be an action and β a cocycle over α. Let $\Gamma \times N \to N$, $(\gamma, y) \to \gamma y$ be an action of Γ on a space N. This yields an action $\widetilde{\alpha}_\beta : G \times (M \times N) \to M \times N$ given by

$$\widetilde{\alpha}_\beta(g)(x,y) = (gx, \beta(g,x)y)\,,$$

for all $x \in M, y \in N$, which is called a *skew-product*. The cocycle equation (1) is equivalent to the fact that $\widetilde{\alpha}_\beta$ is an action, i.e., $\widetilde{\alpha}_\beta(g_1 g_2) = \widetilde{\alpha}_\beta(g_1)\widetilde{\alpha}_\beta(g_2)$.

Particular Γ-actions that will be of interest for us include the action of an arbitrary group Γ on itself by left multiplication, and the action of the diffeomorphism group $\mathrm{Diff}(N)$ on the manifold N.

If $\beta_1, \beta_2 : G \times M \to \Gamma$ are two cohomologous cocycles via a transfer map P, then $\widetilde{\alpha}_{\beta_1}$ and $\widetilde{\alpha}_{\beta_2}$ are equivalent actions, i.e., there exists a conjugacy $h : M \times N \to M \times N$ such that

$$\widetilde{\alpha}_{\beta_1}(g) = h \circ \widetilde{\alpha}_{\beta_2}(g) \circ h^{-1}$$

for all $g \in G$. The conjugacy h is given by $h(x,y) = (x, P(x)y)$.

A cocycle β is cohomologous to a constant cocycle if there exists a function $P : M \to \Gamma$ and a homomorphism $\pi : G \to \Gamma$ such that

$$\beta(g,x) = P(gx)\pi(g)P(x)^{-1}\,.$$

In particular, if π is the trivial homomorphism, β is said to be cohomologous to the trivial cocycle. We call the equation

$$\beta(g,x) = P(gx)P(x)^{-1} \tag{3}$$

the *Livsic cohomological equation*.

A cocycle β satisfies the *closing conditions* if for any $g \in G$ and $x \in M$ such that $gx = x$, one has $\beta(g,x) = 1_\Gamma$. Note that the closing conditions are necessary conditions for a cocycle to be cohomologous to the trivial cocycle.

In the case $G = \mathbb{Z}$, the cocycle is determined by the function $f : M \to \Gamma$, $f(x) := \beta(1,x)$. If $T : M \to M$ is the map which implements the $\mathbb{Z}$-action, then the closing conditions become

$$T^n x = x \text{ implies } f(T^{n-1}x)f(T^{n-2}x)\dots f(x) = 1_\Gamma\,. \tag{4}$$

For brevity, we will use the term *small* for a cocycle whose values are close to the identity on a compact generating set in the group acting on the manifold.

The following questions turned out to be important for applications in the rigidity theory of hyperbolic and partially hyperbolic actions:

1. When is a cocycle cohomologous to a constant (trivial) cocycle?
2. If two C^K-cocycles, $1 \le K \le \omega$, are cohomologous through a continuous transfer map P, what can be said about the regularity of P?

2 Partially Hyperbolic and Anosov Actions

We recall the definition of a partially hyperbolic diffeomorphism.

Let M be a compact manifold. A C^1-diffeomorphism $T : M \to M$ is called *partially hyperbolic* if there exists a continuous invariant splitting of the tangent bundle $TM = E^s(T) \oplus E^0(T) \oplus E^u(T)$ such that the derivative Tf expands $E^u(T)$ much more than $E^0(T)$, and contracts $E^s(T)$ much more than $E^0(T)$. See [3] and [7] for the theory of partially hyperbolic diffeomorphisms. The sub-bundles $E^s(T)$ and $E^u(T)$ are called the *stable*, respectively *unstable*, distributions. These distributions are integrable. We denote by $W^s(x;T)$ and $W^u(x;T)$ the stable, respectively unstable, manifolds at the point $x \in M$. The stable and unstable foliations are Hölder foliations. If the diffeomorphism $T \in C^K(M)$, then the leaves of the stable and unstable foliations are C^K as well.

If $E^0 = \{0\}$ then the diffeomorphism T is called *Anosov*. The only manifolds which are known to admit Anosov diffeomorphisms are tori, nilmanifolds and infranilmanifolds. It is an outstanding conjecture [6] that these are the only ones supporting Anosov diffeomorphisms.

A *nilmanifold* is the quotient of a connected, simply connected nilpotent Lie group N by a lattice Γ. All such lattices are cocompact, torsion free and finitely generated (see Theorems 2.1 and 2.18 in [39]). An infranilmanifold is finitely covered by a nilmanifold. More precisely, let N be a connected, simply connected nilpotent Lie group and C a compact group of automorphisms of N. Let Γ be a torsion free cocompact discrete subgroup of the semi-direct product NC. Recall that an element (x, c) of NC (where $x \in N$ and $c \in C$) acts on N by first applying c and then left translating by x. By a result of Auslander [1], $\Gamma \cap N$ is a cocompact discrete subgroup of N and $\Gamma \cap N$ has finite index in Γ. The quotient space N/Γ is a compact manifold called an *infranilmanifold*.

Anosov diffeomorphisms on nilmanifolds and infranilmanifolds were introduced in [41], [40] and [6]. Let $\bar{f} : NC \to NC$ be an automorphism for which $\bar{f}(\Gamma) = \Gamma$, $\bar{f}(N) = N$. Then $\bar{f}$ induces a diffeomorphism $f : N/\Gamma \to N/\Gamma$, called an *infranilmanifold automorphism*. If the derivative $D\bar{f}|_N$ at the identity is hyperbolic, i.e., has all eigenvalues of absolute value different from 1, then f is an Anosov diffeomorphism. Note that in this case the stable and unstable distributions are smooth.

An *Anosov action* is a discrete group action containing at least one Anosov diffeomorphism. In this paper we will consider three types of Anosov actions: $\mathbb{Z}$-actions, higher rank abelian actions, and higher rank lattice actions.

3 Livsic Theory

The cohomology of dynamical systems was introduced by Sinai, and the first results for hyperbolic systems were obtained by Livsic [24], [25], and by Livsic and Sinai [26].

Theorem 3.1 (Livsic). *Let M be a compact Riemannian manifold with a $\mathbb{Z}$-action implemented by a topologically transitive C^1-Anosov diffeomorphism T. Then:*

(a) An α-Hölder function $f : M \to \mathbb{R}$ determines a cocycle cohomologous to a trivial cocycle if and only if f satisfies the closing conditions (4). The transfer map P is α-Hölder.
(b) For each Hölder class α and each finite dimensional Lie group Γ, there is a neighborhood U of the identity in Γ such that an α-Hölder function $f : M \to U$ determines a cocycle cohomologous to the trivial cocycle if and only if f satisfies the closing conditions (4). The transfer map P is α-Hölder.

Theorem 3.1 relies on the following result:

Theorem 3.2 (Anosov's Closing Lemma). *Given $T : M \to M$ an Anosov C^1-diffeomorphism, there are constants $\varepsilon > 0$, and $K > 0$ with the following property: for $0 < \delta < \varepsilon$, given any periodic δ-pseudo-orbit $(x_0, x_2, \dots, x_{n-1})$ (i.e., $\mathbf{d}_M(Tx_k, x_{k+1}) < \delta$ for all $k \pmod n$), there is a periodic orbit of the same length $(x_*, Tx_*, \dots, T^{n-1}x_*)$, $T^n x_* = x_*$, such that*

$$\mathrm{d}_M(x_k, T^k x_*) \le K\delta, \quad 0 \le k \le n-1\,.$$

Using Fourier analysis, Veech proved in [43] a Livsic type result for an ergodic endomorphism of the n-dimensional torus, not necessarily hyperbolic:

Theorem 3.3 (Veech). *Let T be an ergodic endomorphism of $\mathbb{T}^n$, and let $\alpha \ge 0$, $\gamma > n$ be given. Then $f \in C^{\alpha+\gamma}(\mathbb{T}^n)$ determines a real-valued cocycle cohomologous to the trivial cocycle if and only if f satisfies the closing conditions (4). The transfer map P is C^α.*

As we will see in Sect. 4, in the hyperbolic case the solution P of the Livsic cohomology equation (3) may enjoy more smoothness than what follows from Theorem 3.1. In the quasi-hyperbolic setting certain restrictions to regularity appear:

Theorem 3.4 (Veech). *Let T be an ergodic endomorphism of $\mathbb{T}^n$, and suppose that T is not hyperbolic. Then there exists $f \in C^1(\mathbb{T}^n)$ such that the real-valued cocycle determined by f satisfies the closing conditions (4), but the Livsic cohomology equation (3) has no solution P of class C^1.*

W. Parry [36] expanded Livsic's criterion from Theorem 3.1 to the cohomology of *two* cocycles with values in compact Lie groups. The proof is based on a strengthening of Theorem 3.2.

Theorem 3.5 (Parry). *Let M be a compact Riemannian manifold with a $\mathbb{Z}$-action implemented by a topologically mixing C^1-Anosov diffeomorphism T.*

Assume that G is a compact Lie group and $f_1, f_2 : M \to G$ are Hölder functions such that the cocycles determined by f_1 and f_2 have identical periodic data. That is, $T^n x = x$ implies

$$f_1(T^{n-1}x)f_1(T^{n-2}x)\dots f_1(x) = f_2(T^{n-1}x)f_2(T^{n-2}x)\dots f_2(x)\,.$$

Then the two cocycles are cohomologous.

The remaining theorems in this section are motivated by certain rigidity problems (see [32] and Theorem 6.4 in Sect. 6).

First, we extend Livsic's Theorem 3.1 to cocycles with values in diffeomorphism groups:

Theorem 3.6. *Let M be a compact Riemannian manifold with a $\mathbb{Z}$-action implemented by a topologically transitive C^1-Anosov diffeomorphism. Let N be a compact smooth manifold with stably trivial tangent bundle. Fix a Hölder class α and a smoothness order $K \geq 4$.*

Then there exists a neighborhood $U \subset \mathrm{Diff}^K(N)$ of Id_N such that the cocycle determined by an α-Hölder map $f : M \to U$ is null cohomologically through a $\mathrm{Diff}^{K-3}(N)$-valued transfer map if and only if f satisfies the closing conditions (4). The transfer map is α-Hölder with respect to the metric on $\mathrm{Diff}^{K-3}(N)$.

The proof of Theorem 3.6 is inspired by that of Theorem 3.1(b). The difficulty is that on the group of C^K-diffeomorphisms the natural metric is neither left- nor right-invariant. Nevertheless, in a small neighborhood of the identity one can show that this metric is quasi-invariant and this is enough for the proof. The technique used to prove Theorem 3.6 can also be applied to deal with cocycles with values in the group of invertible elements of a unital Banach algebra [2].

Several improvement of Theorem 3.6 were obtained by de la Llave [29].

So far we considered results only for $\mathbb{Z}$-actions. Next, we extend these results to nonabelian Anosov actions. A Noncommutative Closing Lemma, which generalizes Anosov's Closing Lemma, is essential in the proof. We state first this Lemma, and then the Livsic type results.

Theorem 3.7 (Noncommutative Closing Lemma). *Let M be a compact Riemannian manifold. Let $A, B \in \mathrm{Diff}^1(M)$, A Anosov. Consider the splitting $TM = E^s \oplus E^u$ induced by DA, the differential of A.*

Assume that

$$P_s(D\,B)|_{E^s} : E^s \to E^s \text{ and } P_u(D\,B^{-1})|_{E^u} : E^u \to E^u$$

can be inverted fiberwise, where $P_s : TM \to E^s$ *and* $P_u : TM \to E^u$ *are the projections along* E^u*, respectively* E^s*.*

Then there are $N_0 \in \mathbb{N}$*,* $\varepsilon > 0$ *and* $K > 0$ *with the following property: for any* $N \geq N_0$ *and* $0 < \delta < \varepsilon$*, given any sequence* $(x_0, x_1, \cdots, x_N)$ *such that* $\mathbf{d}_M(Ax_k, x_{k+1}) < \delta$ *for* $0 \leq k \leq N-1$*, and* $\mathbf{d}_M(Bx_N, x_0) < \delta$*, there is a point* $x \in M$ *whose "orbit" shadows the given sequence. That is:*

(a) $B\,A^N x = x$*;*
(b) the pseudo-orbit $(x_0, x_1, \cdots, x_N)$ *and the orbit of* x *are* $K\delta$*-close, i.e.,*

$$\mathbf{d}_M(x_k, A^k x) < K\delta,\ 0 \leq k \leq N\,.$$

Moreover, in a neighborhood of radius ε *of* x_0*, there is only one point* x *that satisfies both (a) and (b).*

Definition 3.8. If B is as in the previous theorem, then we say that B is in *generic position* with respect to A.

Definition 3.9. Let G be a discrete group and $\alpha : G \times M \to M$ an action of G on a manifold M. Let Γ be a topological group and $\beta : G \times M \to \Gamma$ a continuous cocycle over α. We say that β satisfies the *isolated closing conditions* if $\beta(g, x) = 1_\Gamma$ for any $g \in G$ and any *isolated* fixed point $x \in M$ of α_g.

For an Anosov $\mathbb{Z}$-action, the closing conditions are equivalent to the isolated closing conditions. In general, the isolated closing conditions are weaker than the closing conditions, and therefore easier to verify in applications. As we will see in Sect. 6, this can be done for some natural examples.

Theorem 3.10. *Let* M *be a compact Riemannian manifold. Let* $A, B \in \mathrm{Diff}^1(M)$*, where* A *topologically transitive and Anosov and* B *in generic position with respect to* A*. Let* $G = \langle A, B\rangle$ *be the (discrete) group generated by* A *and* B *in* $\mathrm{Diff}^1(M)$*. Let* Γ *be a topological group and* $\beta : G \times M \to \Gamma$ *a continuous cocycle which satisfies the isolated closing conditions. Denote by* $\langle A\rangle$ *the group generated by* A *in* $\mathrm{Diff}^1(M)$ *and assume that* $\beta|_{\langle A\rangle \times M}$ *is cohomologically trivial, i.e., there is* $P : M \to \Gamma$ *continuous such that* $\beta(A, x) = P(Ax)P(x)^{-1}$ *for* $x \in M$*. Then* β *is cohomologically trivial on* G *as well, being trivialized by* P*.*

Proof. For any $g \in G$ and $x \in M$ define $\mu(g, x) = P(gx)^{-1}\beta(g, x)P(x)$. Then $\mu(A, x) = 1_\Gamma$ and we have to show that $\mu(B, x) = 1_\Gamma$ for any $x \in M$.

Pick x_0 close to Bx such that $A^n x_0$ is close to x for some large n. Then, by the continuity of B, $BA^n x_0$ is close to x_0. Apply now Theorem 3.7 to find x_* such that $BA^n x_* = x_*$, x_* is close to x_0 and $A^n x_*$ is close to $A^n x_0$. Hence $A^n x_*$ is also close to x. But

$$\mu(B, A^n x_*) = \mu(BA^n, x_*) \cdot \mu(A^n, x_*)^{-1} = \mu(BA^n, x_*) = 1_\Gamma\,,$$

because the isolated closing conditions hold. By Theorem 3.7, $A^n x_*$ approaches x when x_0 approaches Bx and $A^n x_0$ approaches x. Using this and the continuity of μ, we conclude that $\mu(B,x) = 1_\Gamma$. □

It is not hard to see that for many Anosov actions of finitely generated groups we can apply Theorem 3.10 repeatedly for the generators and extend Theorems 3.1 and 3.6 to more general Anosov actions. This motivates the following definition:

Definition 3.11. We say that a smooth action $G \times M \to M$ is *generic Anosov* if G contains a set of generators $\{A, B_1, \ldots, B_n\}$ such that A is transitive Anosov and each B_k is in generic position with respect to A, $1 \leq k \leq n$.

An example of generic Anosov action is the canonical linear action of $\mathrm{SL}(n,\mathbb{Z})$ on the n-dimensional torus $\mathbb{T}^n$.

Combining the above results, one obtains:

Theorem 3.12. *Assume the action of G on M is generic Anosov and the Hölder cocycle $\beta : G \times M \to \Gamma$ is small, where Γ is one of the groups to which either Theorem 3.1 or Theorem 3.6 applies.*

Then the cocycle β is cohomologically trivial if and only if the isolated closing conditions hold.

4 Regularity Results

As we mentioned in Sect. 1, the following question turned out to be important for many rigidity results:

> What can be said about the regularity of the transfer map, provided that the cocycles and the action are C^K?

For cohomologically trivial cocycles, the first results were obtained by Livsic. He proved that if a cohomologically trivial real valued cocycle β over an Anosov $\mathbb{Z}$-actions is C^1, then the trivialization P is C^1 as well [24]. For hyperbolic toral automorphisms he also showed that if the cocycle is C^∞, respectively C^ω, then so is the solution [25]; this was obtained by studying the decay of the Fourier coefficients. Later Guillemin and Kazhdan [12], [13] showed the C^∞-regularity of the solutions in the case of geodesic flows on negatively curved surfaces. Collet, Epstein and Gallavotti [5] proved a C^ω version for geodesic flows of constant negative curvature. The final conclusion for the C^∞ case appears in the paper by de la Llave, Marco and Moriyón [30]. One of the technical results involved is that if a function is smooth along two transverse foliations which are absolutely continuous and whose Jacobians have some regularity properties, then it is smooth globally. This was proved in [30] using properties of elliptic operators. Later a more general result was obtained by Journé [14], relying mainly on Taylor expansions and the estimate of the error. Another approach is presented in Hurder and Katok [10], based on an unpublished idea of C. Toll. Here the decay of the Fourier coefficients is used to characterize smoothness. The method can be applied for spanning families of foliations which have the same property as those used in [30]. Foliations arising from Anosov diffeomorphisms have this property. Using the approach in [10], de la Llave proved the analytic case in [27].

Using completely different methods, Parry and Pollicott [37] extended to cohomologous cocycles taking values in compact Lie groups results of Livsic stating that a measurable trivialization of a Hölder cocycle must coincide a.e. with a Hölder transfer map.

In [33] we consider this question for C^K-cocycles taking values in either a Lie group or a group of diffeomorphisms. To improve the regularity of the transfer map we have to require that it is already Hölder of some order. That this condition is essential follows from a counterexample of de la Llave (see [33, Sect. 4]).

Theorem 4.1. *Let M and N be compact Riemannian manifolds, and let $\beta, \widetilde{\beta} : \mathbb{Z} \times M \to \mathrm{Diff}^K(N)$ be two C^K-cocycles (i.e., the induced maps $\mathbb{Z} \times M \times N \to N$ are C^K) over the C^K Anosov action $T : \mathbb{Z} \times M \to M$. Assume that β and $\widetilde{\beta}$ are cohomologous through a continuous transfer map $P : M \to \mathrm{Diff}^K(N)$. Denote*

$$\begin{aligned}
\lambda_- &= \lim_{n\to\infty} \|DT^n|_{E^s}\|^{1/n} ,\\
\lambda_+ &= \lim_{n\to\infty} \|DT^{-n}|_{E^u}\|^{-1/n} ,\\
\mu_+ &= \lim_{n\to\infty} \sup_{x\in M} \|D_N\beta(n,x)\|^{1/n} ,\\
\mu_- &= \lim_{n\to\infty} \inf_{x\in M} \|D_N\beta(n,x)^{-1}\|^{-1/n} ,
\end{aligned}$$

where $TM = E^s \oplus E^0 \oplus E^u$ is the $\mathbb{Z}$-invariant splitting of TM. Note that $\mu_- \le \mu_+$. Assume that $\lambda_- < \mu_- \le \mu_+ < \lambda_+$, and set

$$\alpha_0 = \max\left\{ \frac{\ln \mu_+}{\ln \lambda_+}, \frac{\ln \mu_-}{\ln \lambda_-} \right\} .$$

If $\alpha_0 = 0$ (possible only if $\mu_- = \mu_+ = 1$) and

$$\limsup_{n\to\infty} \sup_{x\in M} \|D_N\beta(n,x)\| < \infty , \qquad \limsup_{n\to\infty} \sup_{x\in M} \|D_N\beta(n,x)^{-1}\| < \infty ,$$

then $P : M \to \mathrm{Diff}^K(N)$ is $C^{K-\varepsilon}$ for any small $\varepsilon > 0$.

In general, if $P : M \times N \to N$ is α-Hölder for some $\alpha > \alpha_0$, then $P : M \to \mathrm{Diff}^K(N)$ is $C^{K-\varepsilon}$ for any small $\varepsilon > 0$. For $K \in \{1, \infty, \omega\}$, $K - \varepsilon := K$.

Corollary 4.2. *Let K, N and $T : \mathbb{Z} \times M \to M$ be as in Theorem 4.1. Assume that $\widetilde{\beta} : \mathbb{Z} \times M \to \mathrm{Diff}^K(N)$ is a C^K-cocycle which is cohomologically trivialized by the continuous transfer map $P : M \to \mathrm{Diff}^K(N)$.*

Then P is $C^{K-\varepsilon}$ for any small $\varepsilon > 0$. □

Theorem 4.1 implies a similar result for cocycles taking values in a closed subgroup of a finite dimensional Lie group Γ_0 that admits a cocompact lattice (in particular, Γ_0 can be any semisimple Lie group).

5 Higher Rank Abelian Actions

Cohomological results for higher rank abelian groups were first obtained by Katok and Spatzier in [21], where real valued cocycles over certain hyperbolic actions of $\mathbb{Z}^k$ and $\mathbb{R}^k$ ($k \geq 2$) are shown to be cohomologous to constant cocycles:

Theorem 5.1 (Katok–Spatzier). *Every real valued C^∞ (respectively Hölder) cocycle over a standard hyperbolic action of $\mathbb{R}^k$ is C^∞ (respectively Hölder) conjugate to a constant cocycle, provided $k \geq 2$.*

A typical example of standard action is the action of the diagonal group in $SL(n,\mathbb{R})$ on $SL(n,\mathbb{R})/\Gamma$ by left translations, where Γ is a uniform lattice in $SL(n,\mathbb{R})$ and $n > 2$. Other examples are generated by commuting Anosov automorphisms of a torus. The proof of Theorem 5.1 relies on harmonic analysis (abelian and non-abelian), more specifically on the exponential decay of Fourier coefficients for smooth functions on a torus, and exponential decay of matrix coefficients for irreducible representations of semisimple Lie groups.

A. and S. Katok [15] obtained vanishing results for higher cohomologies of higher rank abelian automorphism groups of the torus. Katok and Schmidt [20] extended Theorem 5.1 to automorphisms of compact abelian groups other than tori.

In [19], using a geometric approach, the results of [21] are generalized to cocycles with values in a Lie group. The crucial property on which the rigidity results for abelian actions are based is that of a TNS-action, introduced by A. Katok:

Definition 5.2. Let $\alpha : \mathcal{A} \times M \to M$ be an action of $\mathcal{A} \cong \mathbb{Z}^k$ on a compact manifold M. We say that the action α is *totally non-symplectic*, or *(TNS)*, if there is a family S of Anosov diffeomorphisms in $\mathcal{A}$ and a continuous splitting of the tangent bundle $TM = \oplus_{i=1}^{\ell} E_i$ into $\mathcal{A}$-invariant distributions such that:

(i) the stable and unstable distributions of any element in S are direct sums of a sub-family of the E_i's;
(ii) each pair of distributions E_i and E_j, $1 \leq i, j \leq \ell$, is included in the stable distribution of some element in S.

Here are a few examples of TNS-actions:

Example 5.3. Consider the following two matrices in $\mathrm{SL}(4,\mathbb{Z})$:

$$A = \begin{pmatrix} 6 & 13 & 1 & -4 \\ 4 & 10 & 1 & -3 \\ 3 & 7 & 1 & -2 \\ 2 & 5 & 1 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & -1 & 2 & 1 \\ -1 & -2 & 2 & 3 \end{pmatrix}.$$

One can check that A and B are hyperbolic and $AB = BA$. Therefore A and B generate an Anosov $\mathbb{Z}^2$-action on $\mathbb{T}^4$. There is a basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^4$ in which both A and B are diagonalizable. The signs of the Lyapunov exponents are $(+,-,-,-)$ for A, $(-,-,-,+)$ for B and $(+,-,-,+)$ for AB. Denote $E_1 = \mathrm{span}\{e_1\}$, $E_2 = \mathrm{span}\{e_2, e_3\}$, $E_3 = \mathrm{span}\{e_4\}$. Then E_1, E_2, E_3 induce a splitting of the tangent bundle $T\mathbb{T}^4$ which satisfies the definition of a (TNS) action. The set S is $\{A, B, B^{-1}A^{-1}\}$.

Example 5.4. Let $\mathrm{SL}(n,\mathbb{R})$, $n \geq 3$, be the group of invertible matrices of determinant one. Let $\mathbf{T} \subset \mathrm{SL}(n,\mathbb{R})$ be a maximal torus such that $\mathbf{T} \cong \mathbb{R}^{n-1}$. It follows from a theorem of Prasad–Raghunathan (see [38, Theorem 7.1]), that there exists an element $g \in \mathrm{SL}(n,\mathbb{R})$ such that $\mathcal{A} := g\mathbf{T}g^{-1} \cap \mathrm{SL}(n,\mathbb{Z})$ is a cocompact lattice in $g\mathbf{T}g^{-1}$. Therefore the natural action of $\mathcal{A}$ on $\mathbb{T}^n$ is a (TNS) $\mathbb{Z}^{n-1}$-action. This example was investigated by Katok and Lewis in [16].

Example 5.5. By the aforementioned theorem of Prasad and Raghunathan, given a lattice Γ in a semisimple Lie group G, the existence of a (TNS) sub-action of a linear action $\alpha : \Gamma \to \mathrm{Diff}(\mathbb{T}^m)$ coming from a representation $\pi : G \to \mathrm{SL}(m,\mathbb{Z})$ can be decided by computing the weights of π on an $\mathbb{R}$-split Cartan subgroup of G. The irreducible representations of $\mathrm{SL}(n,\mathbb{R})$ are described by Young tableaux. For example, for $1 \leq k \leq n-1$ and $n \geq 3$, the representations corresponding to the tableaux with one column of height $k \neq n/2$, or two equal columns of height $k \neq n/4, n/3, n/2, 2n/3, 3n/4$, or one line of length $k < n/2$ contain a (TNS) sub-representation. However, for each given semisimple Lie group, only finitely many irreducible representations do so.

Example 5.6. We describe now an example of a (TNS) $\mathbb{Z}^3$-action on a nilmanifold. Let $\mathfrak{n}$ be the 2-step nilpotent Lie algebra generated by $\{e_i; 1 \leq i \leq 10\}$, with the relations $[e_1, e_2] = e_5$, $[e_1, e_3] = e_6$, $[e_1, e_4] = e_7$, $[e_2, e_3] = e_8$, $[e_2, e_4] = e_9$, $[e_3, e_4] = e_{10}$, and all the other brackets between the generators are zero. Let $C = \mathrm{span}_{\mathbb{Z}}\{e_i\}$. Denote $N = \exp(\mathfrak{n})$ and $\Gamma = \exp(2C)$. Then N is a connected, simply connected nilpotent Lie group, and Γ is a cocompact lattice in N.

Consider the canonical representation of $\mathrm{SL}(4,\mathbb{Z})$ on $\mathrm{span}\{e_i; 1 \leq i \leq 4\}$. Then, using the relations between e_i's, we find a representation of $\mathrm{SL}(4,\mathbb{Z})$ on $\mathrm{span}\{e_i; 5 \leq i \leq 10\}$. Thus we have a representation of $\mathrm{SL}(4,\mathbb{Z})$ on $\mathfrak{n}$, and therefore an action on N, which preserves Γ. By the theorem of Prasad–Raghunathan, there are abelian subgroups generated by three hyperbolic matrices in $\mathrm{SL}(4,\mathbb{Z})$. One can verify that the $\mathbb{Z}^3$-action induced on the nilmanifold N/Γ is (TNS).

Cocycles over TNS-actions exhibit strong rigidity properties. One should contrast this with Theorems 3.1 and 3.5: while in the case of a $\mathbb{Z}$-action the cohomological information is described by *all* the periodic orbits, for some higher rank abelian actions it is determined by only *one* periodic orbit (or a few; we have to find the value of finitely many constants).

Theorem 5.7. *Let M be a torus and $\alpha : \mathcal{A} \times M \to M$ an Anosov (TNS) action. Let $\beta : \mathcal{A} \times M \to GL(n, \mathbf{R})$ be a small δ-Hölder cocycle over α. Then β is cohomologous to a constant cocycle, i.e., there is a δ-Hölder function $P : M \to GL(n, \mathbf{R})$ and a representation $\pi : \mathcal{A} \to GL(n, \mathbf{R})$ such that*

$$\beta(a, x) = P(ax)^{-1}\pi(a)P(x)\,.$$

Moreover, if α and β are C^K, $K = 1, 2, \ldots, \infty, \omega$, then P is $C^{K-\varepsilon}$.

The idea of the proof is to construct an invariant foliation by putting together the stable and unstable foliations of the generators of the skew-product action determined by the cocycle. For small cocycles, the holonomy of such a foliation is trivial. This implies that the cocycle is cohomologous to a constant cocycle.

For Hölder cocycles over toral TNS-actions with values in a compact Lie group, one has a complete classification [35]. Note that as a consequence of Theorem 4.1, a continuous transfer map between two such Hölder (smooth) cocycles is automatically Hölder (respectively smooth).

By a *Γ-extension* of an action $\alpha : G \times M \to M$ we mean a lift of α to an action on a principal Γ-bundle over M.

Theorem 5.8. *Assume that M is a torus and that the TNS-action α of $\mathcal{A} \cong \mathbb{Z}^k$ has a fixed point x_0.*

(a) The continuous cohomology classes of $\mathbb{T}^\kappa$-valued Hölder cocycles are in one-to-one correspondence with pairs of homomorphisms

$$\bar{H} : V \to \mathbb{T}^\kappa, \rho : \mathcal{A} \to \mathbb{T}^\kappa\,,$$

where

$$V := \pi_1(M, x_0)/\mathrm{Span}\{a_*(\omega)\omega^{-1} \mid \omega \in \pi_1(M, x_0), a \in \mathcal{A}\}$$

is a finite group (determined completely by the action).
In particular, there are TNS-actions that admit Hölder (actually, smooth) cocycles not cohomologous to a constant.

(b) Let Γ be a compact connected Lie group. The continuous cohomology classes of Hölder Γ-extensions of α are in one-to-one correspondence with conjugacy classes of pairs of homomorphisms (H, ρ), where

$$H : \pi_1(M, x_0) \to \Gamma, \quad \rho : \mathcal{A} \to \Gamma,$$

and H intertwines the actions $\alpha_* : \mathcal{A} \to \mathrm{Aut}(\pi_1(M, x_0))$ and $\rho_* : \mathcal{A} \to \mathrm{Aut}(\Gamma)$, $\rho_* := \mathrm{Int} \circ \rho$, $\mathrm{Int}_g(h) := ghg^{-1}$. That is,

$$H(\alpha_*(a)(\omega)) = \rho(a) H(\omega) \rho(a)^{-1}, \tag{5}$$

for all $a \in \mathcal{A}, \omega \in \pi_1(M, x_0)$.
For each such pair (H, ρ), the image of H is a finite group. Therefore there is a finite cover of the Γ-extension which is a trivial Γ-bundle and on which the lifted $\mathcal{A}$-action corresponds to a cocycle that is cohomologous to a constant.
Moreover, up to conjugacy, there are only finitely many homomorphisms H that satisfy (5) for some choice of ρ, and therefore only finitely many Γ-bundles to which the action α lifts.

Theorem 5.8 can be generalized to *smooth* cocycles over TNS-actions of infranilmanifolds and to the case when the action has no fixed points [35].

6 Applications to the Rigidity of Higher Rank Lattice Actions

The study of the rigidity of higher rank lattice actions on compact manifolds was initiated by Zimmer [46], [47]. We say that a smooth action π of a group Γ on a manifold M is $C^{k,l}$*-locally rigid* ($k, l \geq 0$) if any smooth action close (on a set of generators) to π in the C^k-topology is conjugate to π through a C^l-diffeomorphism. We say that π is $C^{k,l}$*-deformation rigid* if any C^k-small path deformation of π is conjugate to π through a path of C^l-diffeomorphisms.

Zimmer [48] proved *infinitesimal rigidity* (i.e., the cohomological triviality of the action on vector fields) of cocompact lattice actions on certain homogeneous manifolds. Lewis [23] proved the infinitesimal rigidity of the action of $SL(n, \mathbb{Z})$ on the n-torus. This latter result was subsequently simplified by Hurder [9]. Hurder proved the $C^{1,\infty}$-deformation rigidity of certain Anosov actions in [8]. Katok, Lewis and Zimmer obtained local and even global results for certain Anosov actions [16], [17], [18]. One such result is:

Theorem 6.1 (Hurder, Katok, Lewis, Zimmer). *The canonical linear action of $SL(n, \mathbb{Z})$, or of any of its subgroup of finite index, on the n-torus is $C^{1,\infty}$-locally rigid.*

Katok and Spatzier [22] extended this to a large class of actions:

Theorem 6.2 (Katok–Spatzier). *Let G be a linear semisimple Lie group all of whose simple factors have real rank at least 2. Let Γ be an irreducible lattice in G. Then any algebraic Anosov action of Γ on a nilmanifold is $C^{1,\infty}$-locally rigid, through a conjugacy which is C^1-close to the identity.*

In the same paper they also prove the local rigidity of some algebraic Anosov actions of $\mathbb{Z}^k$ ($k \geq 2$) on infranilmanifolds.

Global rigidity results for *Cartan* actions were obtained by Goetze and Spatzier [11]. These Anosov actions were first considered by Hurder [8].

Results for weakly hyperbolic higher rank lattice actions were obtained by Margulis and Qian [31]:

Theorem 6.3 (Margulis–Qian). *Let G be the connected component of a semisimple algebraic group without compact factors, with each simple factor of real rank at least 2. Let Γ be a lattice in G.*

(a) Let H be an algebraic group, Λ a cocompact lattice in H, $\pi_0 : G \to H$ a continuous (hence C^∞) homomorphism such that the centralizer $Z_H(\pi_0(G))$ of $\pi_0(G)$ in H is discrete (and therefore finite). Denote by ρ_0 the action of either G or Γ on H/Λ by left translations via π_0.
Then ρ_0 is $C^{1,\infty}$ rigid.
(b) Let N/Λ be a nilmanifold, $\pi_0 : \Gamma \to \mathrm{Aut}(N/\Lambda)$ a homomorphism that extends to a continuous homomorphism $\pi_0 : G \to \mathrm{Aut}(N)$, and let ρ_0 be the action of Γ on N/Λ determined by π_0. Denote by $d\pi_0$ the induced representation of G on the Lie algebra $\mathfrak{n}$ of N. Assume that $d\pi_0$ does not contain the trivial representation.
Then ρ_0 is $C^{1,\infty}$ rigid.

One ingredient in many of these proofs is the Cocycle Superrigidity Theorem of Zimmer [45], which under certain conditions implies that the derivative cocycle is measurably constant.

Note that the case when the action contains the trivial action is not covered by the above results. No complete answer to the question of local rigidity of lattice actions on compact manifolds can be given without understanding this case. The first rigidity result for such actions is proved in [32] (see [34] for this version):

Theorem 6.4. *Let $SL(n, \mathbb{Z})$ act on the torus $\mathbb{T}^n \times \mathbb{T}^d$ ($d \geq 1$) by $\rho(A)(x, y) = (Ax, y)$, where $A \in SL(n, \mathbb{Z})$, $x \in \mathbb{T}^n$, $y \in \mathbb{T}^d$. Then, for $n \geq 3$ and any integer $K \geq 1$, the action ρ is $C^{5,K-\varepsilon}$-deformation rigid.*

The action ρ is neither irreducible nor ergodic. In order to show the deformation rigidity of ρ one uses techniques different from the case of Anosov actions. A major role is played by Diff-valued cocycles and their cohomology.

In order to produce, from a perturbation of ρ, a cocycle which is Hölder in the C^K-metric we prove a variant of the Hirsch–Pugh–Shub structural stability theorem. Namely, we show that for some partially hyperbolic automorphism of the torus, the leafwise conjugation between the automorphism and a $C^{K+\mathrm{Lip}}$-small perturbation is Hölder, differentiable of order K along the leaves, and these derivatives are Hölder. One needs a Livsic type result for a cocycle over an $\mathrm{SL}(n,\mathbb{Z})$-action, instead of a $\mathbb{Z}$-action. The range of the cocycle is the group of diffeomorphisms of the torus, rather than $\mathbb{R}$ or a Lie group. We use Theorem 3.6, and show that the isolated closing conditions are satisfied. Since the canonical linear action of $SL(n,\mathbb{Z})$ on $\mathbb{T}^n$ is generic Anosov, we can apply Theorem 3.10 and obtain the triviality of the cocycle.

In [34] we improve Theorem 6.4 in the case of one-dimensional center fibers:

Theorem 6.5. *Let $\Gamma < SL(n,\mathbb{Z})$ be a subgroup of finite index and let ρ be the action of Γ on $\mathbb{T}^n \times \mathbb{T}$ given by $\rho(A)(x,y) = (Ax,y)$, $x \in \mathbb{T}^n, y \in \mathbb{T}, A \in \Gamma$. Then, for $n \geq 3$ and any integer $K \geq 1$, the action ρ is $C^{2,K-\varepsilon}$-locally rigid.*

One fact proved in [34] is that for certain higher rank *abelian* partially hyperbolic actions of product type, the sum of the stable and unstable distributions of any regular element of a perturbation is integrable. In view of the situation for Lie group valued cocycles [19], one would expect the leaves of the integral lamination to be closed manifolds covering the base simply, thus obtaining a conjugacy between the perturbation and a product action. This amounts to showing that the only small solutions of a particular system of equations in the homeomorphism (or diffeomorphism) group of the fiber, determined by the holonomy of the integral lamination, are the trivial ones. We obtain the above conclusion only under the supplementary assumption that one regular element of the perturbation has a pointwise fixed center leaf. For a higher rank *lattice* action on a compact manifold, this is a consequence of Stowe's theorem [42]. To show that the conjugacy we obtain for the perturbation of the abelian group is a conjugacy for the whole Γ action, we use Kazhdan's property (T).

Similar results for analytic actions were obtained by Zeghib [44]:

Theorem 6.6 (Zeghib). *Let Γ be a finite-index subgroup of $\mathrm{SL}(n,\mathbb{Z})$ with $n \geq 3$, let N be a compact, real-analytic manifold. Extend the canonical action of Γ on $\mathbb{T}^n$ to an action ρ_0 of Γ on $\mathbb{T}^n \times N$ by letting Γ act trivially on the second factor. Then any analytic action ρ of Γ on $\mathbb{T}^n \times N$ that is C^1-close to ρ_0 must be analytically conjugate to ρ_0.*

The approach is completely different: the theorem of Stowe [42] implies that ρ has fixed points, and a theorem of Cairns and Ghys [4] states that ρ can be linearized near any fixed point. The key to the proof is a study of maximal open sets to which the linearization can be extended by analytic continuation.

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