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RIGIDITY OF PARTIALLY HYPERBOLIC ACTIONS OF PROPERTY (T) GROUPS

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Abstract. We show that volume-preserving perturbations of some product actions of property (T) groups exhibit a "foliation rigidity" property, which reduces the partially hyperbolic action to a family of hyperbolic actions. This is used to show that certain partially hyperbolic actions are locally rigid.

1. **Introduction and Main Results.** The purpose of this paper is to prove the local rigidity of a wider class of measure preserving partially hyperbolic product actions of higher rank lattices. It extends the results of [11]. We prove here that C^1-small perturbations of certain partially hyperbolic actions are conjugated to the original action. This is an improvement over the previous results, which required at least C^2-closeness.

The new tool is our "foliation-rigidity" result for actions of property (T) groups, Theorem 1.3.

We combine it with the local rigidity results for hyperbolic actions obtained by A. Katok and R. Spatzier [6] (see Theorem 1.6 below) to obtain the following:

Theorem 1.1. *Let Γ be an irreducible higher rank lattice, and $\alpha : \Gamma \to \mathrm{Diff}(M)$ a linear Anosov action on the infranilmanifold M. Consider the product action $\rho_0 : \Gamma \to \mathrm{Diff}(M \times S^1)$, $\rho_0(\gamma) = \alpha(\gamma) \times \mathrm{Id}_{S^1}$. Fix a smooth ρ_0-invariant volume μ on $M \times S^1$ and $K \geq 1$.*

Then ρ_0 is C^{1,K^-}-locally rigid in $\mathrm{Diff}^\infty_\mu(M \times S^1)$.

See Corollary 1.5 for a more general statement.

Let us recall the rigidity properties we are considering.

Unless specified otherwise, we assume that all manifolds and maps are smooth.

Notations. 1. Throughout this paper by a C^r*-lamination* we mean a topological foliation whose leaves are immersed C^r-submanifolds that vary continuously in the C^r-topology. A (continuous) *foliation* stands for a C^0-foliation.

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2. By C^{k^-} we denote the class of functions that are $C^{k-\varepsilon}$ for any $\varepsilon > 0$. The C^{k-}-topology stands for the coarsest topology for which the inclusions $C^{k-} \subset C^{k-\varepsilon}$ are continuous for each $\varepsilon > 0$. However, by C^{1^-} we mean C^1.

Definition 1.2. Let Γ be a finitely generated discrete group, M a compact manifold, and $\rho, \widetilde{\rho} : \Gamma \times M \to M$ two C^∞-actions. Fix a finite set of generators $\{\gamma_i\}$ of Γ. We say that ρ is C^L*-close* to $\widetilde{\rho}$ if the C^∞-diffeomorphisms $\rho(\gamma_i)$ and $\widetilde{\rho}(\gamma_i)$ are close in the C^L-topology for all i. A C^L*-perturbation* of the action ρ is a C^∞-action C^L-close to ρ. A C^L*-deformation* of the action ρ is a C^L-continuous path of C^∞-actions ρ_t, $0 \le t \le 1$, with $\rho_0 = \rho$. An action ρ is said to be $C^{L,K}$*-locally rigid* if any C^L-perturbation of ρ contained in a sufficiently small C^L-neighborhood of ρ is conjugated to ρ by a C^K-diffeomorphism which is C^0-close to the identity. An action ρ is said to be $C^{L,K}$*-deformation rigid* if any C^L-deformation of ρ contained in a sufficiently small C^L-neighborhood of ρ is conjugated to ρ by a continuous path of C^K-diffeomorphisms C^0-close to the identity.

Here are a few previous rigidity results for non-hyperbolic actions. Fix $K \ge 1$. C^{2,K^-}-local rigidity results for actions similar to ρ_0 were obtained in [11], where the actions are more particular, but there is no need to require the existence of an invariant volume. The center direction is still one-dimensional. C^{5,K^-}-deformation rigidity results for product actions having center direction of arbitrary dimension were obtained in [10, 11]. Local rigidity results in the analytic category were obtained by Zeghib [19]. G. Margulis and N. Qian proved $C^{1,\infty}$-local rigidity for weakly-hyperbolic actions in [8].

The main new ingredient is our result dealing with actions of groups having Kazdahn's property (T). One result used in its proof is the sufficient conditions given by C. Pugh and M. Shub [15] for the ergodicity of a partially hyperbolic diffeomorphism (see Theorem 4.3).

Theorem 1.3. *Let Γ be a discrete group having Kazdahn's property (T). Assume that each of the finite index subgroups of Γ has vanishing cohomology for any finite dimensional representation. Let $\alpha : \Gamma \to \mathrm{Diff}^K(M)$ be an action generated by finitely many Anosov diffeomorphisms. [Note that under generic conditions, this holds provided there is one Anosov diffeomorphism in $\alpha(\Gamma)$.]*

Denote by $\rho_0 : \Gamma \to \mathrm{Diff}^K(M \times S^1)$ the action $\alpha \times \mathrm{Id}_{S^1}$. Fix a smooth ρ_0-invariant volume μ on $M \times S^1$ and $K \ge 2$.

If α has a periodic point (i.e., a point whose Γ-orbit is finite), then any volume preserving action $\rho : \Gamma \to \mathrm{Diff}^K_\mu(M \times S^1)$ which is C^1-close to ρ_0 is conjugated to a "foliated" action. That is, there is a C^{K^-}-lamination $\mathcal{H} = \{\mathcal{H}_y\}_{y \in S^1}$ each leaf of which is preserved by ρ. On each leaf, the induced action is hyperbolic. This invariant lamination is close to $\{M \times \{y\}\}_{y \in S^1}$.

The lamination $\mathcal{H}$ is spanned by the stable and unstable foliations of a partially hyperbolic map $\rho(\gamma_*)$, where γ_* is a fixed element with $\alpha(\gamma_*)$ Anosov. Therefore, the $\mathcal{H}$-holonomy between the center leaves of $\rho(\gamma_*)$ is C^{K-1} (see 2 of Theorem 2.1). The $\mathcal{H}$-holonomy between the verticals $\left\{\{x\} \times S^1\right\}_{x \in M}$ is Lipschitz (see 4 of Lemma 3.3).

Examples of groups Γ that satisfy the hypothesis of the Theorem are lattices in higher rank Lie groups.

One actually needs the vanishing cohomology condition only for a normal subgroup of finite index $\Gamma_0 \lhd \Gamma$ for which $\alpha|_{\Gamma_0}$ has a fixed point.

A more precise result can be obtained if the action α is rigid. We need a slight strengthening of the local rigidity property:

Definition 1.4. Let Γ be a discrete group, M a compact manifold, $K, L \geq 1$, and $\alpha : \Gamma \to \mathrm{Diff}^K(M)$ an action. The action α is called *continuously* $C^{L,K}$-locally rigid if it is $C^{L,K}$-locally rigid, and the conjugacy varies continuously in the C^{K^-} topology when the perturbation varies continuously within a compact set in the C^K topology.

Corollary 1.5. *Let $L \geq 1$ be fixed and assume that the hypothesis of Theorem 1.3 holds.*

If α is also $C^{L^-,K}$-locally rigid, then any action $\rho \in \mathrm{Diff}^\infty_\mu(M \times S^1)$ which is C^L-close to ρ_0 is conjugated to ρ_0 by a homeomorphism $\Phi : M \times S^1 \to M \times S^1$ such that $\Phi(\cdot, y) \in C^K(M, M \times S^1)$ for each $y \in S^1$ and $x \in M \mapsto \Phi(x, \cdot) \in C^{K-1}(S^1, M \times S^1)$ is continuous.

If, moreover, α is continuously *$C^{L^-,K-1}$-locally rigid, then ρ_0 is $C^{L,(K-1)^-}$-locally rigid in $\mathrm{Diff}^\infty_\mu(M \times S^1)$.*

Remark. One can state a similar result for finitely differentiable actions, provided the local-rigidity of α holds in that class. This is the case for many of the known rigidity results for Anosov actions (including the actions considered in Theorem 1.1).

Theorem 1.1 is a special case of Corollary 1.5 (take $L = 1$), in view of the following result (the continuity of the local rigidity follows from the proof):

Theorem 1.6 (Katok-Spatzier, [6]). *Any linear Anosov action on an infranilmanifold of an irreducible lattice in a linear semisimple Lie groups G all of whose factors have real rank at least 2 is (continuously) $C^{1,\infty}$-locally rigid.*

This paper is organized as follows: in §2 we recall a few basic results about partially hyperbolic maps. In §3 we prove Theorem 1.3 and its corollary, using a few lemmas whose proof is given in §4.

2. **Preliminaries.**

We recall first several standard facts about partially hyperbolic diffeomorphisms.

Let X be a compact, connected, boundaryless manifold. Denote by TX the tangent bundle of X. A C^1-diffeomorphism $f : X \to X$ is called *partially hyperbolic* if the derivative $Tf : TX \to TX$ leaves invariant a continuous splitting $TX = E^s \oplus E^c \oplus E^u$, $E^s \neq 0 \neq E^u$, such that, with respect to a fixed Riemannian metric on TX:

$$\|T^u f^{-1}\| < 1, \quad \|T^s f\| < 1, \tag{2.1}$$

$$\|T^s_p f\| < m(T^c_p f), \quad \|T^c_p f\| < m(T^u_p f) \quad \text{for all } p \in X, \tag{2.2}$$

where $m(L) := \inf\{\|Lv\| \mid \|v\| = 1\} = \|L^{-1}\|^{-1}$ is the *conorm* of the linear transformation L.

E^s, E^c and E^u are called the *stable*, *center*, respectively *unstable* distributions. If the center distribution $E^c = 0$, then f is called an *Anosov* (or *hyperbolic*) diffeomorphism.

The distributions E^s and E^u are tangent to unique laminations W^s_f and W^u_f which have C^1 leaves. If the diffeomorphism f is C^K, $1 \leq K \leq \infty$, then the laminations W^s_f and W^u_f are C^K as well [1, 13]. These are called the *stable* and *unstable* foliations. [To be precise, we should call these *laminations*, but this is not the standard terminology. We will adhere to this convention when speaking about the *center foliation* as well.]

Let $f : X \to X$ be a partially hyperbolic diffeomorphism. f is *r-normally hyperbolic* if the center distribution E^c is integrable to a C^r-boundaryless leaf immersion (see [4, §6]) and

$$m(T_p^u f) \geq \|T_p^c f\|^k, \qquad \|T_p^s f\| \leq m(T_p^c f)^k, \quad k = 0, \dots, r. \tag{2.3}$$

Roughly speaking, the center distribution integrates to a "lamination" that can have self-intersections; its leaves are C^r. This set-up is necessary in order to assure that r-normal hyperbolicity is a C^1-open condition.

We recall the results of [4, Theorems 6.1, 6.8, 7.1, 7.2] about partially hyperbolic diffeomorphisms and their small perturbations. We describe only the case that will be of interest in the sequel. In the case of hyperbolic diffeomorphisms, these are the classical results of Anosov ([1]). See also the Remark following the Theorem.

The partially hyperbolic diffeomorphism $f \in \mathrm{Diff}(X)$ is said to satisfy the *r-th order center-bunching conditions* if for all $p \in X$ and $0 \leq \ell \leq r$

$$\|T_p^s f\| \|T_p^c f\|^\ell < m(T_p^c f) \text{ and } \|T_p^c f\| < m(T_p^u f) m(T_p^c f)^\ell.$$

Theorem 2.1 (Hirsch-Pugh-Shub, [4]). *Let X be a compact manifold and $f \in \mathrm{Diff}^r(X)$, $r \geq 1$, a diffeomorphism which is r-normally hyperbolic at a C^r-lamination W_f^c having compact leaves.*

1. *Through each leaf of W_f^c there exists a C^r* center-stable *manifold. The center-stable manifold through $x \in X$ consists of those points whose forward f-orbit does not stray away from the orbit of $W_f^c(x)$. Hence, since the leafs of the lamination W_f^c are compact, each center-stable manifold is a union of center leafs; the center-stable manifolds form the* center-stable *lamination W_f^{cs}. A similar statement holds for the* center-unstable *lamination, W_f^{cu}.*
2. *There exists a C^r* stable *lamination W_f^s whose leafs lie in those of W_f^{cs}. The points of a stable leaf are characterized by sharp forward asymptoticity. If f satisfies the $(r-1)$-th order center-bunching conditions then the stable distribution is C^{r-1} on each center-stable leaf. In particular, the holonomy maps determined by the stable lamination inside the center-stable leafs are C^{r-1}. A similar statement holds for the* unstable *lamination, W_f^u.*
3. *If $g \in \mathrm{Diff}^r(X)$ is C^1-close to f, then g is r-normally hyperbolic at a unique C^r-lamination W_g^c, and the stable, unstable and center laminations of g converge in C^r to those of f as g converges to f in the C^r-topology. The stable (unstable) holonomy maps within the center-stable (respectively, center-unstable) leafs of g converge in C^{r-1} to those of f, as g converges in C^r to f.*
4. *Moreover, if W_f^c is a C^r-foliation, then in the case 3 there exists a* leaf-conjugacy *$H \in \mathrm{Homeo}(X)$ between (f, W_f^c) and (g, W_g^c): H maps the leaves of W_f^c to those of W_g^c and $W_g^c(H \circ f(x)) = W_g^c(g \circ H(x))$. H is a C^r diffeomorphism of each leaf of W_f^c onto its image, varying continuously in C^r with the leaf. For $x \in X$, $W_g^c(H(x))$ is uniquely characterized by the fact that its g-orbit does not stray away from the f-orbit of $W_f^c(x)$. Modulo the choice of a normal bundle to W_f^c, H is uniquely determined. If g converges to f in the C^r-topology then H converges to the identity in the C^r-topology along the leafs of W_f^c and to Id_X in C^0.*

Here "never strays away" means that $g^n(W^c_g(H(x)))$ stays within a tubular neighborhood of predetermined small size of $f^n(W^c_f(x))$, for each $n \in \mathbb{Z}$.

Remark. The statement in 2 above about the smoothness of the stable distribution within the center-stable leaves follows from the C^r-section theorem [4, Theorem 3.5] (applied in this case for C^{r-1}). The compactness of the base space can be replaced by the appropriate uniformities. The continuous dependence of these holonomies described in 3 follows from a straight-forward generalization of the similar continuity contained in the C^r-section Theorem. Theorem B of [16] proves that the holonomy of W^s inside W^{cs} is C^{r-1} under milder conditions.

We introduce now a few notions related to Theorem 4.3 of Pugh and Shub [15].

If the distributions $E^u \oplus E^c$, E^c, and $E^c \oplus E^u$ of a partially hyperbolic diffeomorphism f are tangent to continuous foliations with C^1 leaves W^{cu}, W^c, respectively W^{cs}, and if W^c and W^u subfoliate W^{cu}, while W^c and W^s subfoliate W^{cs}, then f is said to be *dynamically coherent.*

The *center bolicity* of f is the ratio $b = \|T^c f\|/m(T^c f)$. The map f is said to be *center bunched* if b is close to 1 (see [15, §4] for the precise meaning of "close"). This is a stricter condition that the "relative" partial hyperbolicity introduced in (2.2).

By Theorem 2.1, r-normal hyperbolicity to a C^r lamination with compact leaves is a C^1-open property in $\mathrm{Diff}^r(X)$ (see [4, Theorem 6.1] for the general case). The property of being center bunched is preserved by C^1-small perturbations. By [14, Proposition 2.3], and [4, Theorem 7.2], dynamical coherence is stable under C^1-small perturbations, provided the center lamination of the original diffeomorphism is a C^1-foliation.

In conclusion: for each $\gamma \in \Gamma$ such that $\alpha(\gamma)$ is Anosov, $\rho_0(\gamma) = \alpha(\gamma) \times \mathrm{Id}_{S^1}$ is r-normally hyperbolic, where r is limited only by the smoothness of the map. $\rho_0(\gamma)$ satisfies the center-bunching conditions of any order. A C^1-small perturbation $f \in \mathrm{Diff}^r(M \times S^1)$ of $\rho_0(\gamma)$ is r-normally hyperbolic, center bunched, dynamically coherent, and leaf-wise conjugated to $\rho_0(\gamma)$.

3. Proof of the Main Theorem.

In this section we are going to prove Theorem 1.3 and its corollary, based on a few Lemmas. These Lemmas will be proven in §4.

Definition 3.1. Consider a partially hyperbolic diffeomorphism $f \in \mathrm{Diff}^1(X)$, where X is a compact manifold. Denote by W^s_f and W^u_f its stable, respectively unstable foliations. We say that $x \in X$ and $y \in X$ are (u,s)*-accessible for* f if there is a continuous, piecewise-C^1 path connecting x and y each segment of which is in either a stable or an unstable leaf of f. Introduce the equivalence relation

$$x \sim_f y \iff x \text{ and } y \text{ are } (u,s)\text{-accessible for } f.$$

Definition 3.2. Assume $X = M \times S^1$ and $f \in \mathrm{Diff}^1(M \times S^1)$ is C^1-close to $A \times \mathrm{Id}_{S^1}$, where $A \in \mathrm{Diff}^1(M)$ is Anosov.

Denote the center foliation of f by W^c and let $q : M \times S^1 \to \widehat{M} := (M \times S^1)/W^c$ be the quotient map. By the Hirsch-Pugh-Shub Theorem 2.1, $\widehat{M}$ is homeomorphic to M and the action $\widehat{f}$ induced by f is conjugated to A.

Let $\mathcal{C}_0$ be a center leaf of f. For $x, y \in \mathcal{C}_0$ define the equivalence relation $x \sim_{f,0} y$ if $x \sim_f y$ through a path whose image in $(\widehat{M}, q(\mathcal{C}_0))$ is contractible with fixed end-points.

We might drop from the notations the reference to the diffeomorphism, when it is clear which one we have in mind.

Notation. For an equivalence relation $\cong$, we will denote by $[x]_\cong$ the $\cong$-equivalence class of x.

Lemma 3.3. *Let $K \geq 1$ and assume $A \in \mathrm{Diff}^K(M)$ is an Anosov diffeomorphism, $f \in \mathrm{Diff}^K(M \times S^1)$ is C^1-close to $A \times \mathrm{Id}_{S^1}$, $\mathcal{C}_0$ is a center leaf of f, and $f|_{\mathcal{C}_0} = Id_{\mathcal{C}_0}$. Then:*

1. *Each $\sim$-equivalence class is f-invariant and projects onto $\widehat{M}$ via q.*
2. *Any $\sim_0$-equivalence class that contains more than one point is open. In particular, if $\mathcal{C}_0$ is not a single equivalence class, then there are classes consisting of only one point.*
3. *If $M \times S^1$ is not a single $\sim$-equivalence class, then for each $y \in \mathcal{C}_0$ the equivalence class $[y]_\sim$ is either a compact C^{K^-} submanifold $\mathcal{H}_y \subset M \times S^1$ for which $q|_{\mathcal{H}_y} : \mathcal{H}_y \to \widehat{M}$ is a homeomorphism, or an open set $\mathcal{U}_y$ bounded by submanifolds described above. In particular, $\mathcal{C}_0/\sim = \mathcal{C}_0/\sim_0$.*

   We call the equivalence classes of the form $\mathcal{H}_y$ horizontal leaves *of f.*
4. *Assume that $K \geq 2$ and f preserves a smooth volume μ.*
   - *With respect to the measure μ, the diffeomorphism f is ergodic on each $\mathcal{U}_y$.*
   - *If there is an open set $\widetilde{I} \subset M \times S^1$ laminated by $\{\mathcal{H}_y\}_{y \in I}$, $I \subset \mathcal{C}_0$, then for each $y \in I$ there is an f-invariant volume form μ_y on $\mathcal{H}_y$. In particular, f is ergodic on each $\mathcal{H}_y$ with respect to the measure μ_y. Moreover, the lamination by $\{\mathcal{H}_y\}$ of $\widetilde{I}$ is absolutely continuous, in the following sense. Label the leaves by their intersection with a vertical segment $V = (\{x_*\} \times S^1) \cap \widetilde{I}$. Then any measurable set $U \subset \widetilde{I}$ has zero μ-measure if and only if for almost each leaf (measured on V with respect to the Lebesgue measure), the μ_y measure of $U \cap \mathcal{H}_y$ is zero. This follows from the fact that the holonomy maps between vertical segments are uniformly Lipschitz (and therefore absolutely continuous).*

The result of Pugh and Shub [15] (see Theorem 4.3 below) is needed to prove the first statement in part 4 above.

The next two lemmas are used in connection with the property (T) of the group Γ.

Notations. 1. Given a smooth measure μ on the manifold X which is positive on open sets, there is a homomorphism from the group of diffeomorphisms on X to the unitaries of $L^2(X, \mu)$,

$$f \in \mathrm{Diff}^1(X) \mapsto U_f \in \mathcal{U}(L^2(X, \mu)), U_f(\phi) = \phi \circ f^{-1} \cdot \Delta_f^{1/2},$$

where $\Delta_f = \Delta_f(\mu) = \frac{\partial f^* \mu}{\partial \mu}$ is the Radon-Nikodym derivative of $f^*\mu(\Omega) := \mu(f^{-1}(\Omega))$ with respect to μ.

Although in the end we will deal with volume preserving actions, in which case $U_f(\phi) = \phi \circ f^{-1}$, some of the results are more general.

2. It will be convenient to describe subsets of S^1 by inequalities. To do this we are going to specify the sets in $\mathbb{R}$ and use the quotient map $\mathbb{R} \to \mathbb{R}/\mathbb{Z} \cong S^1$ (sometimes without mentioning it).

3. For a C^1 function $w : M \to S^1$, denote by $\|w\|_{(1)} := \|d\,w\|_{C^0}$ the norm of its differential (considered with respect to some fixed metrics on M and S^1).

Definition 3.4. Call a set $\widetilde{I} \subset M \times S^1$ *horizontal* if there are $\mathrm{bot}, \mathrm{top} : M \to \mathbb{R}$ such that their images in S^1, $\mathrm{bot}_0, \mathrm{top}_0 : M \to S^1$ are C^1 mappings and

$$\begin{gathered}\widetilde{I} = \{(x,y) \in M \times S^1 \mid x \in M, \mathrm{bot}(x) \le y \le \mathrm{top}(x)\}, \\ \text{where} \quad \mathrm{bot}(x) < \mathrm{top}(x) \le \mathrm{bot}(x) + 1,\end{gathered} \tag{3.1}$$

We call such a set C-*flat* if $\|\mathrm{bot}_0\|_{(1)} \le C, \|\mathrm{top}_0\|_{(1)} \le C$.

Associate to a horizontal set $\widetilde{I}$ the function

$$\phi_{\widetilde{I}}(x,y) := \chi_{\widetilde{I}}(x,y) \cdot \frac{2\min\{|y - \mathrm{top}(x)|, |y - \mathrm{bot}(x)|\}}{\mathrm{top}(x) - \mathrm{bot}(x)}, \quad (x,y) \in M \times S^1 \tag{3.2}$$

(the difference is meant in $\mathbb{R}$).

Lemma 3.5. *Let $A \in \mathrm{Diff}^1(M)$ be hyperbolic and assume given an $A \times \mathrm{Id}_{S^1}$-invariant smooth volume μ on $M \times S^1$ which is positive on open sets.*

Fix $\varepsilon > 0$. Then there is a $\delta = \delta(A, \mu) > 0$ such that if $f \in \mathrm{Diff}^1(M \times S^1)$ satisfies $\mathrm{dist}_{C^1}(f, A \times \mathrm{Id}_{S^1}) \le \delta$ and $\widetilde{I}$ is any set of the form (3.1) *such that $f(\widetilde{I})$ is also of the form* (3.1)*, then*

$$\|U_f(\phi_{\widetilde{I}}) - \phi_{f(\widetilde{I})}\|_{L^2} \le \varepsilon \|\phi_{\widetilde{I}}\|_{L^2}.$$

Remark. It is not hard to see that given A as above, for any $\delta > 0$ there is $c_1 = c_1(A) > 0$ such that if $\mathrm{dist}_{C^1}(f, A \times \mathrm{Id}_{S^1}) \le c_1$ then $f(\widetilde{I})$ is horizontal and δ-flat for any horizontal c_1-flat set $\widetilde{I}$.

Moreover, by 3 of Theorem 2.1 used for $r = 1$, any horizontal leaf of f is c_1-flat provided $\mathrm{dist}_{C^1}(f, A \times \mathrm{Id}_{S^1})$ is small enough.

Lemma 3.6. *Assume $\widetilde{I} \subset M \times S^1$ is described by* (3.1) *and μ is a smooth volume on $M \times S^1$. Then for any $\varepsilon > 0$ there is a $\delta = \delta(\mu) > 0$ such that if*

$$\|\psi - \phi_{\widetilde{I}}\|_{L^2} \le \delta \|\phi_{\widetilde{I}}\|_{L^2} \tag{3.3}$$

then for any $\lambda \in \mathbb{C}$

$$\mu(\psi^{-1}(\lambda) \cap \widetilde{I}) \le \varepsilon \mu(\widetilde{I}). \tag{3.4}$$

With these preparations we are ready to prove the main Theorem.

Proof of Theorem 1.3.

We will specify the C^1-distance between ρ and ρ_0 along the way.

Let $x_0 \in M$ be a periodic point of α, and denote by Γ_0 the finite index normal subgroup of Γ that fixes each point of the (finite) orbit of x_0.

First we obtain a fixed center leaf (needed in Lemma 3.3). Because α contains Anosov elements, $\{x_0\} \times S^1$ is an isolated set of fixed points for $\rho_0|_{\Gamma_0}$. In view of the vanishing cohomological condition imposed on Γ_0 we can apply Stowe's Theorem [18, Thm. 2.1]. Hence, there is a C^1-neighborhood $\mathcal{U}$ of ρ_0 such that for each ρ in $\mathcal{U}$ the restriction $\rho|_{\Gamma_0}$ has a set of fixed points $\mathcal{C}_0$ diffeomorphic to $\{x_0\} \times S^1$. We assume from now on that $\rho \in \mathcal{U}$.

Pick now one of the Anosov elements $\alpha(\gamma_0)$. There is an integer $p_0 \ne 0$ such that $\gamma_0^{p_0} \in \Gamma_0$. Let $A_0 := \alpha(\gamma_0^{p_0})$, $f_0 := \rho(\gamma_0^{p_0})$. Assume ρ is C^1-close enough to ρ_0 so that Theorem 2.1 applies to $A_0 \times \mathrm{Id}_{S^1}$ and f_0. Since f_0 is partially hyperbolic and leaf-wise conjugated to $A_0 \times \mathrm{Id}_{S^1}$, we conclude that $\mathcal{C}_0$ is a center leaf of f_0.

The other assumption we made about Γ is Kazhdan's property (T). Recall its definition [3, Proposition 14 in Ch. 1]:

Definition 3.7. A discrete group Γ has the *property (T)* if there exists a finite set $S \subset \Gamma$ and $\delta > 0$ such that any unitary representation $\pi : \Gamma \to \mathcal{U}(H)$ that has a nonzero (δ, S)-invariant vector has a nonzero invariant vector as well. [A vector $\xi \in H$ is (δ, S)-invariant if $\|\pi(a)\xi - \xi\| \leq \delta\|\xi\|$ for all $a \in S$.]

Remarks. 1. The following is a consequence of the property (T) [3, Proposition 16 in Ch. 1]:

Assume the group Γ has the property (T) relative to the finite set $S \subset \Gamma$. Then, for any $\varepsilon > 0$ there is a $\delta_T = \delta_T(\varepsilon) > 0$ such that given a unitary representation $\pi : \Gamma \to \mathcal{U}(H)$ and a (δ_T, S)-invariant vector $\xi \in H$, $\xi \neq 0$, there exists an invariant vector $\xi' \neq 0$ with $\|\xi - \xi'\| \leq \varepsilon\|\xi\|$.

2. By reducing the value of δ_T above, one can replace the set S by any finite generating set.

3. Finite index subgroups of property (T) groups have property (T) as well.

Pick a finite family $\{\widetilde{\gamma}_j\}_{1\leq j\leq\widetilde{\kappa}}$ of Anosov generators of $\alpha(\Gamma)$. Then there is a finite family $\{\gamma_i\}_{1\leq i\leq\kappa} \subset \Gamma_0$ of (finite) products of the $\widetilde{\gamma}_j$'s that generate Γ_0. Let $A_j := \alpha(\widetilde{\gamma}_j)$, $f_j := \rho(\widetilde{\gamma}_j)$ and $g_i := \rho(\gamma_i)$.

Take $\varepsilon = 1/2$ in Lemma 3.6. Use the $\delta_{L3.6}$ provided by that Lemma as the ε in 1. of the previous Remark, applied to Γ_0 and $S = \{\gamma_i\}$, the generator set described above. The δ_T obtained this way is used below in (3.5).

If ρ is C^1-close enough to ρ_0, we can apply successively Lemma 3.5 (note the Remark following the Lemma) to conclude that there is a $c_1 > 0$ such that:

$$\begin{aligned} &\text{if } \widetilde{I} \text{ is any } c_1\text{-flat set of the form (3.1), then} \\ &\|U_{g_i}(\phi_{\widetilde{I}}) - \phi_{g_i(\widetilde{I})}\|_{L^2} \leq \delta_T \|\phi_{\widetilde{I}}\|_{L^2}, 1 \leq i \leq \kappa \end{aligned} \tag{3.5}$$

We first show that ρ is not ergodic. See $M \times S^1$ as $\widetilde{I}$ with top $\equiv 1$, bot $\equiv 0$ (viewed in $\mathbb{R}$), and consider the corresponding function given by (3.2). This function is invariant under ρ_0, hence it is almost invariant for ρ under a set of generators (in the sense of Lemma 3.5; since now $\widetilde{I}$ is a large set, the almost invariance is immediate). Therefore, by property (T), ρ has an invariant L^2-function, which, in view of Lemma 3.6, cannot be constant.

We apply Lemma 3.3 to the element $f_0 \in \rho(\Gamma_0)$ defined earlier. Since ρ is not ergodic, the decomposition $\{\mathcal{F}_\xi\}_{\xi\in\mathcal{C}_0/\sim_{f_0}}$ of $M \times S^1$ into $\sim_{f_0}$ classes cannot be a single class.

By reducing the C^1-distance between f_0 and $A_0 \times \mathrm{Id}_{S^1}$, we can assure that each connected subset of $M \times S^1$ bounded by horizontal leaves of f_0 is a horizontal c_1-flat set.

Abbreviate $\sim_{f_0}$ to $\sim$, $\sim_{f_0,0}$ to $\sim_0$, and define

$$B = \{y \in \mathcal{C}_0 \mid \rho(\gamma)\mathcal{F}_\xi \neq \mathcal{F}_\xi \text{ for some } \gamma \in \Gamma_0, \text{ where } \xi = [y]_{\sim_0}\},$$

$$\widetilde{B} = \left\{z \in M \times S^1 \mid z \sim y, y \in B\right\}.$$

Assume $\widetilde{B}$ is non-empty. Then $\widetilde{B}$, the union of $\sim$-equivalence classes that are not preserved by Γ_0, is $\rho|_{\Gamma_0}$-invariant, because so is its complement. $\widetilde{B}$ is also open, because its complement is closed (the boundary of $M \times S^1 \setminus \widetilde{B}$ consists of horizontal leaves $\mathcal{H}_y$; these leaves are in $M \times S^1 \setminus \widetilde{B}$ because the Γ_0-action on $\mathcal{C}_0$ is trivial).

Let $\widetilde{I}$ be a connected component of $\widetilde{B}$. Since $\rho|_{\Gamma_0}$ acts as the identity on $\mathcal{C}_0$, $\widetilde{I}$ is Γ_0-invariant. In particular, it is a union of $\sim$-equivalence classes and thus has to be c_1-flat. Therefore, (3.5) shows that $\phi_{\widetilde{I}}$ is a $(\delta_T, \{\gamma_i\})$-invariant vector for $\rho|_{\Gamma_0}$.

Property (T) implies that there is an invariant function $\psi \in L^2(M \times S^1, \mu)$ such that $\|\psi - \phi_{\widetilde{I}}\|_{L^2} \le \delta_{L3.6}\|\phi_{\widetilde{I}}\|_{L^2}$. By Lemma 3.6, $\psi|_{\widetilde{I}}$ cannot be a constant, and its preimages split $\widetilde{I}$ into $\rho|_{\Gamma_0}$-invariant sets of volume at most $\mu(\widetilde{I})/2$. One can prove the following (see the proof later):

Claim. *There is an open horizontal subset of $\widetilde{I}$, say $\widetilde{I}_0$, on which ψ is μ-a.e. constant.*

Take the union of all connected open sets containing $\widetilde{I}_0$ on which ψ is a.e. constant. Since $\rho|_{\Gamma_0}$ fixes $\mathcal{C}_0$, this set has to be $\rho|_{\Gamma_0}$-invariant; hence, so is its boundary, which must consist of horizontal leaves of f_0. We conclude that there is a $\rho|_{\Gamma_0}$-invariant horizontal leaf $\mathcal{H}_y$ of f_0 lying inside $\widetilde{I}$. But this contradicts the fact that $\widetilde{I}$ was a connected component of $\widetilde{B}$.

Thus, we conclude that $\widetilde{B}$ is empty, and therefore each $\sim$-equivalence classes of f_0 is $\rho|_{\Gamma_0}$-invariant. Assume that there is an equivalence class of the type $\mathcal{U}_y$ (i.e., which is not a horizontal leaf; see Lemma 3.3, part 3). Apply the above argument to the set $\widetilde{I} := \mathcal{U}_y$. We obtain that there is a ($\rho|_{\Gamma_0}$-invariant) horizontal leaf inside $\mathcal{U}_y$, thus contradicting the definition of $\mathcal{U}_y$.

In conclusion, the $\sim$-equivalence classes are all horizontal leaves $\mathcal{H}_y$, $y \in \mathcal{C}_0$, and each of them is $\rho|_{\Gamma_0}$-invariant. Denote this foliation by $\mathcal{H} = \{\mathcal{H}_y \mid y \in \mathcal{C}_0\}$.

It remains to show that each leaf of this foliation is actually preserved by the whole Γ-action ρ. Denote $F = \Gamma/\Gamma_0$ and let r be the order of the finite group F. Note that the group F is determined by the unperturbed action α.

Let $\widetilde{\gamma}_j$ be one of the Anosov generators of Γ. Then $\widetilde{\gamma}_j^r \in \Gamma_0$, hence the stable and unstable foliations of $\rho(\widetilde{\gamma}_j^r)$ span the leaves of the foliation $\mathcal{H}$. Since the foliations of $\rho(\widetilde{\gamma}_j^r)$ and $\rho(\widetilde{\gamma}_j)$ are the same, we conclude that $\mathcal{H}$ is preserved by $\rho(\widetilde{\gamma}_j)$. Repeating this for each generator, we conclude that ρ preserves $\mathcal{H}$. Thus, there is an action $\hat{\rho} : \Gamma \to \mathrm{Homeo}(\mathcal{C}_0)$ given by $\rho(\gamma)\mathcal{H}_y = \mathcal{H}_{\hat{\rho}(\gamma)(y)}$. Since $\hat{\rho}$ is trivial on Γ_0, it induces an action of F. If ρ is C^1-close to ρ_0 then the action of F is C^0-close to the trivial action. But then the only possible action is the trivial action (this is not hard to see for actions on S^1, but for general manifolds one can use a theorem of M. H. A. Newman [9]; see [2, §9] for a version due to P. A. Smith [17]).

It remains to prove the Claim made earlier.

Proof of the Claim. If there is an open component $\mathcal{U}_y$ of f_0 contained in $\widetilde{I}$, we are done because $f|_{\mathcal{U}_y}$ is ergodic (by Theorem 4.3; see the first part in 4 of Lemma 3.3), hence $\psi|_{\mathcal{U}_y}$ is constant a.e.

Otherwise, $\widetilde{I}$ is foliated by horizontal leaves $\mathcal{H}_y$ of f_0, and thus the second part in 4 of Lemma 3.3 applies.

Label each leaf $\mathcal{H}_y$, $y \in I$, by its intersection v with a fixed vertical segment $V = (\{x_*\} \times S^1) \cap \widetilde{I}$ (recall that $\widetilde{I} = \{z \in M \times S^1 \mid z \sim y,\ y \in I\}$). This defines a homeomorphism $v \in V \mapsto y = y(v) \in I$ (which is actually bi-Lipschitz). Then, by 4 of Lemma 3.3, for a.e. $v \in V$, the function $\psi|_{\mathcal{H}_{y(v)}}$ is $\mu_{y(v)}$-a.e. constant. By changing ψ on a set of μ-measure zero, we may assume that ψ is constant on each leaf $\mathcal{H}_{y(v)}$, $v \in V$.

The idea is the following: there is a leaf $\mathcal{H}_y$ mapped across an open set of leaves. Since ψ is constant on each leaf and is (a.e.) Γ_0-invariant, this forces an (a.e.) open set of leaves to carry the same value of ψ. The following details are needed because the invariance holds only a.e.

For each $\gamma \in \Gamma_0$, $\psi \circ \rho(\gamma)|_{\widetilde{I}} = \psi|_{\widetilde{I}}$ μ-a.e., hence there is Γ_0-invariant set $Z \subset \widetilde{I}$ of full μ-measure such that $\psi \circ \rho(\gamma)|_Z = \psi|_Z$, for all $\gamma \in \Gamma_0$.

Notice first that there are $\gamma_0 \in \Gamma_0$ and $y_0 = y(v_0), y_0' = y(v_0') \in I$ such that $\rho(\gamma_0)(\mathcal{H}_{y_0})$ and $\mathcal{H}_{y_0'}$ have a point in common and intersect transversally. Indeed, otherwise each $\rho(\gamma)$ ($\gamma \in \Gamma_0$) preserves the distribution $E^u \oplus E^s$ tangent to $\{\mathcal{H}_y\}$, and therefore it permutes the leaves of $\{\mathcal{H}_y\}$ (because E^u and E^s are uniquely integrable). But $\rho(\gamma)|_{\mathcal{C}_0} = \mathrm{Id}$, hence $\rho(\gamma)$ preserves each leaf of $\{\mathcal{H}_y\}$, which contradicts the fact that $I \subset B$.

Then there are open intervals $v_0 \in V_0, v_0' \in V_0'$ in V such that $\rho(\gamma_0)(\mathcal{H}_{y(v)})$ and $\mathcal{H}_{y(v')}$ have a point in common and intersect transversally, whenever $v \in V_0, v' \in V_0'$.

Pick $v_1 \in V_0$ such that $\mathcal{H}_{y_1} \cap Z$ has full μ_{y_1}-measure in $\mathcal{H}_{y_1}$, where $y_1 = y(v_1)$. By the transversally condition, there is a non-zero vector $\mathbf{u} \in T_w\mathcal{H}_{y_1} = (E^u \oplus E^s)|_w$ such that $d\rho(\gamma_0)\mathbf{u} \notin (E^u \oplus E^s)|_{\rho(\gamma_0)w}$. Consider a 1-dimensional C^1-foliation $\mathcal{O}$ of a neighborhood of w in $\mathcal{H}_{y_1}$ such that the leaf through w is tangent to $\mathbf{u}$. Since C^1-foliations are absolutely continuous, a.e. leaf of $\mathcal{O}$ intersects Z in a set of full (1-dimensional) measure. Therefore, we can find a local leaf, say $\mathcal{O}_0$, together with a regular parametrization $\omega : (-\varepsilon, \varepsilon) \to \mathcal{O}_0 \subset \mathcal{H}_{y_1}$ such that the set $J := \{t \in (-\varepsilon, \varepsilon) \mid \omega(t) \notin Z\}$ has zero measure and

$$t \in (-\varepsilon, \varepsilon) \mapsto \rho(\gamma_0)(\omega(t)) \quad \text{is transversal to the foliation } \{\mathcal{H}_y\}. \tag{3.6}$$

Consider the map $h : (-\varepsilon, \varepsilon) \to V$ defined by $\rho(\gamma_0)(\omega(t)) \in \mathcal{H}_{y(h(t))}$. As in the proof of 4 of Lemma 3.3, one can check that $h : (-\delta, \delta) \to V$ is a Lipschitz (and therefore absolutely continuous) map for small $\delta \in (0, \varepsilon)$. By (3.6), $h(0)$ is an interior point of $h((-\delta, \delta))$. Therefore, $h(J \cap (-\delta, \delta)) \subset V$ has zero measure, and thus $h((-\delta, \delta) \setminus J) \subset V$ contains — up to measure zero — an open interval. Denote such an interval by I_0.

Since Z is ρ_{γ_0}-invariant, $\rho_{\gamma_0}(\omega(t)) \in \rho_{\gamma_0}(\mathcal{H}_{y_1} \cap Z) \cap (\mathcal{H}_{y(h(t))} \cap Z)$ for each $t \in (-\delta, \delta) \setminus J$. But ψ is constant on each horizontal leaf and $\psi|_Z$ is $\rho(\gamma_0)$-invariant, therefore $\psi|_{\mathcal{H}_{y(h(t))}} \equiv \psi(\rho_{\gamma_0}(\omega(t))) = \psi(\omega(t)) \equiv \psi|_{\mathcal{H}_{y_1}}$. We conclude that $\psi|_{\mathcal{H}_{y(v)}}$ has the value $\psi|_{\mathcal{H}_{y_1}}$ for each $v \in h((-\delta, \delta) \setminus J)$. This proves the claim, with $\widetilde{I}_0 := \{z \in M \times S^1 \mid z \sim y,\ y \in I_0\}$. □

This concludes the proof of Theorem 1.3. □

Proof of Corollary 1.5. Choose an element γ_* of Γ such that $\alpha(\gamma_*)$ has a fixed point and $\mathcal{H}$ is spanned by the stable and unstable foliations of the partially hyperbolic diffeomorphism $f_0 := \rho(\gamma_*)$ (e.g., the element denoted $\gamma_0^{p_0}$ in the proof of Theorem 1.3). Denote by $\mathcal{C}$ the center foliation of f_0.

By 4 of Theorem 2.1, there is a homeomorphism H which is a leaf-wise conjugacy between $\rho_0(\gamma_*) = \alpha(\gamma_*) \times \mathrm{Id}_{S^1}$ and $\rho(\gamma_*)$. That is, H maps each $V_x := \{x\} \times S^1$ diffeomorphically onto a center leaf of $\rho(\gamma_*)$ and $\rho_{\gamma_*}(H(V_x)) = H(V_{\alpha_{\gamma_*}(x)})$. Moreover, $H(x, \cdot) : S^1 \to M \times S^1$ is a C^K-diffeomorphism onto its image, and varies continuously with $x \in M$.

Define Φ as follows: pick a point $x_* \in M$ and let

$$\Phi(x, y) := H(V_x) \cap \mathcal{H}(H(x_*, y)).$$

Then $\Phi(x, \cdot) : S^1 \to M \times S^1$ is C^{K-1}, because, by 2 of Theorem 2.1, the $\mathcal{H}$-holonomy between the center leaves of ρ_{γ_*} is C^{K-1}. Moreover, $x \in M \mapsto \Phi(x, \cdot) \in C^{K-1}(S^1, M \times S^1)$ is continuous and Φ approaches $\mathrm{Id}_{M \times S^1}$ in $C^0(M \times S^1)$ as ρ approaches ρ_0 in C^1.

Because H is a leaf-conjugacy and each leaf of $\mathcal{H}$ is preserved by ρ, it follows that $\rho(\gamma_*) \circ \Phi = \Phi \circ \rho_0(\gamma_*)$.

Note that the image of $\Phi(\cdot, y)$ is one of the leaves of $\mathcal{H}$; let us denote it by $\mathcal{H}^y$. This leaf can be described as the image of a map $\widetilde{\phi}_y : M \to \mathcal{H}^y \subset M \times S^1$, $x \mapsto (x, \phi_y(x))$, where $\phi_y : M \to S^1$ is C^{K^-} and $y \in S^1 \mapsto \widetilde{\phi}_y \in C^{K^-}(M, M \times S^1)$ is continuous.

If the diffeomorphism $\rho(\gamma_*)$ is C^∞, then $W^s_{\rho(\gamma_*)}$ and $W^u_{\rho(\gamma_*)}$ are C^∞-laminations, hence $\widetilde{\phi}_y$ is actually smooth and $y \in S^1 \mapsto \widetilde{\phi}_y \in C^\infty(M, M \times S^1)$ is continuous.

Assume first that α is $C^{L^-,K}$-locally rigid. If ρ approaches ρ_0 in C^L then the foliation $\mathcal{H}$ approaches in C^{L^-} the horizontal foliation of ρ_0 (by Theorem 2.1 applied to the element γ_*, and Journé's Theorem 4.1; see §4), hence the Γ-action induced from $\mathcal{H}^y$, $\widetilde{\phi}_y^{-1} \circ \rho \circ \widetilde{\phi}_y$, approaches α in C^{L^-}. By the rigidity of α, there is a diffeomorphism $h_y \in \mathrm{Diff}^K(M)$, C^0 close to the identity, such that $\widetilde{\phi}_y^{-1}\rho_\gamma\widetilde{\phi}_y h_y = h_y \alpha_\gamma$ on M for each $\gamma \in \Gamma$. Considering this relation for γ_*, we conclude that $\Phi(\cdot, y) = \widetilde{\phi}_y h_y$ because $\alpha(\gamma_*)$ is hyperbolic and the centralizer in Homeo of a hyperbolic diffeomorphism is discrete.

We conclude that $\Phi(\cdot, y)$ is a C^K diffeomorphism from M onto its image $\mathcal{H}^y$ for each $y \in S^1$, and $\rho(\gamma) \circ \Phi = \Phi \circ \rho_0(\gamma)$ for all $\gamma \in \Gamma$.

If α is also *continuously* $C^{L^-,K-1}$-locally rigid, then $y \in S^1 \mapsto h_y \in C^{(K-1)^-}$ is continuous, hence $y \in S^1 \mapsto \Phi(\cdot, y) \in \mathrm{Diff}^{(K-1)^-}$ is continuous. By Journé's Theorem 4.1, we conclude that $\Phi \in \mathrm{Diff}^{(K-1)^-}(M \times S^1)$, as desired. □

4. **Proof of the Lemmas.** In this section we prove Lemmas 3.3, 3.5 and 3.6.

Proof of Lemma 3.3. Denote $f_0 = A \times \mathrm{Id}_{S^1}$.

From the continuous dependence of W^s, W^c, W^u on the diffeomorphism (3 of Theorem 2.1) it follows that the foliations W^s and W^u of f are close to those of f_0. The dynamical coherence of f implies that the projection $q : M \times S^1 \to \widehat{M}$ along the center foliation takes the stable and unstable foliations of f to foliations of $\widehat{M}$, which we will still call stable and unstable. By 4 of Theorem 2.1 we conclude that these foliations are conjugated to the foliations of $A : M \to M$.

Notice that any "(u, s)-path" in $\widehat{M}$ can be lifted to a (u, s)-path for f, starting at any point of the corresponding center leaf.

We consider now one-by-one the statements of the Lemma.

1. By [12, Lemma 3.1] we see that the $\sim$-equivalence classes cover $\widehat{M}$.

The f-invariance of each class follows from the f-invariance of W^u and W^s, and the fact that in each equivalence class there is a point in $\mathcal{C}_0$.

2. Assume that there are points $x \neq y$ in $\mathcal{C}_0$ connected through a (u, s)-path γ whose projection to $\widehat{M}$ is contractible. We will show that y is an interior point of its $\sim_0$-equivalence class. One can shrink the path γ (within the class of (u, s)-paths) while keeping its initial point fixed. This shows that one of the intervals determined by x and y in $\mathcal{C}_0 \cong S^1$ is contained in $[y]_{\sim_0}$. Denote this interval by I_1. To reach the "other side" of y in $\mathcal{C}_0$, start from a point $x' \in I_1$ close to x and follow the (u, s)-path $\widetilde{\gamma}$ that projects onto $q(\gamma)$ (such a path exists by the dynamical coherence of f). Since holonomies of the foliations W^u and W^s induce orientation preserving local homeomorphisms of the central leafs (due to the leaf-wise conjugacy of $(M \times S^1, W^c)$ with the center foliation of f_0, all central leafs of f can be oriented consistently), the end point of $\widetilde{\gamma}$ lies outside I_1 and near y. Shrinking $\widetilde{\gamma}$, we conclude that a neighborhood of y in $\mathcal{C}_0$ is in $[y]_{\sim_0}$.

In conclusion, each $\sim_0$-equivalence class of $\mathcal{C}_0$ is either a point or an open subset of $\mathcal{C}_0$.

3. Consider now the $\sim$-equivalence relation restricted to $\mathcal{C}_0$. We claim that the equivalence classes coincide with the $\sim_0$-equivalence classes.

Note first that if $[y]_{\sim_0}$ consists of a single point, then the foliations W^u and W^s commute along leaves that start at y, hence (u,s)-paths emanating from y fit into a topological (actually, C^{K-}, as we will see later) manifold which we denote by $\mathcal{H}_y$ and $q : \mathcal{H}_y \to \widehat{M}$ is a covering. Since $f(y) = y$, we conclude that $\mathcal{H}_y$ is f-invariant. We have to show that $\mathcal{H}_y \cap \mathcal{C}_0 = \{y\}$. Since $q|_{\mathcal{H}_y}$ is a covering, by the definitions of $\sim$ and $\sim_0$, there is an onto map

$$H : \pi_1(\widehat{M}, q(y)) \to \mathcal{H}_y \cap \mathcal{C}_0,$$

defined by letting $H(\omega)$ be the endpoint of the (u,s)-lift starting at y of $\omega \in \pi_1(\widehat{M}, q(y))$.

Moreover, H induces an f-equivariant action of $\pi_1(\widehat{M}, q(y))$ on $\mathcal{H}_y \cap \mathcal{C}_0$.

In view of the orientation preservation mentioned above, H is monotonic: the images $\{H(\omega^k)\}_{k\in\mathbb{Z}}$ travel around $\mathcal{C}_0$ in a fixed direction.

The f-invariance of $\mathcal{H}_y$ implies that $H(\widehat{f}_*\omega) = H(\omega)$ for $\omega \in \pi_1(\widehat{M}, q(y))$, hence

$$H(\eta \widehat{f}_*(\omega)\omega^{-1}) = H(\eta). \tag{4.1}$$

Here $\widehat{f}$ is the map induced on $\widehat{M}$ by f.

This implies that the image of H is only the point y (provided f is C^1-close to f_0). The idea is that for an infranilmanifold M the image of "$\widehat{f}_* - \mathrm{Id}$" $: \pi_1(\widehat{M}, q(y)) \to \pi_1(\widehat{M}, q(y))$ has finite index.

More precisely, note that, as long as f is close to f_0, $\pi_1(\widehat{M}, q(y)) \cong \pi_1(M)$ such that $\widehat{f}_*$ corresponds to A_* and that there is an A_*-invariant exact sequence

$$1 = N_0 \hookrightarrow N_1 \hookrightarrow N_2 \hookrightarrow \ldots \hookrightarrow N_r \hookrightarrow \pi_1(M) \to F \to 1$$

where F is finite, N_{k+1}/N_k are abelian, and the images of $A_* - \mathrm{Id} : N_{k+1}/N_k \to N_{k+1}/N_k$ have finite indexes (by the Franks–Manning classification of hyperbolic diffeomorphisms on infranilmanifolds; see [7, proof of Lemma 4.5]).

We will prove by induction that $H(N_k) = \{y\}$. This is clear for N_0; assume it holds for N_{k-1}.

Pick a finite family $\{\omega_i\} \subset N_k$ whose H-images cover $H(N_k)$. Since there is an order p_k such that $\omega_i^{p_k} \in \mathrm{Ran}[A_* - \mathrm{Id} : N_k/N_{k-1} \to N_k/N_{k-1}]$, then, by (4.1) $H(\omega_i^{p_k}) = y$. Thus the points $, H(\omega_i), H(\omega_i^2), \ldots, H(\omega_i^{p_k})$ travel around the circle from y and y. But if f is C^1-close to f_0 then each step is too small to cover the circle in p_k steps, therefore $H(\omega_i) = y$. This proves that $H(N_k) = \{y\}$.

Hence $H(N_r) = \{y\}$. Therefore one can consider $H : \pi_1(M)/N_r \cong F \to \mathcal{H}_y \cap \mathcal{C}_0$. Since F is finite, the same argument as above shows that $H(\pi_1(M)) = \{y\}$ for f close to f_0.

We proved therefore that for $y \in \mathcal{C}_0$, if $[y]_{\sim_0}$ is a single point then $[y]_{\sim} \cap \mathcal{C}_0 = \{y\}$, hence that H_y is a simple cover of $\widehat{M}$. Consider the complement U in $\mathcal{C}_0$ of these points. If this is the whole $\mathcal{C}_0$, then it has to be a single $\sim_0$-equivalence class (because in that case each equivalence class is open). As in [12, Lemma 3.1], this implies that $M \times S^1$ is a single $\sim$-equivalence class. Otherwise, again by 2 of this Lemma, each connected component V of U is an $\sim_0$-equivalence class, and its endpoints $y_0, y_1 \in \mathcal{C}_0$ do not belong to U; since the fibers are S^1, this implies that $\sim$-equivalence class of V is the connected component of $M \times S^1 \setminus (\mathcal{H}_{y_0} \cup \mathcal{H}_{y_1})$ which intercepts V.

The fact that the leaves $\mathcal{H}_y$ are C^{K^-} follows the fact that the leaves of W^u and W^s are C^K and the following theorem of Journé (see [11, Theorem 3.1 (a)] for more details).

Theorem 4.1 (Journé, [5])**.** *Assume given on a manifold two continuous transverse laminations, $\mathcal{F}_s$ and $\mathcal{F}_u$, with uniformly smooth (or C^{k+1}) leaves. If a function f is uniformly $C^{k+\delta}$–smooth along the leaves of $\mathcal{F}_s$ and $\mathcal{F}_u$, then f is $C^{k+\delta}$–smooth ($1 \leq k \leq \infty, \delta \in (0,1)$).*

Moreover, if $\mathcal{F}'_s \to \mathcal{F}_s$, $\mathcal{F}'_u \to \mathcal{F}_u$, $f'|_{\mathcal{F}'_u} \to f|_{\mathcal{F}_u}$, $f'|_{\mathcal{F}'_s} \to f|_{\mathcal{F}_s}$ in the $C^{k+\delta}$-topology, then $f' \to f$ in the $C^{k+\delta}$-topology.

4. Recall the result of Pugh and Shub [15]:

Definition 4.2. By *(essential) accessibility* of a partially hyperbolic diffeomorphism $f \in \mathrm{Diff}(X)$ we mean that (almost) each pair of points $x, y \in X$ is (u,s)-accessible (with respect to the stable and unstable foliations of f).

Theorem 4.3 (Pugh-Shub, [15])**.** *Assume X is a compact manifold endowed with a smooth volume μ.*

If $f \in \mathrm{Diff}^2_\mu(X)$ is a center bunched and dynamically coherent partially hyperbolic diffeomorphism with the essential accessibility property then f is ergodic.

This shows that on each $\mathcal{U}_y$ the measure μ is ergodic for f.

Assume now that the manifolds $\{\mathcal{H}_t\}_{t\in I}$ foliate their union $\widetilde{I}$, where $I \subset \mathcal{C}_0$ is an open set.

Denote by ω the volume form determined by μ (with respect to a Riemannian structure on $M \times S^1$).

We will show that there is a (continuous) f-invariant vector field X^c on $\widetilde{I}$ which spans the center distribution of f. Then $i_{X^c}(\omega)$, the interior product of X^c with ω, restricts to a non-degenerate continuous f-invariant volume form on each $\mathcal{H}_y$, which determines an invariant measure μ_y. Since the diffeomorphism f is Anosov on each leaf $\mathcal{H}_y$, it will be ergodic on each $(\mathcal{H}_y, \mu_y)$.

To show that such a vector field exists, recall that the $\mathcal{H}_y$-holonomies between center leaves of f are C^1 (since one can write the holonomy between two center leaves as a composition of W^s-holonomies within W^{cs} leaves and W^u-holonomies within W^{cu} leaves, and these holonomies are C^1 by part 2 of Theorem 2.1). Choose a C^1-parametrization of one of the center leaves in $\widetilde{I}$ and extend it to the the other leaves using the $\mathcal{H}_y$-holonomies. Since $\{\mathcal{H}_y\}$ is f-invariant, these parametrizations are f-equivariant. Therefore, the tangent field X^c they determine along each center leaf is f-invariant as well.

It remains to prove the absolute continuity of the lamination $\{\mathcal{H}_y\}_{y\in I}$. This follows from the fact that holonomies between the vertical segments $\left\{V_x := (\{x\} \times S^1) \cap \widetilde{I} \mid x \in M\right\}$ are uniformly Lipschitz. [We are not using the holonomies between center leaves because the center foliation need not be absolutely continuous.]

Indeed, let $V := V_x$ and $V' := V_{x'}$ be two vertical segments, $w_1, w_2 \in V$ close to each other, $\mathcal{H}_i$ the leaf that contains w_i, $(i = 1,2)$, and $w'_i := V' \cap \mathcal{H}_i$. Denote

$u_2 = \mathcal{H}_2 \cap W^c(w_1)$, $u_2' = \mathcal{H}_2 \cap W^c(w_1')$. Then the ratios

$$\frac{\operatorname{dist}_{W^c(w_1)}(w_1, u_2)}{\operatorname{dist}_{W^c(w_1')}(w_1', u_2')}, \quad \frac{\operatorname{dist}_{W^c(w_1)}(w_1, u_2)}{\operatorname{dist}_V(w_1, w_2)}, \quad \text{and} \quad \frac{\operatorname{dist}_{W^c(w_1')}(w_1', u_2')}{\operatorname{dist}_{V'}(w_1', w_2')}$$

are bounded away from both 0 and ∞: the first ratio because the holonomy between the center leaves is uniformly C^1, the last two ratios because the tangent space of $\{\mathcal{H}_y\}$, $E^s \oplus E^u$, is close to the "constant" horizontal distribution $TM \oplus 0 \subset T(M \times S^1)$ while $E^c = TW^c$ is close to the "constant" vertical distribution $0 \oplus TM \subset T(M \times S^1)$. Thus the ratio $\operatorname{dist}_V(w_1, w_2)/\operatorname{dist}_{V'}(w_1', w_2')$ is uniformly bounded away from zero and infinity, which proves our assertion. □

We now proceed to the Lemmas related to the property (T).

Notations. 1. In order to simplify the exposition we are not going to write explicitly the small constants that are "uniform". Instead, we will use either $\ll$ or $o(1)$. For example, $\|U_f(\phi_{\widetilde{I}}) - \phi_{\widetilde{I}}\|_{L^2} \ll \|\phi_{\widetilde{I}}\|_{L^2}$ means that $\|U_f(\phi_{\widetilde{I}}) - \phi_{\widetilde{I}}\|_{L^2} \le \varepsilon \|\phi_{\widetilde{I}}\|_{L^2}$ where $\varepsilon > 0$ depends only on the quantities "specified" beforehand, and can be made arbitrarily small by choosing these quantities correspondingly.

In order to show that two quantities are comparable, we are going to use $\approx$. That is, $u \approx v$ means that $1/c \le u/v \le c$ for some constant $0 < c < \infty$ which is again uniform in the sense specified above. For inequalities that hold up to a multiplicative uniform constant we use $\lessapprox$ and $\gtrapprox$.

2. Given a smooth volume μ on $M \times S^1$, we can write it as

$$\int_{M \times S^1} \phi(x,y)\, d\mu = \int_M \int_{S^1} \phi(x,y) m(x,y)\, dy\, d\nu(x),$$

where ν is a smooth measure on M and m is a smooth positive function. If $A \in \operatorname{Diff}(M)$ is Anosov and $A \times \operatorname{Id}_{S^1}$ preserves μ, then we can choose ν to be A-invariant (take the measure induced via the projection $M \times S^1 \to M$) and then m depends only on the y-variable (because A is ergodic).

Proof of Lemma 3.5. We consider fixed the measure μ and the diffeomorphism A. All the constants (explicit or not) used in the rest of the proof depend only on A, μ and $\delta = \operatorname{dist}_{C^1}(f, A \times \operatorname{Id}_{S^1})$.

Since μ is $A \times \operatorname{Id}_{S^1}$-invariant and f is C^1-close to $A \times \operatorname{Id}_{S^1}$, $\|\Delta_f - 1\|_{C^0} = o(1)$. Therefore

$$\|U_f(\phi_{\widetilde{I}}) - \phi_{\widetilde{I}} \circ f^{-1}\|_{L^2} = \|\phi_{\widetilde{I}} - U_f^{-1}(\phi_{\widetilde{I}} \circ f^{-1})\|_{L^2} \ll \|\phi_{\widetilde{I}}\|_{L^2},$$

thus it is enough to deal with $\|\phi_{\widetilde{I}} \circ f^{-1} - \phi_{f(\widetilde{I})}\|_{L^2}$.

We are going to check that

$$\|\phi_{\widetilde{I}} \circ f^{-1} - \phi_{f(\widetilde{I})}\|_{C^0} = o(1) \tag{4.2}$$

and

$$\|\phi_{\widetilde{I}}\|_{L^2} \approx (\mu(\widetilde{I}))^{1/2}, \tag{4.3}$$

which imply our assertion.

Denote $\widetilde{I}' = f(\widetilde{I})$. Given $\xi = (x,y) \in \widetilde{I}$, let $W^c_{\widetilde{I}}(\xi)$ be the connected component of $W^c_f(\xi) \cap \widetilde{I}$ containing ξ. Denote the corresponding subset of $\xi' = f(\xi) \in \widetilde{I}'$ by $W^c_{\widetilde{I}'}(\xi')$. Then $f(W^c_{\widetilde{I}}(\xi)) = W^c_{\widetilde{I}'}(f(\xi))$, by the invariance of W^c_f.

Consider $a = a(\xi) := \operatorname{top}(x) - y$, $b = b(\xi) := y - \operatorname{bot}(x)$, $u = u(\xi) := \operatorname{dist}_{W^c}(\xi, W^c_{\widetilde{I}}(\xi) \cap \operatorname{top}(\widetilde{I}))$, $v = v(\xi) := \operatorname{dist}_{W^c}(\xi, W^c_{\widetilde{I}}(\xi) \cap \operatorname{bot}(\widetilde{I}))$ where $\operatorname{top}(\widetilde{I}) :=$

$\{(x, \mathrm{top}(x)) \in M \times S^1 \mid x \in M\}$, $\mathrm{bot}(\widetilde{I}) := \{(x, \mathrm{bot}(x)) \in M \times S^1 \mid x \in M\}$ and by dist_{W^c} we mean the distance along the center leaves within $\widetilde{I}$.

Since $\mathrm{dist}_{C^1}(f, A \times \mathrm{Id}_{S^1})$ is small, 3 of Theorem 2.1 implies that $\|Df|_{E^c_f}\| = 1 + o(1)$, and the center, stable and unstable distributions of f are C^0-close to those of $A \times \mathrm{Id}_{S^1}$. Therefore, as $\delta \to 0$,

$$\left|\frac{a}{u} - 1\right| = o(1), \quad \left|\frac{b}{v} - 1\right| = o(1), \tag{4.4}$$

which imply that

$$\left|\frac{a}{a+b} - \frac{u}{u+v}\right| = o(1). \tag{4.5}$$

Denote by a', b', u', v' the quantities a, b, u, v corresponding to $\xi' = f(\xi)$. Since f preserves the orientation of W^c_f (being close to $A \times \mathrm{Id}_{S^1}$; however, with the obvious changes in the proof, (4.2) holds even if f reverses the orientation of W^c_f), the segment of the center manifold whose length is u is mapped onto the one whose length is u', and similarly for v and v'. But the differential of f along the center distribution is almost an isometry, therefore $\left|\frac{u}{u'} - 1\right| = o(1)$, $\left|\frac{v}{v'} - 1\right| = o(1)$, and hence

$$\left|\frac{u}{u+v} - \frac{u'}{u'+v'}\right| = o(1). \tag{4.6}$$

Since $\phi_{\widetilde{I}}(\xi) = 2\min\left\{\frac{a(\xi)}{a(\xi)+b(\xi)}, \frac{b(\xi)}{a(\xi)+b(\xi)}\right\}$, the relation (4.5) and its equivalent for $\widetilde{I}'$, together with (4.6), show that $\left|\phi_{\widetilde{I}}(\xi) - \phi_{f(\widetilde{I})}(f(\xi))\right| = o(1)$, thus proving (4.2).

In order to prove (4.3), notice first that

$$\|y \in [0,\ell] \mapsto 2\min\{\frac{y}{\ell}, \frac{\ell - y}{\ell}\}\|^2_{L^2(\,dy)} = \frac{\ell}{3}$$

and therefore

$$\|\phi_{\widetilde{I}}\|^2_{L^2(\mu)} = \int_M \int_{\mathrm{bot}(x)}^{\mathrm{top}(x)} |\phi_{\widetilde{I}}(x,y)|^2 m(x,y)\, dy\, d\nu(x) \approx$$

$$\approx \int_M \int_{\mathrm{bot}(x)}^{\mathrm{top}(x)} |\phi_{\widetilde{I}}(x,y)|^2\, dy\, d\nu(x) = \frac{1}{3}\int_M (\mathrm{top}(x) - \mathrm{bot}(x))\, d\nu(x) \approx \mu(\widetilde{I})$$

□

Proof of Lemma 3.6. It is enough to prove relation (3.4) for the real part of ψ, hence we may assume that ψ takes only real values and $\lambda \in \mathbb{R}$.

Fix $\lambda \in \mathbb{R}$ and denote $\widetilde{I}(\lambda) := \psi^{-1}(\lambda) \cap \widetilde{I}$, $\widetilde{I}(\lambda)_x := \{y \in S^1 \mid (x,y) \in \widetilde{I}(\lambda)\}$, and $\widetilde{I}(\lambda)^- := \widetilde{I}(\lambda) \setminus \{(x,y) \in M \times S^1 \mid |\phi_{\widetilde{I}}(x,y) - \lambda| < \omega\}$, $\widetilde{I}(\lambda)^-_x := \widetilde{I}(\lambda)_x \cap \widetilde{I}(\lambda)^-$, where ω will be specified later.

Then, using (4.3) and assuming (3.3):

$$\delta^2 \mu(\widetilde{I}) \approx \delta^2 \|\phi_{\widetilde{I}}\|^2_{L^2} \geq \|\psi - \phi_{\widetilde{I}}\|^2_{L^2} \geq \int_{\widetilde{I}(\lambda)^-} |\psi - \phi_{\widetilde{I}}|^2\, d\mu =$$

$$= \int_M \int_{\widetilde{I}(\lambda)^-_x} |\lambda - \phi_{\widetilde{I}}(x,y)|^2 m(x,y)\, dy\, d\nu(x) \approx \int_M \int_{\widetilde{I}(\lambda)^-_x} |\lambda - \phi_{\widetilde{I}}(x,y)|^2\, dy\, d\nu(x) \geq$$

$$\geq \omega^2 \int_M \int_{\widetilde{I}(\lambda)^-_x} dy\, d\nu(x) \approx \omega^2 \mu(\widetilde{I}(\lambda)^-)$$

and $\mu(\{(x,y) \in \widetilde{I} \mid |\phi_{\widetilde{I}}(x,y) - \lambda| \leq \omega\}) \lessapprox 2\omega\mu(\widetilde{I})$, as can be easily checked by integrating first along the vertical direction. Therefore

$$\mu(\widetilde{I}(\lambda)) \lessapprox \mu(\widetilde{I}(\lambda)^-) + 2\omega\mu(\widetilde{I}) \lessapprox \left(\frac{\delta^2}{\omega^2} + 2\omega\right)\mu(\widetilde{I}).$$

This shows that, given ε, one can choose δ and ω to obtain the desired conclusion. □

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