REGULARITY RESULTS FOR THE SOLUTIONS OF THE LIVSIC COHOMOLOGY EQUATION WITH VALUES IN DIFFEOMORPHISM GROUPS

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§1. Introduction and the Main Result

In this paper we obtain regularity results for the Livsic cohomology equation for cocycles with values in diffeomorphism and Lie groups.

For certain real-valued cocycles over Anosov systems, Livsic ([L1], [L2]) found Hölder and, in some special cases, C^∞ or C^ω, solutions. Livsic also found Hölder solutions for certain Hölder cocycles with values in Lie groups, provided the cocycles are close to identity. Major progress was done later in the study of the regularity of the solutions for real valued cocycles. Guillemin and Kazhdan ([G-K1], [G-K2]) found C^∞-solutions for the case of geodesic flows on negatively curved surfaces. Collet, Epstein and Gallavotti ([C-E-G]) proved a C^ω version for geodesic flows of constant negative curvature. The final conclusion for the C^∞ case appears in the paper of de la Llave, Marco and Moriyon ([L-M-M]). Since then two new proofs for the case of a C^∞ Anosov system were given by Journé ([J]), Katok and Hurder ([K-H]). Using the approach in [K-H], de la Llave proved the analytic case in [L].

Motivated by the study of the rigidity of the actions of higher rank lattices on tori, we considered in [N-T] cocycles with values in diffeomorphism groups. We were able to prove the triviality of some cocycles close to identity. Here we show that trivializations of regular cocycles are regular. In particular, we obtain C^∞–regularity for cocycles with values in matrix Lie groups. Other applications appear in the study of extensions of various group actions on compact manifolds.

Let M be a compact manifold and $T : M \to M$ an Anosov diffeomorphism. By definition there is a continuous splitting of the tangent bundle into DT-invariant subbundles $TM = E^s \oplus E^u$ together with a metric such that:

$$\|DTv^s\| \le \lambda \|v^s\|, \quad v^s \in E^s,$$
$$\|DT^{-1}v^u\| \le \lambda \|v^u\|, \quad v^u \in E^u,$$

for some $\lambda < 1$. Note that a diffeomorphism of M corresponds to a $\mathbb{Z}$–action.

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Let G be a discrete group acting on a closed Riemannian manifold M, and let Γ be some topological group. A cocycle β is a continuous function $\beta(g,x)$ which maps $G \times M$ into Γ such that

$$\beta(g_1g_2,x)=\beta(g_1,g_2x)\beta(g_2,x)$$

for all $g_1, g_2 \in G$, $x \in M$. Two cocycles β_1 and β_2 are called cohomologous if there exists a continuous map $P : M \to \Gamma$ such that

$$\beta_1(g,x)=P(gx)\beta_2(g,x)P(x)^{-1}$$

for all $g \in G$, $x \in M$. If the cocycle β is cohomologous to the identity, then

$$\beta(g,x)=P(gx)P(x)^{-1}$$

and P is called a trivialization of β.

If $G=\mathbb{Z}$, the cocycle is determined by the function $\beta(1,\cdot)$.

If X is a manifold and $K=1,2,\dots,\infty$, we denote by $\mathrm{Diff}^K(X)$ the group of C^K–diffeomorphisms of X. Let $\alpha \in [0,1)$.

Let $f : M \to \mathrm{Diff}^K(X)$ be a function. One can identify f with a function $\bar{f} : M \times X \to X$. We say that f is continuous if the first K derivatives in the X direction of $\bar{f}$ are continuous. We say that f is $C^{K+\alpha}$ if $\bar{f}$ is $C^{K+\alpha}$.

We say that a $\mathbb{Z}$–cocycle $\beta : \mathbb{Z} \times M \to \mathrm{Diff}^K(X)$ is C^K if $\beta(1,\cdot) : M \to \mathrm{Diff}^K(X)$ is a C^K–function.

The main result is the following:

Theorem 1. *Let M be a compact manifold and $T : M \to M$ a C^∞-Anosov diffeomorphism. Fix a differentiability order K, $K=1,2,\dots,\infty$. Let U be an open set in $\mathbb{R}^d$ and let $\beta : \mathbb{Z} \times M \to \mathrm{Diff}^K(U)$ be a C^K-cocycle. Let $P : M \to \mathrm{Diff}^K(U)$ be a trivialization of β through a continuous function.*

Then P is $C^{K-1+\alpha}$ for any $\alpha \in [0,1)$.

Remark. It is likely that such a result is true for any manifold U. See Theorem 2 for a step in this direction.

§2. Proof of the Main Result

The method we use is similar to the one used in [L-M-M]. The proof there had two parts. One first proved the regularity along the stable and unstable foliations, and then showed that such a function is regular on the whole manifold. The second part was done initially in [L-M-M] using the theory of elliptic operators and the absolute continuity of the Jacobian along the stable and unstable foliations. Other proof was found by Journé ([J]), using only the fact that the stable and unstable foliations are continuous and transverse. Namely:

Theorem. *([J]) Let F_s and F_u be two continuous transverse foliations with uniformly smooth leaves of some manifold. If F is uniformly $C^{n+\alpha}$ smooth along the leaves of F_s and F_u, then f is $C^{n+\alpha}$ smooth $(1 \le n \le \infty,\ \alpha \in (0,1))$.*

Since we can use this result in our case, we have to deal only with the first part.

The proof will consist of a sequence of comments and lemmas. We list below the main steps.

The existence of the first derivatives along the stable leaves is proved as follows:

– P is differentiable along the stable leaves $W^s_{x_*} \times \{y\}$, for $x_* \in M$ a periodic point for T and any $y \in U$.

– The formula found for the above derivative makes sense on $E^s_{x_*} \times U$ even if x_* is not a periodic point.

– P is differentiable along any stable leaf and the derivative is continuous transversally.

We obtain a formula for the first derivative along the stable leaves. Using appropriate coordinates, we differentiate it further. Since mixed derivatives are involved, we have to deal with them too. This way we show the regularity along the weak stable leafs. The same strategy works for the weak unstable leaves, therefore, by Journé's theorem, P is globally differentiable.

Notation. *For β as above define $\gamma : M \to \mathrm{Diff}^K(U)$ by $\gamma(x) := \beta(1,x)^{-1}$. We will sometimes see P and γ as functions depending on the parameter $x \in M$ and denote them by $P_x, \gamma_x : U \to U$. Then*

$$\gamma_x = P_x \circ P^{-1}_{Tx}. \tag{1}$$

The First Order Derivative

Pick a periodic point $x_* \in M$ for T. Hence $T^p(x_*) = x_*$ for some positive integer p. Consider x on $W_* := W^s_{x_*}$, the stable leaf of x_*. Using the triviality condition (1), one has

$$P_x = \gamma_x P_{Tx} = \cdots = \gamma_x \gamma_{Tx} \cdots \gamma_{T^n x} P_{T^{n+1}x} = (\gamma_x \gamma_{Tx} \cdots \gamma_{T^n x} P_{x_*})(P^{-1}_{x_*} P_{T^{n+1}x}).$$

Denote

$$\Phi_{k,x} := \gamma_x \gamma_{Tx} \cdots \gamma_{T^{k-1}x} P_{x_*} (= P_x P^{-1}_{T^k x} P_{x_*}).$$

Since $T^{np}x$ converges to x_* for $x \in W_*$, from the continuity of $x \in M \mapsto P_x \in \mathrm{Diff}^K(U)$ it follows that

$$\lim_{n\to\infty} P^{-1}_{T^{np}x} P_{x_*} = Id_U \qquad \text{(uniform covergence on compact subsets of } U). \tag{2}$$

Therefore

$$P_x = \lim_{n\to\infty} \Phi_{np,x} \qquad \text{(uniform covergence on compact subsets of } U).$$

We will show that in the last limit we have C^1–convergence with respect to $x \in W_*$. We first compute the derivative of $\Phi_{np,x}$ along W_*.

Notation. *Let X, Y, Z be manifolds, $X_1 \subset X$ a submanifold and $F : X \times Y \to Z$ be a smooth function. Then $D_{X_1}F$ will denote the "partial derivative" of $F|_{X_1\times Y}$ in the direction X_1, i.e. $D_{X_1}F : TX_1 \times Y \to TZ$.*

Also:

- *We use the symbol "$\circ$" to show the action on the derivatives in the variable $y \in U$, and "$\cdot$" to indicate the composition of the derivatives.*
- *In order to simplify the notation, we write $D_X g_{h(x)}$ for $(D_X g)|_{h(x)}$. However, by $D_X(g_{h(x)})$ we mean the derivative of the composition $x \mapsto g_{h(x)}$.*

For example, if $f_x = \gamma_x\gamma_{Tx}$ (i.e. $f(x,y) = \gamma(x, \gamma(Tx, y))$), then we write

$$D_M f_x = D_M\gamma_x \circ \gamma_{Tx} + D_U\gamma_x \circ \gamma_{Tx} \cdot D_M\gamma_{Tx} \tag{3}$$

instead of

$$D_M f|_{(x,y)} = D_M\gamma|_{(x,\gamma(Tx,y))} + D_U\gamma|_{(x,\gamma(Tx,y))} \cdot D_M\gamma|_{(Tx,y)}.$$

Differentiating relation (1) we have

$$D_U\gamma_x = D_U P_x \circ P_{Tx}^{-1} \cdot (D_U P_{Tx})^{-1} \circ P_{Tx}^{-1}. \tag{4}$$

Using (3) repeatedly, together with (1) and (4), we obtain

$$\begin{aligned}
&D_{W_*}(\Phi_{np,x}) = \\
&= D_{W_*}\gamma_x \circ (\gamma_{Tx} \circ \cdots \circ \gamma_{T^{np-1}x} \circ P_{x_*}) \\
&\qquad + D_U\gamma_x \circ (\gamma_{Tx} \circ \cdots \circ \gamma_{T^{np-1}x} \circ P_{x_*}) \cdot D_{W_*}(\gamma_{Tx} \circ \cdots \circ \gamma_{T^{np-1}x} \circ P_{x_*}) \\
&= \cdots \\
&= \sum_{k=0}^{np-1} D_U\gamma_x \circ \Phi_{np-1,Tx} \cdot \ldots \cdot D_U\gamma_{T^{k-1}x} \circ \Phi_{np-k,T^k x} \\
&\qquad \cdot D_{W_*}\gamma_{T^k x} \circ \Phi_{np-k-1,T^{k+1}x} \cdot D_{W_*}T^k|_x \\
&= \sum_{k=0}^{np-1} D_U P_x \circ P_{T^{np}x}^{-1} \circ P_{x_*} \cdot (D_U P_{T^k x})^{-1} \circ P_{T^{np}x}^{-1} \circ P_{x_*} \\
&\qquad \cdot (D_{W_*}\gamma_{T^k x}) \circ P_{T^{k+1}x} \circ (P_{T^{np}x}^{-1} \circ P_{x_*}) \cdot D_{W_*}T^k|_x.
\end{aligned} \tag{5}$$

Let us remark that the points where the derivatives in (5) are estimated lie in a compact subset of $M \times U$ which is independent of n, provided that y stays in a compact subset of U. This follows from the compactness of M and the continuity of $P : M \times U \to U$.

Take $y \in U$ in a compact set. Since $\|D_{W_*}T^k\| \le \lambda^k$ and because of the uniform convergence of the factors involving $P_{T^{np}x}^{-1} \circ P_{x_*}$ (by (2)), it is clear that $\{D_{W_*}\Phi_{np,x}\}_{n\ge 1}$ converges uniformly with respect to $x \in W_*$.

In conclusion, for each $y \in U$ the map $x \in W_* \mapsto P_x(y)$ is C^1 and

$$D_{W_*} P_x = \sum_{k=0}^{\infty} (D_U P_x) \cdot (D_U P_{T^k x})^{-1} \cdot (D_{W_*} \gamma_{T^k x}) \circ P_{T^{k+1} x} \cdot D_{W_*} T^k|_x.$$

We remark that each term in the last sum is a linear map from $T_x W_* = E^s_x$ into $T_{P(x,y)} U$, hence the summation makes sense.

Moreover, the sum converges to a map $\Psi : E^s \times U \to TU$ which is continuous. (The sum is convergent because the norm of the terms decreases exponentially.)

As in [L-M-M], one can now show that P is differentiable along the stable direction on the closure of the union $\mathcal{W}$ of the stable manifolds of the periodic points, and the derivative is Ψ. But $\overline{\mathcal{W}} = M$ because the periodic points are dense in the non-wandering set and the stable manifolds of points in the non-wandering set are dense in M (see [Sm]), hence P is differentiable along the stable direction and

$$D_{W^s_x} P_x = \sum_{k=0}^{\infty} (D_U P_x) \cdot (D_U P_{T^k x})^{-1} \cdot (D_{W^s_x} \gamma_{T^k x}) \circ P_{T^{k+1} x} \cdot D_{W^s_x} T^k|_x. \qquad (6)$$

Indeed, for $y \in U$ fixed, $x \in \overline{\mathcal{W}}$ and $\xi \in E^s_x$, take $c : (-1,1) \to W^s_x$ a C^1-path with $c(0) = x$, $\dot{c}(0) = \xi$. By the uniformity of the stable foliation, $W^{s,loc}_x$ is the limit of a sequence of stable local leaves $W^{s,loc}_{x_n}$ which are included in the stable leaves of some periodic points. Therefore one can choose $c_n : (-1,1) \to W^s_{x_n}$ such that:

$$\begin{aligned} c_n &\to c && \text{uniformly,} \\ \dot{c}_n &\to \dot{c} && \text{uniformly,} \\ c_n(0) &= x_n. \end{aligned}$$

Then

$$P(c_n(t), y) - P(x_n, y) = \int_0^t \Psi(\dot{c}_n(s), y)\, ds$$

for every n, and by continuity, letting $n \to \infty$, one obtains

$$P(c(t), y) - P(x, y) = \int_0^t \Psi(\dot{c}(s), y)\, ds.$$

Hence

$$\frac{\partial}{\partial \xi} P|_{(x,y)} = \Psi(\xi, y).$$

Higher Order Derivatives

In what follows, in order to simplify the notation, we denote $D^k_{W^s_x}$ by D^k_s $(k \geq 0)$.

The basic idea is that the representation of $D_s P$ obtained in (6) can be differentiated termwise, to give the higher derivatives of P in the stable direction. However, we notice that mixed derivatives appear too. We will proceed by induction on the number of derivatives in the stable direction which are involved.

We introduce first local coordinates along the stable leaves of M and reduce the problem to matrix-valued functions, defined on open sets in linear spaces. These coordinates are given in Lemma 2.7 in [L-M-M], which we reproduce below for completeness (see also Theorem 6.1 in [H-P-S] and [Sh], p. 96). Here E^s_δ is the bundle of balls of radius δ centered on the zero section of the stable bundle E^s.

Lemma 1. *There exists $\delta > 0$ and a continuous map $w^s : E^s_\delta \to M$ such that for each $x \in M$, $w^s_x := w^s|_{E^s_{\delta,x}}$ is a C^∞ embedding onto a neighborhood of x in W^s_x with $w^s_x(0) = x$. Moreover, for x in M, $x \mapsto w^s_x$ is continuous with respect to the C^∞-topology on $\{w^s_x\}_x$.* $\square$

In fact, w^s_x will be the exponential map of W^s_x with the adapted metric.

Definition. *Let X be a manifold and $k \geq \ell \geq 0$ integers. We say that a function $f : M \times U \to X$ is of class $\mathcal{S}^k_\ell$ if for any $x \in M$ the restriction of f to $W^s_x \times U$ admits all the partial derivatives up to order k which contain no more than ℓ derivatives along the W^s_x direction, and these derivatives are continuous on $M \times U$.*

More precisely, f belongs to $\mathcal{S}^k_\ell$ if for any integer $a, b \geq 0$ with $a + b \leq k$, $b \leq \ell$, the functions $f \circ (w^s_x \times Id_U) : E^s_{\delta,x} \times U \to X$ admit all the partial derivatives of order $(a + b)$ which have exactly b derivatives along the $E^s_{\delta,x}$ direction, and these derivatives are continuous on $E^s_\delta \times U$.

If $E' \to U$ and $E'' \to X$ are C^∞ fiber bundles and $F : E^s \times E' \to E''$ is a morphism of fiber bundles, we say that $F \in \mathcal{S}^k_\ell$ if we get $\mathcal{S}^k_\ell$-functions after locally trivializing E^s through Lemma 1 and E', E'' with C^∞ coordinates.

Using the unstable direction, we can define similarly the classes $\mathcal{U}^k_\ell$.

Remark. By our initial condition $P \in \mathcal{S}^K_0$, and (6) shows that $P \in \mathcal{S}^1_1$.

Note that, due to the following elementary result, we do not need to calculate all the mixed partial derivatives:

Lemma 2. *Let $\Omega \subset \mathbb{R}^2$ be an open set and (x_0, y_0) a point in Ω. Let $f : \Omega \to \mathbb{R}$ be a function such that:*

(i) $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} : \Omega \to \mathbb{R}$ *exists;*

(ii) $\frac{\partial^2 f}{\partial x \partial y} : \Omega \to \mathbb{R}$ *exists and is continuous in (x_0, y_0).*

Then $\frac{\partial^2 f}{\partial y \partial x}$ exists in (x_0, y_0) and coincides with $\frac{\partial^2 f}{\partial x \partial y}(x_0, y_0)$. $\square$

Hence, in order to show that $P \in \mathcal{S}^k_\ell$, it is enough to prove the existence and continuity of the derivatives $D^a_U D^b_s P$, with $a, b \geq 0$, $b \leq \ell$, $a + b \leq k$. This will follow from the next lemma used for

$$\phi_x = (D_U P_x)^{-1} \cdot D_s \gamma_x \circ P_{Tx} \quad \text{and} \quad \psi_x = (D_U P_x)^{-1} \cdot D_s P_x. \tag{7}$$

Lemma 3. *Let $\phi : E^s \times U \to TU$ be a continuous homomorphism of fiber bundles, and define $\psi : E^s \times U \to TU$ by*

$$\psi_x := \sum_{\ell=0}^{\infty} \phi_{T^\ell x} \cdot D_s T^\ell|_x. \tag{8}$$

Then $\phi \in \mathcal{S}^q_r$ implies $\psi \in \mathcal{S}^q_r$.

Remark. By expressions like ϕ_x we mean the restriction to $E^s_x \times U \to TU$.

Since T is Anosov, the right-hand side of (8) converges uniformly for y in a compact subset of U.

Proof. Using on M the coordinate charts given by Lemma 1, ϕ becomes a matrix valued function with entries in $\mathcal{S}^q_r$.

Pick a point x in M. Write the maps ψ, ϕ, T^ℓ in coordinates, and denote the result by Ψ, Φ^ℓ and G_ℓ respectively (the index ℓ indicates that one of the coordinates in which the given function is expressed is around $T^\ell x$). Note that these are now functions of $\xi \in E^s_{\delta,x}$ (respectively $\xi \in E^s_{\delta,T^\ell x}$), and D_s becomes the derivative with respect to ξ.

Then, for $a, b \geq 0$ such that $a+b \leq q$ and $b \leq r$, we can write formally

$$D^a_U D^b_s \Psi_\xi = \sum_{\ell=0}^{\infty} \sum_{k=0}^{b} \binom{b}{k} F^{\ell;a,b-k}_\xi \cdot D^{k+1}_s G_\ell(\xi), \tag{9}$$

where

$$F^{\ell;a,c}_\xi := D^a_U D^c_s (\Phi^\ell_{G_\ell(\xi)}).$$

We are done once we show that the series (9) converges uniformly on $E^s_{\delta,x}$, for y in a compact subset of U, and for all the considered values of a and b.

Since

$$\|D^k_s G_\ell\| \leq (k+1)! C_k \cdot \ell^{k+1} \cdot \lambda^{\ell-k+1} \tag{10}$$

([L-M-M], the comment before Lemma 2.3), it is enough to show that $F^{\ell;a,c}$ is uniformly bounded independently of ℓ. Using the formula for the derivative of a composite function, it is clear that $\|D^a_U D^c_s(\Phi^\ell_{G_\ell(\cdot)})\|$ can be bounded, independently of ℓ, with a polynomial in the quantities $\|D^\alpha_U D^{\beta_1}_s \Phi^\ell\|$ and $\|D^{\beta_2}_s G_\ell\|$, $0 \leq \alpha \leq a+c$, $0 \leq \beta_1, \beta_2 \leq c$ (note that all these derivatives exist, by the hypothesis). Of these quantities, the latter is bounded by (10). For the former, if we restrict y to a bounded subset of U, it is evaluated in a compact set which does not depend on ℓ, hence it is uniformly bounded too, as desired.

The continuity on $M \times U$ of the derivatives follows from the formula (9), owing to the uniform convergence of the series. $\square$

End of the Proof of Theorem 1

From Lemma 3 used according to (7), one obtains the following conclusion:

If $P \in \mathcal{S}^K_\ell$ $(0 \leq \ell < K)$ then $D_s P \in \mathcal{S}^{K-1}_\ell$, and therefore $P \in \mathcal{S}^K_{\ell+1}$.

(Note that by (8) we actually computed $(D_U P_x)^{-1} \cdot D_s P_x$, but $D_U P \in \mathcal{S}^{K-1}_\ell$.)

By hypothesis, $P \in \mathcal{S}^K_0$. The above remark used iteratively yields $P \in \mathcal{S}^K_K$. Repeating the whole process for T^{-1} instead of T gives that $P \in \mathcal{U}^K_K$.

Now we can apply Journé's theorem ([J]) to conclude that P is $C^{K-1+\alpha}$ for any $\alpha \in [0,1)$. $\square$

§3. Some Applications

We use Theorem 1 to show regularity of other classes of cocycles.

Definition. *Let N be a connected C^∞ manifold without boundary. N is called a good manifold if there is an embedding of N into an Euclidean space $\mathbb{R}^d$ such that N has a tubular neighborhood U which is an open set in $\mathbb{R}^d$ and which is C^∞-diffeomorphic to $N \times D$, where D is an open bounded disk in $\mathbb{R}^{d-\dim N}$.*

Remark. Any good manifold has stably trivial tangent bundle and any closed manifold with stably trivial tangent bundle is good. Examples of such manifolds are the n–torus and the n–sphere.

Take now $K = 1, 2, \ldots, \infty$ and let N be a good manifold. Then any diffeomorphism a in $\mathrm{Diff}^K(N)$ can be extended to a diffeomorphism $\tilde{a}$ in $\mathrm{Diff}^K(U)$ by $\tilde{a}(x, y) := (a(x), y)$, for $x \in N$ and $y \in D$ (here $U \cong N \times D$ is the open set given by the previous definition).

Theorem 2. *Let $T : M \to M$ be a C^∞-Anosov diffeomorphism on the compact manifold M and let N be a good manifold. Let $\beta : \mathbb{Z} \times M \to \mathrm{Diff}^K(N)$ be a C^K-cocycle ($K = 1, 2, \ldots, \infty$). Let $P : M \to \mathrm{Diff}^K(N)$ be a trivialization of β through a continuous function.*

Then P is $C^{K-1+\alpha}$ for any $\alpha \in [0, 1)$.

Proof. Let U be the open set in $\mathbb{R}^d$ such that $N \subset U \cong N \times D$. Extend β to a cocycle $\tilde{\beta} : \mathbb{Z} \times M \to \mathrm{Diff}^K(U)$ by $\tilde{\beta}(g, x) := \widetilde{\beta(g, x)}$, and extend P to $\tilde{P} : M \to \mathrm{Diff}^K(U)$ by $\tilde{P}_x := \widetilde{P_x}$. Then $\tilde{\beta}$ and $\tilde{P}$ satisfy the requirements of Theorem 1. Since P_x is the restriction of $\tilde{P}_x$ to the smooth manifold N, the conclusion follows. $\square$

As a special case of Theorem 1, one has regularity of the trivialization for cocycles with values in a matrix Lie group (i.e., a Lie subgroup of $GL(d, \mathbb{R})$). This extends Theorem 2.1 in [L-M-M].

Theorem 3. *Let $T : M \to M$ be a C^∞-Anosov diffeomorphism on the compact connected manifold M. Let Γ be a matrix Lie group and let $\beta : \mathbb{Z} \times M \to \Gamma$ be a C^K-cocycle over T ($K = 1, 2, \ldots, \infty$) which admits a continuous trivialization $P : M \to \Gamma$. Then P is $C^{K-1+\alpha}$ for any $\alpha \in [0, 1)$. In particular, if β is C^∞, so is P.*

Proof. Multiplying P on the right by some element of Γ, we may assume that 1_Γ is in the range of P. Then, since M is connected, P takes values in $\Gamma^0 := \Gamma \cap GL^0(d, \mathbb{R})$. Therefore $\beta(n, x) = P_{T^n x} P_x^{-1}$ ($n \in \mathbb{Z}$, $x \in M$) takes also values in Γ^0. Any element in Γ^0 induces by left multiplication a diffeomorphism of $GL^0(d, \mathbb{R})$. We extend β to a cocycle $\overline{\beta}$ with range in $\mathrm{Diff}^K(GL^0(d, \mathbb{R}))$ by

$$\overline{\beta}(n, x)(h) := \beta(n, x)h \quad \text{for} \quad n \in \mathbb{Z},\ h \in GL^0(d, \mathbb{R}),\ x \in M,$$

and we extend P to $\overline{P} : M \to \mathrm{Diff}^K(GL^0(d, \mathbb{R}))$ by

$$\overline{P}_x(h) := P(x)h, \quad \text{for} \quad h \in GL^0(d, \mathbb{R}),\ x \in M.$$

Note that $GL^0(d, \mathbb{R})$ is an open set in $\mathbb{R}^{d^2}$. Since $\Gamma^0 \subset GL^0(d, \mathbb{R})$ as a smooth manifold, $\overline{\beta}$ and $\overline{P}$ satisfy the requirements of Theorem 1, and we can recover P through $P(x) = \overline{P}_x(1_\Gamma)$, $x \in M$. $\square$

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