Stable transitivity of Heisenberg group extensions of hyperbolic systems

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Abstract

We consider skew-extensions with fiber the standard real Heisenberg group $\mathcal{H}_n$ of a uniformly hyperbolic dynamical system.

We show that among the C^r extensions ($r > 0$) that avoid an obvious obstruction, those that are topologically transitive contain an open and dense set. More precisely, we show that an $\mathcal{H}_n$-extension is transitive if and only if the $\mathbb{R}^{2n}$-extension given by the abelianization of $\mathcal{H}_n$ is transitive.

A new technical tool introduced in the paper, which is of independent interest, is a diophantine approximation result. We show, under general conditions, the existence of an infinite set of approximate positive integer solutions for a diophantine system of equations consisting of a quadratic indefinite form and several linear equations. The set of approximate solutions can be chosen to point in a certain direction. The direction can be chosen from a residual subset of full measure of the set of real directions solving the system of equations exactly.

Another contribution of the paper, which is used in the proof of the main result, but it is also of independent interest, is the solution of the so called semigroup problem for the Heisenberg group. We show that for a subset $S \subset \mathcal{H}_n$, that avoids any maximal semigroup with non-empty interior, the closure of the semigroup generated by S is actually a group.

1 Introduction

Consider a continuous transformation $f : \mathcal{X} \to \mathcal{X}$, a Lie group Γ, and a continuous map $\beta : \mathcal{X} \to \Gamma$ called a *cocycle*. These determine a *skew product with fiber* Γ, or a Γ-*extension*,

$$f_\beta : \mathcal{X} \times \Gamma \to \mathcal{X} \times \Gamma, \quad f_\beta(x, \gamma) = (fx, \gamma\beta(x)).$$

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The Γ-extension f_β is called *topologically transitive*, or simply *transitive*, if it has a dense forward orbit $\{f_\beta^n(x_0)|\ n \in \mathbb{N}\}$. The problem of interest to us is whether noncompact Lie group extensions of a hyperbolic basic set are typically topologically transitive.

Let (M, d_M) be a smooth manifold endowed with a Riemannian metric. Let $f : M \to M$ be a smooth diffeomorphism and $\mathcal{X} \subset M$ a compact and f-invariant subset of M. Let Df be the derivative of f. We recall that $\mathcal{X}$ is said to be *hyperbolic* if there exists a continuous Df-invariant splitting $E^s \oplus E^u$ of the tangent bundle $T_{\mathcal{X}}M$ and constants $C_1 > 0, 0 < \lambda < 1$, such that for all $n \geq 0$ and $x \in \mathcal{X}$ we have:

$$\|(Df^n)_x v\| \leq C_1 \lambda^n \|v\|, \ \ v \in E^s_x$$
$$\|(Df^{-n})_x v\| \leq C_1 \lambda^n \|v\|, \ \ v \in E^u_x.$$

We say that $\mathcal{X}$ is *locally maximal* if there exists an open neighborhood U of $\mathcal{X}$ such that every compact f-invariant set of U is contained in $\mathcal{X}$. A locally maximal hyperbolic set $\mathcal{X}$ is a *hyperbolic basic set* if $f : \mathcal{X} \to \mathcal{X}$ is transitive and $\mathcal{X}$ is not a single periodic orbit.

By a *C^r function on $\mathcal{X}$*, $r \geq 1$, we mean a function that is the restriction of a C^r function on an open neighborhood of $\mathcal{X}$.

Given a connected Lie group Γ and a C^r cocycle $\beta : \mathcal{X} \to \Gamma$, $r > 0$ [1], we consider the Γ-extension $f_\beta : \mathcal{X} \times \Gamma \to \mathcal{X} \times \Gamma$ given by $f_\beta(x, \gamma) = (fx, \gamma\beta(x))$. For brevity, we say that the cocycle β is *transitive* if the Γ-extension f_β is transitive.

In [7] we proposed a general conjecture about transitivity: modulo obstructions appearing from the fact that the range of the cocycle is included in a maximal semigroup with non-empty interior, the set of C^r transitive cocycles contains an open and dense subset.

The conjecture is proved for various classes of Lie groups Γ, mostly semidirect products of compact and Euclidean, in [2, 6, 7, 8, 10, 14]. An important test case is presented by the special Euclidean group $\Gamma = SE(n) = SO(n) \ltimes \mathbb{R}^n$. It is shown in [6, 7, 8] that when n is even the set of cocycles that are transitive is open and dense. In [9] we showed that for $SE(n)$, $n \geq 3$ odd, the transitive C^r cocycles form a residual subset of the space of all C^r cocycles for all $r > 0$. More generally, we considered *Euclidean-type* groups of the form $\Gamma = G \ltimes \mathbb{R}^n$ where G is a compact connected Lie group. The general case of the conjecture remains unsolved for $SE(n)$ if $n \geq 3$ odd.

More recently, [13] obtained examples of groups that are compact extensions of nilpotent (not abelian) Lie groups for which transitivity is open and dense. Recall that a *compact element* g in a Lie group is one for which the closure of the cyclic group generated by g is compact. The method used in [13] borrows from [7] as it relies on the existence of an open and dense set of compact elements in Γ. This approach cannot be applied to nilpotent Lie groups due to the lack of a large set of compact elements. In [10] it is shown that the conjecture holds generically for extensions with fiber the standard real Heisenberg group $\mathcal{H}_n$

[1] That is: Hölder or C^r, $r \geq 1$. The Lipschitz class of functions can be considered as well, but for the simplicity of the notation we do not emphasize it.

of dimension $2n+1$. Moreover, if one considers extensions with fiber a connected nilpotent Lie group with compact commutator subgroup (for example, $\mathcal{H}_n/\mathbb{Z}$), among those that avoid the obvious obstruction, topological transitivity is open and dense.

In this paper we prove the conjecture for extensions with fiber the standard real Heisenberg group $\mathcal{H}_n$. Our approach builds on the techniques developed in [14, 7, 9, 10]. In particular, in [10] results from the classical theory of Diophantine approximation, such as Minkowski Diophantine result for indefinite quadratic forms, come for the first time into play. These techniques are further employed and developed here.

More precisely, a new technical tool introduced in the paper, Theorem 6.2, is a diophantine approximation result that gives, under general conditions, an infinite set of approximate positive integer solutions for a diophantine system of equations consisting of a quadratic indefinite form and several linear equations. The set of approximate solutions can be chosen to point in a certain direction, which can be chosen from a residual subset of full measure of the set of real directions solving the associated algebraic system of equations exactly.

Definition 1.1 For $n \geq 1$, let $\mathcal{H}_n$ denote the Heisenberg group, consisting of matrices of the form

$$(a, b, c) := \begin{pmatrix} 1 & a^T & c \\ 0 & I_n & b \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{Mat}_{n+2}(\mathbb{R}).$$

where $a, b \in \mathbb{R}^n, c \in \mathbb{R}$ and I_n is the n-dimensional identity matrix.

Remark 1.2

(a) Note that $\mathcal{H}_1$ is the standard 3-dimensional Heisenberg group.

(b) If we identify $\mathcal{H}_n$ with $\mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R}$, then the multiplication is

$$(a, b, c)(a', b', c') = (a + a', b + b', c + c' + a^T b'). \tag{1.1}$$

(c) One can also identify, via the exponential map, $\mathcal{H}_n$ with its Lie algebra $\mathrm{Lie}(\mathcal{H}_n) = \mathbb{R}^n \oplus \mathbb{R}^n \oplus \mathbb{R}$; the group operation is given by the Baker-Campbell-Hausdorff formula:

$$X * Y = X + Y + \frac{1}{2}[X, Y], \tag{1.2}$$

where for $X = (a, b, c), Y = (A, B, C) \in \mathrm{Lie}(\mathcal{H}_n)$ the Lie bracket is

$$[(a, b, c), (A, B, C)] = a \cdot B - A \cdot b = \omega((a, b), (A, B)), \tag{1.3}$$

the symplectic form on $\mathbb{R}^{2n}$. By abuse of notation, we will write

$$\omega(u, v) = [u, v] \text{ for } u, v \in \mathbb{R}^{2n}.$$

Using the Baker-Campbell-Hausdorff formula, X^n is given by nX, X^{-1} by $-X$, and

$$A_1 * A_2 * A_3 * \cdots * A_m = \sum_{i=1}^{m} A_i + \frac{1}{2} \sum_{1 \le i < j \le m} [A_i, A_j].$$

(d) We will denote by $\zeta : \mathcal{H}_n \cong \mathbb{R}^{2n} \oplus \mathbb{R} \to \mathbb{R}$ the projection on the center component and by $\pi : \mathcal{H}_n \cong \mathbb{R}^{2n} \oplus \mathbb{R} \to \mathbb{R}^{2n}$ the projection on the abelian component of $\mathcal{H}_n$.

(e) Let $\{X_1, \dots, X_n, Y_1, \dots, Y_n, Z\}$ be the canonical basis in $\mathrm{Lie}(\mathcal{H}_n)$. Note that the only nontrivial relations are $[X_i, Y_i] = Z, 1 \le i \le n$.

The center of $\mathcal{H}_n$ is $[\mathcal{H}_n, \mathcal{H}_n] = \{(0,0,c)\} = \mathbb{R}$. Denote $\widehat{\mathcal{H}}_n = \mathcal{H}_n/\mathbb{R} \cong \mathbb{R}^{2n}$. If $\beta : \mathcal{X} \to \mathcal{H}_n$ is a cocycle, denote by $\widehat{\beta} : \mathcal{X} \to \mathbb{R}^{2n}$ the corresponding quotient cocycle. There is an obvious obstruction to the transitivity of β, namely that $\widehat{\beta} : \mathcal{X} \to \mathbb{R}^{2n}$ takes values in a half-space bounded by a hyperplane passing through the origin. To avoid repetition, we assume from now on that a *half-space* in a linear space is always bounded by a hyperplane passing through the origin. More generally, if $\widehat{\beta}$ is cohomologous to a cocycle with values in a half-space, then f_β is not transitive.[2]

Remark 1.3 By [1], a Hölder $\mathbb{R}^d$-valued cocycle is cohomologous to one that takes values in a half-space if and only if its periodic data is in a half-space.

If $r > 0$, let $\mathcal{S}^r(\mathcal{X}, \mathcal{H}_n)$ be the set of C^r cocycles $\beta : \mathcal{X} \to \mathcal{H}_n$ for which $\widehat{\beta}$ is *not* cohomologous to a cocycle with values in a half-space. These are exactly the cocycles which do not have the range included in a maximal semigroup of $\mathcal{H}_n$ with non-empty interior, see Section 8. Our main result is:

Theorem 1.4 *Assume that $\mathcal{X}$ is a hyperbolic basic set for $f : \mathcal{X} \to \mathcal{X}$. Let $r > 0$. Then $\mathcal{S}^r(\mathcal{X}, \mathcal{H}_n)$ contains a dense and open set of transitive cocycles.*

More precisely, we prove:

Theorem 1.5 *Let $\mathcal{X}$ be a hyperbolic basic set for $f : \mathcal{X} \to \mathcal{X}$ and $\beta : \mathcal{X} \to \mathcal{H}_n$ a Hölder cocycle. If $\widehat{\beta} : \mathcal{X} \to \mathbb{R}^{2n}$ is transitive, then so is β.*

This implies Theorem 1.4 because, by [14, 2]:

Theorem 1.6 *For $r > 0$, there is an open and dense set $\mathcal{U}_r \subset \mathcal{S}^r(\mathcal{X}, \mathbb{R}^d)$ consisting of transitive cocycles.*

[2] We recall that $\beta, \beta' : \mathcal{X} \to \mathbb{R}^d$ are *cohomologous* if there exists a map $P : \mathcal{X} \to \mathbb{R}^d$ such that for all $x \in \mathcal{X}$, $\beta'(x) = P(fx) + \beta(x) - P(x)$.

Here $\mathcal{S}^r(\mathcal{X}, \mathbb{R}^d)$ is the set of C^r cocycles $\beta : \mathcal{X} \to \mathbb{R}^d$ that are *not* cohomologous to a cocycle with values in a half-space in $\mathbb{R}^d$, equivalently, the set of C^r cocycle whose periodic data is not on one side of a hyperplane.

For many Lie groups Γ it is not hard to show that, for $p > 0$ big enough, there is a large open set $U \subset \Gamma^p$ such that if $F \in U$ then the family F generates Γ, that is, the group generated by F is dense in Γ [16]. The proof of transitivity of an extension f_β (see Theorem 2.5) is based on showing that the set $\mathcal{L}_\beta(x)$ of "heights" of β over a periodic point x is the whole fiber Γ. To obtain the condition $\mathcal{L}_\beta(x) = \Gamma$, we have to prove that for a typical family $F \in \Gamma^p$ that generates Γ as a group, if F is not contained in a maximal semigroup with non-empty interior, then F generates Γ as a semigroup as well. We refer to this question as the *Semigroup Problem.* The problem was solved for $\Gamma = \mathbb{R}^n$ [14] and more generally for groups of the form $\Gamma = K \times \mathbb{R}^n$ where K is a compact Lie group [7, Theorem 5.10]. It is also solved for $\Gamma = SE(n)$ [7, Theorem 6.8].

The remainder of the paper is organized as follows. In Section 2 we recall some general results from [7], in particular a criterion for transitivity of extensions of hyperbolic systems. In Section 3 we recall a technical result from [9] that allows to obtain elements of $\mathcal{L}_\beta(x)$. In Section 4 we present a strategy to prove Theorem 1.5. In Section 5 we review and extend results for $\mathbb{R}^d$-extensions and their periodic data. In Section 6 we prove Diophantine approximation results for indefinite quadratic forms needed for our estimates. In Section 7 we construct a special type of indefinite quadratic form that is bounded along infinite sequences. In Section 8 we solve the semigroup problem for the Heisenberg group. This is used in the proof of the main result. In Section 9 we show some preparatory results about ε-dense semigroups in $\mathbb{R}^n$. In Section 10 we specialize to the setting of nilpotent Lie groups and prove Theorem 1.5, and consequently the main result Theorem 1.4.

2 Criterion for transitivity

Let Γ be a connected Lie group with Lie algebra $L\Gamma$. We denote by e_Γ the identity element of Γ. Let Ad denote the adjoint representation of Γ on $L\Gamma$. Let $\|\cdot\|$ be a norm on $L\Gamma$. It is known that there is a metric d on Γ with the following properties for any $\gamma, \gamma_1, \gamma_2 \in \Gamma$:

(a) $d(\gamma\gamma_1, \gamma\gamma_2) = d(\gamma_1, \gamma_2)$;

(b) $d(\gamma_1\gamma, \gamma_2\gamma) \le \|\mathrm{Ad}(\gamma)\| d(\gamma_1, \gamma_2)$.

Definition 2.1 Let $f : \mathcal{X} \to \mathcal{X}$ be a map and $\beta : \mathcal{X} \to \Gamma$ a cocycle. We write $f^k_\beta(x, \gamma) = (f^k x, \gamma\beta(k, x))$ where, for $k \ge 1$,

$$\beta(k, x) = \beta(x)\beta(fx)\cdots\beta(f^{k-1}x) = \prod_{j=0}^{k-1} \beta(f^j x).$$

The meaning of the product notation in the last term above is the middle expression.

If Q is a trajectory of f of length k (i.e. $Q = \{x, f(x), \dots, f^{k-1}(x)\}$ for some x), then we define the *height of* β *over* Q to be $\beta(Q) = \beta(k, x)$. In particular, if x is a periodic point of period ω, then the height of the corresponding periodic orbit P is $\beta(P) = \beta(\omega, x)$. The set of heights of β over all periodic orbits of f is referred to as *the periodic data* of β.

Definition 2.2 Given a cocycle $\beta : \mathcal{X} \to \Gamma$ over $f : \mathcal{X} \to \mathcal{X}$, define $\mu \geq 1$ to be

$$\mu = \max\Big\{ \lim_{n\to\infty} \sup_{x\in X} \|\mathrm{Ad}(\beta(n,x))\|^{1/n}, \lim_{n\to\infty} \sup_{x\in X} \|\mathrm{Ad}(\beta(n,x))^{-1}\|^{1/n} \Big\}.$$

We say that the cocycle β has *subexponential growth* if $\mu = 1$.

Remark 2.3 The subexponential growth condition is automatically satisfied for any cocycle if the group Γ is compact, nilpotent, or a semidirect product of compact and nilpotent.

One of the key notions used in this paper was introduced in [7]:

Definition 2.4 Let Γ be a connected Lie group, $\mathcal{X}$ a hyperbolic basic set for $f : \mathcal{X} \to \mathcal{X}$, $\beta : \mathcal{X} \to \Gamma$ a cocycle, and $f_\beta : \mathcal{X} \times \Gamma \to \mathcal{X} \times \Gamma$ the skew-extension. Given $x \in \mathcal{X}$, let

$$\mathcal{L}_\beta(x) = \{\gamma \in \Gamma : \exists x_k \in \mathcal{X}, n_k > 0 \text{ such that } x_k \to x, f_\beta^{n_k}(x_k, e_\Gamma) \to (x, \gamma)\}.$$

The set $\mathcal{L}_\beta(x)$ consists of possible limits $\lim_{k\to\infty} \beta(n_k, x_k)$, subject to $x_k \to x$ and $f^{n_k}(x_k) \to x$. We do not require that $n_k \to \infty$ or $x_k \neq x$. Clearly $\mathcal{L}_\beta(x) \subseteq \Gamma$ is closed.

The following theorem is a special case of [7, Lemma 3.1, Theorem 3.3].

Theorem 2.5 *Assume that $\mathcal{X}$ is a hyperbolic basic set for $f : \mathcal{X} \to \mathcal{X}$, that Γ is a connected Lie group and that $\beta : \mathcal{X} \to \Gamma$ is a Hölder cocycle that has subexponential growth. Then*

(a) $\mathcal{L}_\beta(x)$ is a closed semigroup of Γ for each $x \in \mathcal{X}$.

(b) β is a transitive cocycle if and only if there exists a point $x_0 \in \mathcal{X}$ such that $\mathcal{L}_\beta(x_0) = \Gamma$.

The next lemma is a consequence of [15, Appendix A].

Lemma 2.6 *Assume that $\mathcal{X}$ is a hyperbolic basic set for $f : \mathcal{X} \to \mathcal{X}$, that Γ is a connected Lie group and that $\beta : \mathcal{X} \to \Gamma$ is an α-Hölder cocycle that has subexponential growth. Then the Γ-extension $f_\beta : \mathcal{X} \times \Gamma \to X \times \Gamma$ admits stable and unstable foliations which are α-Hölder and invariant under right multiplication by elements of Γ. The stable and unstable leaves of f_β through $(x, e_\Gamma) \in \mathcal{X} \times \Gamma$ are the graphs of the functions*

$$\gamma_x^s : W^s(x) \to \Gamma, \quad \gamma_x^s(y) = \lim_{n\to\infty} \beta(n,x)\beta(n,y)^{-1},$$
$$\gamma_x^u : W^u(x) \to \Gamma, \quad \gamma_x^u(y) = \lim_{n\to\infty} \beta(-n,x)\beta(-n,y)^{-1}.$$

These functions are α-Hölder and vary continuously with the cocycle β in the following sense: if $\beta_k \to \beta$ in C^0-topology and β_k remains C^α-bounded, then $\gamma^s_{k,x} \to \gamma^s_x$ on $W^s_{loc}(x)$ and $\gamma^u_{k,x} \to \gamma^u_x$ on $W^u_{loc}(x)$ in C^0-topology.

We call the values of the functions γ^s_x, γ^u_x *holonomies along stable/unstable leaves.*

3 Elements of $\mathcal{L}_\beta$

In this section we recall a method to obtain elements of $\mathcal{L}_\beta$ introduced in [9]. Throughout, (M, d_M) is a Riemannian manifold, $\mathcal{X} \subset M$ a hyperbolic basic set for $f : \mathcal{X} \to \mathcal{X}$, Γ a connected Lie group and $\beta : \mathcal{X} \to \Gamma$ a Hölder cocycle that has subexponential growth.

Denote by $W^s(x)$ and $W^u(x)$ the stable and unstable leaves of f through x.

Definition 3.1 By a *periodic heteroclinic cycle* we mean a cycle consisting of points $p_1, \dots, p_k$ that are periodic for the map f, have disjoint trajectories, and such that p_j is transverse heteroclinic to p_{j+1} through a point $\zeta_j \in W^u(p_j) \cap W^s(p_{j+1})$, for $j = 1, \dots, k$ (where $p_{k+1} = p_1$).

Let $P_1, \dots, P_k$ be the corresponding periodic orbits and denote the periods by $\omega_1, \dots, \omega_k$. Denote by O_j the heteroclinic trajectory from p_j to p_{j+1} of the point ζ_j chosen above, and by H_j the holonomy along this heteroclinic connection (that is, along $W^u(p_j)$ from p_j to ζ_j and then along $W^s(p_{j+1})$ from ζ_j to p_{j+1}).

Replace the heteroclinic orbit O_j from p_j to p_{j+1} by the trajectory Q_j of length $\omega_j M_j + \omega_{j+1} M_{j+1}$ that spends time $\omega_j M_j$ in the first half of O_j and time $\omega_{j+1} M_{j+1}$ in the second half of O_j; that is, $Q_j = \{f^n(\zeta_j) \mid -\omega_j M_j \le n < 0\} \cup \{f^n(\zeta_j) \mid 0 \le n < \omega_{j+1} M_{j+1}\})$. For the trajectory connecting p_k to p_{k+1}, we allow M_1 and M_{k+1} to be distinct.

Definition 3.2

(a) For $N = (M_1, \dots, M_{k+1})$ a vector of positive integers, consider the *periodic pseudo-orbit* $Q(N)$ constructed as follows: start at p_1, cover M_1 times the periodic orbit P_1, follow Q_1 to p_2, cover M_2 times the orbit P_2, and so on; at p_k cover M_k times the orbit P_k, follow Q_k to $p_{k+1} = p_1$, and finally cover M_{k+1} times the orbit P_1.

Denote by $\widetilde{Q}(N)$ the *periodic* orbit that shadows $Q(N)$.

(b) For a sequence of vectors $N(1), N(2) \ldots \in \mathbb{N}^{k+1}$ whose entries are positive integers, $N(i) = (M_1(i), \dots, M_{k+1}(i))$, say that the sequence is *admissible* if there is a constant $C_2 \ge 1$ such that $M_p(i)/M_q(i) \le C_2$ for all $p, q = 1 \dots, k+1$ and all $i \ge 1$.

(c) If $N = (M_1, \dots, M_{k+1})$ is a sequence of vectors, we write $N \to \infty$ if $M_s \to \infty$ for each $s = 1, \dots, k+1$.

The following result is [9, Theorem 3.4 and Proposition 3.6].

Theorem 3.3 *For* $N = (M_1, \ldots, M_{k+1}) \in \mathbb{N}^{k+1}$ *define*

$$A(N) = \beta(P_1)^{M_1} H_1\, \beta(P_2)^{2M_2} H_2 \cdots \beta(P_k)^{2M_k} H_k\, \beta(P_1)^{M_{k+1}}.$$

Then $\operatorname{dist}(\widetilde{Q}(N(\ell)), A(N(\ell))) \to 0$ *whenever* $N(1), N(2), \ldots$ *is an admissible sequence converging to infinity.*

Therefore, if $A = \lim_{\ell\to\infty} A(N(\ell))$ *exists, then* $A \in \mathcal{L}_\beta(p_1)$.

Compute now $A(N)$ for $\Gamma = \mathcal{H}_n$ (or, more generally, a step-2 nilpotent group). Identify Γ with $\mathrm{Lie}(\mathcal{H}_n)$ and use the Baker-Campbell-Hausdorff (1.2) formula for the group operation.

Denote

$$A_1 = A_{k+1} = \beta(P_1) \text{ and } A_\ell = \beta(P_\ell)^2 \text{ for } 2 \le \ell \le k. \tag{3.1}$$

Then

$$\begin{aligned} A(N) &:= \beta(P_1)^{M_1} H_1\, \beta(P_2)^{2M_2} H_2 \cdots \beta(P_k)^{2M_k} H_k\, \beta(P_1)^{M_{k+1}} \\ &= A_1^{M_1} H_1\, A_2^{M_2} H_2 \cdots A_k^{M_k} H_k\, A_{k+1}^{M_{k+1}} \\ &= \sum_{i=1}^{k+1} M_i A_i + \tfrac{1}{2}\sum_{j=1}^{k}\left(\sum_{1\le i\le j} M_i[A_i, H_j] - \sum_{j<i\le k+1} M_i[A_i, H_j]\right) + \sum_{i=1}^{k} H_i \\ &+ \tfrac{1}{2}\sum_{1\le i<j\le k+1} M_i M_j [A_i, A_j] + \tfrac{1}{2}\sum_{1\le i<j\le k} [H_i, H_j] \\ &= (L(N) + E, Q(N) + L_Z(N) + e) \in \mathbb{R}^{2n} \oplus \mathbb{R} \end{aligned} \tag{3.2}$$

where the last line describes the components of $A(N)$:

- $L(N) = (L_1(N), L_2(N), \ldots, L_{2n}(N)) = \sum_i M_i \pi(A_i)$ is a linear $Ab(\mathcal{H}_n)$-valued map;
- $Q(N) = \frac{1}{2}\sum_{1\le i<j\le k+1} M_i M_j [A_i, A_j]$ is $\mathbb{R}$-valued quadratic;
- $L_Z(N)$ is $\mathbb{R}$-valued linear;
- E and e do not depend on N:
- the holonomies H_i appear only in the map L_Z and the constants E and e;
- the map L_Z contains also the sum $\sum_i M_i \zeta(A_i)$, and this is the only place where $\zeta(A_i)$ appear in $A(N)$.

4 Strategy for proving Theorem 1.5

Assume that $\widehat{\beta}$ is transitive. Let p_1 be a periodic point. We will show that the projection of the semigroup $\mathcal{L}_\beta(p_1) \subset \mathcal{H}_n$ onto $\mathrm{Ab}(\mathcal{H}_n)$ is dense. From Theorem 8.6, addressing the Semigroup Problem, it then follows that $\mathcal{L}_\beta(p_1) = \mathcal{H}_n$. By Theorem 2.5, this implies that β is transitive.

The main difficulty is to obtain elements of $\mathcal{L}_\beta(p_1)$. For this we use the shadowing of heteroclinic cycles, as described in Theorem 3.3 and Section 3. As the computation in (3.2) shows, in order for $A(N)$ to converge we have to control simultaneously certain linear forms and a quadratic form.

From the transitivity of $\widehat{\beta}$ one can easily see that the periodic data of $\widehat{\beta}$ is dense in $Ab(\mathcal{H}_n)$, see e.g. [10, Proposition 3.2(a)]. This data determines the linear forms in the expression of $\pi(A(N))$, and the quadratic form in $\zeta(A(N))$.

Obtaining convergence of $\pi(A(N))$ is not difficult (this was achieved already in [14]). The problem is to simultaneously keep the center component bounded. Here the (unknown) holonomies contribute to the linear part; to counterbalance them we use Theorem 5.1.

The main new tool is Theorem 6.2, which, under certain conditions, allows to keep a quadratic form bounded and a number of linear forms small, on *integer points* converging to infinity along a given direction $\mathbf{v}$. In our application $\mathbf{v}$ will be a vector with positive coordinates. On this direction all these forms have to vanish, so Theorem 6.2 says that if there is a solution to these equation over $\mathbb{R}$, then there are approximating integer solutions too. Thus, it remains to show that one can arrange for these equations to have solution over the reals. This is done in Section 7.

5 $\mathbb{R}^d$-valued cocycles and periodic data

In this section, we use the transitivity of $\widehat{\beta}$ to derive information about the periodic data for β. Namely, we expand [10, Proposition 3.2] to prove:

Theorem 5.1 *Let $\beta : \mathcal{X} \to \mathcal{H}_n$ be a Hölder cocycle such that $\widehat{\beta} : \mathcal{X} \to \mathrm{Ab}(\mathcal{H}_n) \cong \mathbb{R}^{2n}$ is transitive.*

(a) For any open set $U \subset \mathcal{X}$, the periodic data of $\widehat{\beta}$ corresponding to periodic orbits of f that intersect U is dense in $Ab(\mathcal{H}_n)$.

(b) Moreover, for each open set $V \subset \mathbb{R}^{2n}$ and for each open set $U \subset \mathcal{X}$, there are periodic orbits P of f starting in U such that $\widehat{\beta}(P) \in V$ and the component of $\beta(P)$ in the center of $\mathcal{H}_n$ is arbitrarily large (that is, approaching $+\infty$), or arbitrarily small (that is, approaching $-\infty$).

Proof

Part (a) is proven in [10, Proposition 3.2 (a)], using shadowing.

For (b), recall the first statement in Theorem 3.3: $\mathrm{dist}(\widetilde{Q}(N(\ell)), A(N(\ell))) \to 0$ provided $N(1), N(2), \ldots$ converges to infinity along an admissible sequence (for the notation, see the discussion preceding Theorem 3.3). Therefore, it suffices to prove the statement for the products $A(N)$. These are computed in formula (3.2).

By (a) and Theorem 9.2 for the last requirement, one can select $k = 2n + 1$ distinct periodic points in U and such that the corresponding periodic data A_i of β (see the notation in (3.1)) satisfy

- the origin is in the (interior of the) convex hull of the values $\widehat{A}_i := \pi(A_i)$: there are $\alpha_i > 0$ such that $\sum_{i=1}^{k} \alpha_i = 1$ and $\sum_{i=1}^{k} \alpha_i \widehat{A}_i = 0$;
- $[A_2, A_3] = [\widehat{A}_2, \widehat{A}_3] \neq 0$;
- the semigroup generated by their projection $\widehat{A}_i$ on $\mathrm{Ab}(\mathcal{H}_n)$ is ε-dense in $\mathrm{Ab}(\mathcal{H}_n)$, where $\varepsilon > 0$ is small enough so that the semigroup intersects any translate of V.

As in [8, Lemma 2.12] (see (8.1) and Lemma 8.8), there is a sequence of integers $p_i^{(\ell)} \to \infty$, such that

$$\|(p_1^{(\ell)}, \ldots, p_k^{(\ell)}) - t_{(\ell)}(\alpha_1, \ldots, \alpha_k)\| \to 0, \tag{5.1}$$

with $t_{(\ell)} \to \infty$; so $e_{(\ell)} := \sum_{i=1}^{k} p_i^{(\ell)} \widehat{A}_i \to 0$.

Pick the heteroclinic cycle that visits the above periodic data in the order $A_1, A_2, A_3, \ldots, A_1$. There are integers $n_i \geq 0$ such that $v_0 := \sum_{i=1}^{k} n_i \widehat{A}_i + E \in V$ where E, from formula (3.2), is the contribution of the holonomies.

Since the first periodic point in the heteroclinic cycle is also the last one, $\widehat{A}_1$ appears with the factor $M_1 + M_{k+1}$ in the $\mathrm{Ab}(\mathcal{H}_n)$-component of (3.2); therefore, we split the correct coefficient of $\widehat{A}_1$ into two integers close to each other. Abusing notation, we denote these by the half of the correct value, and define

$$N(\ell) := (n_1/2, n_2, \ldots, n_k, n_1/2) + (p_1^{(\ell)}/2, p_2^{(\ell)}, \ldots, p_k^{(\ell)}, p_1^{(\ell)}/2),$$

which is an admissible sequence because $p_i^{(\ell)} \approx t_{(\ell)} \alpha_i$.

By (3.2) and (5.1),

$$A(N(\ell)) = (v_0 + e_{(\ell)}, \left(t_{(\ell)}\right)^2 Q + O(t_{(\ell)})) \tag{5.2}$$

with

$$Q = \tfrac{1}{2} \sum_{1 \leq i < j \leq k+1} \widetilde{\alpha}_i \widetilde{\alpha}_j [\widehat{A}_i, \widehat{A}_j]$$

where the $\widetilde{\ }$ denotes the split of α_1 into two equal halves: $\widetilde{\alpha}_1 = \widetilde{\alpha}_{k+1} = \alpha_1/2$, $\widetilde{\alpha}_i = \alpha_i$ for $2 \leq i \leq k$.

If $Q = 0$, reroute the heteroclinic cycle to exchange A_2 and A_3 (that is, exchange the order in which the second and third points in the heteroclinic cycle are visited) and redo the above construction. Although this will change E, the only change in the final formula (5.2) is that Q is modified by $\alpha_2\alpha_3[A_2, A_3] \neq 0$.

Once $Q \neq 0$ we are done: to get the $+\infty$ and $-\infty$ limits of the center part, consider the heteroclinic cycle in the original order and in reverse order (which changes Q to its negative). See Section 8, the proof of Theorem 8.6, for a related computation. ∎

6 Diophantine approximation for indefinite quadratic forms

Definition 6.1 By the *rank* of a quadratic form we mean the number of squares (with either positive of negative sign) it has when diagonalized.

The Diophantine approximation result we need is the following:

Theorem 6.2 *For $d \geq 2$ assume given in $\mathbb{R}^d$ a (homogeneous) quadratic form Q and k (homogeneous) linear forms $L_1, L_2, \ldots, L_k$ such that $Q|_{\cap Ker\ L_i}$ is indefinite.*

Assume that $\operatorname{rank} Q \geq 2k+3$. Then for a residual, full measure set (in the induced topology/Lebesgue measure) of vectors $\mathbf{v} \neq \mathbf{0}$ in

$$\{Q = 0\} \cap \{L_i = 0, 1 \leq i \leq k\},$$

for any $\varepsilon > 0$ there are $\mathbf{x}_n \in \mathbb{Z}^d$ such that:

(a) $\|\mathbf{x}_n\| \to \infty$,

(b) $\sup |Q(\mathbf{x}_n)| < \infty$,

(c) $\operatorname{dist}(\mathbf{x}_n, \mathbb{R}_+\mathbf{v}) \leq \varepsilon$.

In particular,

$$|L_i(\mathbf{x}_n)| \leq C\varepsilon, \text{ for all } 1 \leq i \leq k \text{ and all } n,$$

with a constant $C > 0$ determined by the linear forms.

This is a consequence of the following two results.

Theorem 6.3 *For $d \geq 2$ assume given in $\mathbb{R}^d$ a quadratic form Q and k linear forms $L_1, L_2, \ldots, L_k$.*

Assume that $\mathbf{v} \in \mathbb{R}^d$ satisfies the following two conditions:

$$\mathbf{0} \neq \mathbf{v} \in \{Q = 0\} \cap \{L_i = 0, 1 \leq i \leq k\}; \tag{6.1}$$

$$\mathbf{v} \notin \operatorname{span}_{\mathbb{R}}(\mathcal{P}_{\mathbf{v}} \cap \mathbb{Z}^d) \tag{6.2}$$

where $\mathcal{P}_{\mathbf{v}}$ is the tangent plane to Q at $\mathbf{v}$; note that $Q|_{\cap Ker\ L_i}$ must be indefinite.

Then for any $\varepsilon > 0$ there are $\mathbf{x}_n \in \mathbb{Z}^d$ such that properties (a)-(c) of Theorem 6.2 hold.

Remark 6.4 The conclusion obviously holds for rational vectors satisfying (6.1), even if (6.2) fails: take for $\mathbf{x}_n$ multiples of $\mathbf{v}$. We use (6.1) to obtain in other cases a sequence of vectors converging to infinity.

Proposition 6.5 *Assume that $Q, L_1, L_2, \ldots, L_k$ are as in Theorem 6.3, with $Q|_{\cap Ker\ L_i}$ indefinite.*

If $\operatorname{rank} Q \geq 2k+3$, then the set of vectors $\mathbf{v} \in \mathbb{R}^d$ that satisfy (6.1) and do not satisfy (6.2) is contained in a countable union of codimension-$(k+2)$ subspaces of $\mathbb{R}^d$.

Proof of Theorem 6.3 We will use Minkowski's Convex Body Theorem [11, pp. 73-76]: if $K \subset \mathbb{R}^d$ is a closed convex body symmetric about the origin and of volume at least 2^d, then it contains a non-zero integer point.

Without loss we can assume that both $\mathbf{v}$ and $\operatorname{grad} Q|_{\mathbf{v}}$ have length one (for the latter, rescale Q). To make the estimates of volumes and bounds of Q easier, choose orthonormal coordinates $(x_1, \ldots, x_d)$ in $\mathbb{R}^d$ such that $\mathbf{v} = (1, 0, \ldots, 0)$ and $\operatorname{grad} Q|_{\mathbf{v}} = (0, 1, \ldots, 0)$ [3]. Note however that we apply Minkowski's theorem to the original lattice $\mathbb{Z}^d$. Denote $\mathbf{r} = (x_3, \ldots, x_d)$. Then

$$Q(x_1, x_2, \mathbf{r}) = x_2^2 + x_1 x_2 + x_2 \varphi(\mathbf{r}) + \widetilde{Q}(\mathbf{r})$$

where φ is linear and $\widetilde{Q}$ is quadratic. [4]

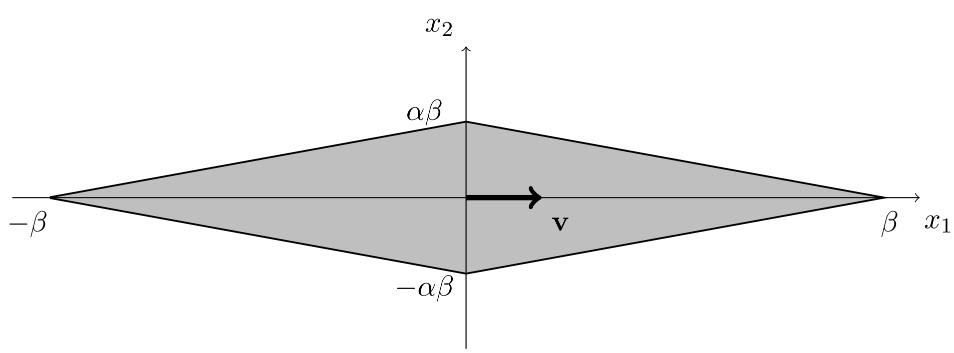

Figure 1: (x_1, x_2) cross-section through $V_{\alpha,\beta,\varepsilon}$.

[3] Thus the tangent plane $\mathcal{P}_{\mathbf{v}}$ has equation $\{x_2 = 0\}$.

[4] The term $x_1\psi(\mathbf{r})$ is missing because of the gradient of Q at $\mathbf{v}$.

Consider the convex body given by the "wedge" $V_{\alpha,\beta,\varepsilon}$ along the $\mathbf{v}$ direction (we will use $\beta \to \infty$, $\alpha \to 0^+$)

$$|x_1| \le \beta, \qquad |x_2| \le \alpha(\beta - |x_1|), \qquad \|\mathbf{r}\| \le \varepsilon,$$
$$\mathrm{vol}(V_{\alpha,\beta,\varepsilon}) = 2C_{r-2}\alpha\beta^2\varepsilon^{d-2}$$

where C_{r-2} is the volume of the unit ball in $\mathbb{R}^{d-2}$. An x_1, x_2 cross-section through $V_{\alpha,\beta,\varepsilon}$ is shown in Figure 1.

We estimate the maximum value of Q on $V_{\alpha,\beta,\varepsilon}$:

$$\begin{aligned}|Q| &\le |x_2|^2 + |x_1 x_2| + |x_2| \cdot \|\varphi\| \cdot \|\mathbf{r}\| + C\|\mathbf{r}\|^2 \\ &\le \alpha^2\beta^2 + \alpha\beta^2 + C(\alpha\beta\varepsilon + \varepsilon^2).\end{aligned}$$

Therefore, if

$$\alpha = \frac{1}{MA^2}, \quad \beta = MA \qquad \text{with} \qquad A > 1, \quad M > 1 \tag{6.3}$$

then

$$\mathrm{vol}(V_{\alpha,\beta,\varepsilon}) = CM\varepsilon^{d-2}, \tag{6.4}$$

and

$$\sup_{V_{\alpha,\beta,\varepsilon}} |Q| \le C(1 + M + \varepsilon + \varepsilon^2). \tag{6.5}$$

We can now apply Minkowski's theorem. In view of (6.2), which we rewrite as

$$\mathbb{R}\mathbf{v} \cap \mathrm{span}_{\mathbb{R}}(\mathcal{P}_{\mathbf{v}} \cap \mathbb{Z}^d) = \{\mathbf{0}\},$$

one can reduce $\varepsilon > 0$ so that the ε-neighborhood of $\mathbb{R}\mathbf{v}$ does not contain any non-zero point of $\mathcal{P}_{\mathbf{v}} \cap \mathbb{Z}^d$. By (6.4), one can select M so large that $\mathrm{vol}(V_{\alpha,\beta,\varepsilon}) > 2^d$. Since there is only one parameter left in (6.3), denote the corresponding wedge $V_{\alpha,\beta,\varepsilon}$ by $V(A)$.

Pick A_1 so that the x_2-height of $V(A)$ satisfies $\alpha\beta = 1/A_1 \le \varepsilon$. By Minkowski's theorem, there is a non-zero integer point $\mathbf{x}_1 \in V(A_1)$; by symmetry, one can take $\mathbf{x_1}$ so that its x_1-coordinate is non-negative, so it is close to the ray $\mathbb{R}_+\mathbf{v}$.

By the choice of ε, $\mathbf{x_1}$ is not in the tangent plane $\mathcal{P}_{\mathbf{v}} = \{x_2 = 0\}$. Therefore, we can select $A_2 > A_1$ such that neither $\mathbf{x_1}$ nor any integer point of norm less than $2\|\mathbf{x}_1\|$ are in $V(A_2)$.[5] Minkowski's theorem applied to $V(A_2)$ yields $\mathbf{x}_2 \in V(A_2)$ along $\mathbb{R}^+\mathbf{v}$. We continue the process by selecting at each step A so large so that none of the previous points is in $V(A)$.

It follows from (6.4) that Q remains bounded during this process. ∎

[5] Here we use (6.2) and the shape of $V(A)$, which becomes closer to the plane $\mathcal{P}_{\mathbf{v}}$ as $A \to \infty$.

Proof of Proposition 6.5 Let $Q(\mathbf{x}) = \mathbf{x}^T M \mathbf{x}$, where $M = M^T$ is a symmetric $d \times d$ matrix.

Assume that $\mathbf{v} \in \mathbb{R}^d$ satisfies (6.1). Then the tangent plane $\mathcal{P}_{\mathbf{v}}$ has normal $\mathbf{a} = (a_1, \ldots, a_d) = \mathbf{v}^T M$ (seen as a row vector).

To find $\operatorname{span}_{\mathbb{R}}(\mathcal{P}_{\mathbf{v}} \cap \mathbb{Z}^d)$, pick a maximal linearly independent set over $\mathbb{Q}$ from the coordinates of $\mathbf{a}$, say the first r coordinates, $\{a_1, \ldots, a_r\}$. Thus

$$\begin{aligned}(a_{r+1}, \ldots, a_d) = (a_1, \ldots, a_r)\Gamma \iff & (a_1, \ldots, a_d) = (a_1, \ldots, a_r) \begin{pmatrix} \mathrm{I}_r & \Gamma \end{pmatrix} \\ \iff & \mathbf{a} \begin{pmatrix} -\Gamma \\ \mathrm{I}_{d-r} \end{pmatrix} = \mathbf{0} \end{aligned} \tag{6.6}$$

for a matrix $\Gamma \in \operatorname{Mat}_{r \times (d-r)}(\mathbb{Q})$. Then $\mathbf{x} = (x_1, \ldots, x_d) \in \mathcal{P}_{\mathbf{v}} \cap \mathbb{Z}^d$ means

$$0 = \mathbf{a}\, \mathbf{x} = (a_1, \ldots, a_r) \begin{pmatrix} \mathrm{I}_r & \Gamma \end{pmatrix} \mathbf{x}$$

which implies

$$(\mathrm{I}_r \ \ \Gamma)\, \mathbf{x} = \mathbf{0} \iff \mathbf{x} = \begin{pmatrix} -\Gamma \\ \mathrm{I}_{d-r} \end{pmatrix} \begin{pmatrix} x_{r+1} \\ \vdots \\ x_d \end{pmatrix}$$

and therefore

$$\operatorname{span}_{\mathbb{R}}(\mathcal{P}_{\mathbf{v}} \cap \mathbb{Z}^d) = \operatorname{Range}\left[\begin{pmatrix} -\Gamma \\ \mathrm{I} \end{pmatrix} : \mathbb{R}^{d-r} \to \mathbb{R}^d \right] = \operatorname{Ker}\left[\begin{pmatrix} \mathrm{I} & \Gamma \end{pmatrix} : \mathbb{R}^d \to \mathbb{R}^r \right]. \tag{6.7}$$

In conclusion, if $\mathbf{v} \neq \mathbf{0}$ satisfies $Q(\mathbf{v}) = 0$ but not (6.2) then, considering equations (6.7) and (6.6) (and a possible permutation of the coordinates),

$$\begin{pmatrix} \mathrm{I} & \Gamma \end{pmatrix} \mathbf{v} = \mathbf{0}$$
$$\mathbf{v}^T M \begin{pmatrix} -\Gamma \\ \mathrm{I} \end{pmatrix} = \mathbf{0}$$

(these imply that $Q(\mathbf{v}) = \mathbf{v}^T M \mathbf{v} = 0$). That is, $\mathbf{v}$ is in the kernel of the matrix

$$A = \begin{pmatrix} \mathrm{I} \ \ \Gamma \\ (-\Gamma^T \ \ \mathrm{I})M \end{pmatrix},$$

and

$$A \begin{pmatrix} \mathrm{I} & -\Gamma \\ 0 & \mathrm{I} \end{pmatrix} = \begin{pmatrix} \mathrm{I}_r & 0 \\ * & B \end{pmatrix} \quad \text{with} \quad B = U^T M U, \quad U = \begin{pmatrix} -\Gamma \\ \mathrm{I}_{d-r} \end{pmatrix}.$$

Since Γ is rational, the desired conclusion follows from

$$\operatorname{rank} A = r + \operatorname{rank} B \geq k + 2.$$

Write $M = L^T D L$ for some invertible L and a diagonal D. Then $B = V^T D V$ for $V = LU : \mathbb{R}^{d-r} \to \mathbb{R}^d$ of rank $d - r$ and

$$\begin{aligned}\operatorname{rank} B &\geq \dim \operatorname{Range} DV - \dim \operatorname{Ker} V^T \geq (\dim \operatorname{Range} V - \dim \operatorname{Ker} D) - \dim \operatorname{Ker} V^T \\ &= (d - r) - (d - \operatorname{rank} D) - r = \operatorname{rank} D - 2r\end{aligned}$$

which means that it suffices to have

$$r + \max\{0, \operatorname{rank} D - 2r\} \geq k + 2.$$

This is indeed true if $\operatorname{rank} D \geq 2k + 3$. ∎

7 Choosing the quadratic form

In this section we show how to construct out of the periodic data a quadratic form Q and linear forms $L_1, \ldots, L_k$ (see (3.1), (3.2) and the comments after them) that satisfy the assumptions of Theorem 6.2. For this we first pick particular values for the $\mathrm{Ab}(\mathcal{H}_n)$-components of the periodic data; these determine the quadratic form Q and the $2n$ linear forms $L_1, \ldots, L_{2n}$. The last linear form that we have to consider, denoted $\widetilde{L}_Z$, appears in the center; it depends on the center-component of the periodic data as well, and we have some control over that by Theorem 5.1. More details are in Section 10, see (10.1) and the comments preceding it.

We will show that for any small perturbations of the $\mathrm{Ab}(\mathcal{H}_n)$-components, if $\widetilde{L}_Z$ is chosen conveniently, then the hypotheses of Theorem 6.2 are satisfied and therefore we can select a vector $\mathbf{v}$ with all entries positive.

To simplify the notation, we specify the image in $\mathrm{Ab}(\mathcal{H}_n)$ of an element in $\mathcal{H}_n$ by its $\{X_i, Y_i \mid 1 \leq i \leq n\}$-components. We discuss first $\mathcal{H}_1$, then $\mathcal{H}_n$.

7.1 The case of $\mathcal{H}_1$.

We consider first the case $n = 1$, which avoids some of the complications that appear in the general case. On one hand this makes our approach easier to follow, on the other hand referring to this case also simplifies the presentation of the proof for general n.

For $k = 9$, assume the periodic data in $\mathrm{Lie}(\mathcal{H}_1)$ has $\mathrm{Ab}(\mathcal{H}_1)$-component given by [6]

$$A_1 = A_3 = A_6 = A_8 = A_{10} = Y_1,\ A_2 = A_5 = A_7 = A_9 = X_1,\ A_4 = -(X_1 + Y_1). \quad (7.1)$$

[6] There is a small abuse in our notation here: A_i here actually correspond to $\pi(A_i)$ in (3.2). But $\pi(A_i)$ is the only part that contributes to Q and L_j, $1 \leq j \leq 2n$.

We compute the quadratic form (ignoring factors of 2 or 1/2):

$$
\begin{aligned}
Q &= \sum_{1\le i<j\le k+1} M_iM_j[A_i,A_j] \\
&= -M_1M_2 + M_1M_4 - M_1M_5 - M_1M_7 - M_1M_9 \\
&+ M_2M_3 - M_2M_4 + M_2M_6 + M_2M_8 + M_2M_{10} \\
&+ M_3M_4 - M_3M_5 - M_3M_7 - M_3M_9 \\
&+ M_4M_5 - M_4M_6 + M_4M_7 - M_4M_8 + M_4M_9 - M_4M_{10} \\
&+ M_5M_6 + M_5M_8 + M_5M_{10} \\
&- M_6M_7 - M_6M_9 + M_7M_8 + M_7M_{10} - M_8M_9 + M_9M_{10}.
\end{aligned} \tag{7.2}
$$

The 10×10 matrix associated to the form is a symmetry matrix and one can read the signature of the quadratic form from the signs of the eigenvalues, which are approximately:

$$
-4.57, -2.63, -1.54, -1.15, 0, .775, 1.09, 1.27, 3.10, 3.66, \tag{7.3}
$$

with the zero solution being an exact solution, thus the rank of the quadratic form is 9. Note that a small perturbation of the A_i's can only increase the rank of Q.

The linear forms $L = (L_1, L_2)$ are given by[7] $\sum M_iA_i$. Assume the form $\widetilde{L}_Z$ is M_1. Setting them equal to zero gives:

$$
\begin{aligned}
L_1 &: & M_1 + M_3 - M_4 + M_6 + M_8 + M_{10} &= 0 \\
L_2 &: & M_2 - M_4 + M_5 + M_7 + M_9 &= 0 \\
\widetilde{L}_Z &: & M_1 &= 0.
\end{aligned} \tag{7.4}
$$

We need to solve the system $Q = 0, L_1 = 0, L_2 = 0, \widetilde{L}_Z = 0$ under a small perturbation of the $\mathrm{Ab}(\mathcal{H}_1)$-part of the periodic data and of $\widetilde{L}_Z$, and obtain real positive solutions. We discuss first the unperturbed system and then show that the required solution exists for the perturbed system provided $\widetilde{L}_Z$ is perturbed appropriately (namely, to the form (7.11)).

Eliminating M_9 and M_{10} via (7.4) gives the reduced quadratic form in the remaining variables:

$$
\begin{aligned}
\bar{Q} = &-2M_2M_3 + 2M_2M_4 - 2M_2M_6 - 2M_2M_8 - M_4^2 \\
&+2M_4M_6 + M_4M_8 - 2M_5M_6 - 2M_5M_8 - 2M_7M_8.
\end{aligned} \tag{7.5}
$$

It happens that M_1 is eliminated as well.

We start solving the reduced quadratic form. We impose the additional conditions:

$$
\begin{aligned}
M_5 = M_7 &= B \\
M_2 = M_3 = M_6 = M_8 &= A.
\end{aligned} \tag{7.6}
$$

[7] L_1 is the coefficient of Y_1, L_2 of X_1.

The reduced form $\bar{Q}$ becomes:

$$M_4^2 - 6AM_4 + 6A^2 + 6AB = 0, \tag{7.7}$$

which gives:

$$M_4 = 3A + \sqrt{3A^2 - 6AB}.$$

For a perturbation small enough compared to the constants A, B below, the quadratic equation becomes

$$1(\varepsilon)M_4^2 - 6(\varepsilon)AM_4 + 6(\varepsilon)A^2 + 6(\varepsilon)AB + C_1(\varepsilon)M_1^2 + C_2(\varepsilon)M_1 + C_3(\varepsilon)M_4M_1 = 0, \tag{7.8}$$

where $C(\varepsilon)$ is a constant ε-close to C, $-.001 < C_1, C_2, C_3 < .001$, and which gives:

$$M_4 = 3(\varepsilon)A + \sqrt{3(\varepsilon)A^2 - [6(\varepsilon)A + C_3(\varepsilon)M_1]B - C_1(\varepsilon)M_1^2 - C_2(\varepsilon)M_1}.$$

Assuming A is much greater than B (say $A = 100, B = 1$) and using (7.6) and (7.4) we obtain:

$$\begin{aligned}
M_5 &= M_7 = 1 \\
M_2 &= M_3 = M_6 = M_8 = 100 \\
M_4 &= 300 + \sqrt{28200} \\
M_9 &= -M_2 + M_4 - M_5 - M_7 \\
&= 198 + \sqrt{28200} \\
M_{10} &= -M_1 - M_3 + M_4 - M_6 - M_8 \\
&= -M_1 + \sqrt{28200}
\end{aligned} \tag{7.9}$$

which in the perturbed case becomes:

$$\begin{aligned}
M_5 &= M_7 = 1 \\
M_2 &= M_3 = M_6 = M_8 = 100 \\
M_4 &= 300(\varepsilon) + \sqrt{28200(\varepsilon) - C_1(\varepsilon)M_1^2 - C_2'(\varepsilon)M_1} \\
M_9 &= -1(\varepsilon)M_2 + 1(\varepsilon)M_4 - 1(\varepsilon)M_5 - 1(\varepsilon)M_7 \\
&= 198(\varepsilon) + \sqrt{28200(\varepsilon) - C_1(\varepsilon)M_1^2 - C_2'(\varepsilon)M_1} \\
M_{10} &= -1(\varepsilon)M_1 - 1(\varepsilon)M_3 + 1(\varepsilon)M_4 - 1(\varepsilon)M_6 - 1(\varepsilon)M_8 \\
&= -1(\varepsilon)M_1 + \sqrt{28200(\varepsilon) - C_1(\varepsilon)M_1^2 - C_2'(\varepsilon)M_1},
\end{aligned} \tag{7.10}$$

where $C_2'(\varepsilon) = C_1(\varepsilon) + C_3(\varepsilon)$.

For the unperturbed case of (7.4), (7.9) gives a solution with $M_1 = 0$ and all other M_i's strictly positive. We will use Theorem 5.1 to produce orbits that have the abelian parts ε-close to those in (7.1) and arrange for the central parts to be such that, up to a large multiplicative factor, the coefficient of M_1 in $\widetilde{L}_Z$ is large and positive while the coefficients of the other M_i's are close to zero and negative, that is $\widetilde{L}_Z = 0$ becomes

$$M_1 - \sum_{i=2}^{10} \alpha_i M_i = 0, \tag{7.11}$$

where α_i are positive and as close to zero as we wish. Replacing the variables $M_2, \ldots, M_{10}$ with the values from (7.10) gives in the perturbed case a nonlinear equation of the type

$$(1+\alpha_{10})M_1 - \alpha_{10}\sqrt{28200(\varepsilon) - C_1(\varepsilon)M_1^2 - C_2(\varepsilon)M_1} = \mu_2, \tag{7.12}$$

where α_{10}, μ_2 are positive constants that can be chosen as small as we wish. The equation has a positive solution. This follows from the intermediate value theorem: for $M_1 = 0$ the left hand side of (7.12) is negative, and for $\varepsilon < .001$, $\alpha_{10} < .001$ and $M_1 > 100$, the right hand side of (7.12) is bigger then 50. By choosing μ_2 small enough, which can be done by choosing $\alpha_2, \alpha_3, \ldots, \alpha_9$ small, one obtains a solution $M_1 \in (0,1)$. Moreover, M_1 can be made as close to 0 as we wish. Therefore M_{10} given by (7.10) is also positive, due to the bounds obtained above for M_1.

Thus for the perturbed case, the values $M_i > 0, 1 \leq i \leq 10$, solve both the perturbed quadratic form and the three perturbed linear forms. We finally observe that the rank of Q is at least 9 and that the existence of the strictly positive solution $M_i, 1 \leq i \leq 10$, implies that the quadratic form Q restricted to the intersection of the kernels of the linear forms is nondegenerate.

To summarize, we have shown the following:

Lemma 7.1 *For $k = 9$, let Q and L_1, L_2 be the quadratic form, respectively the linear forms defined in* (3.2)*. Consider the following system in the variables $M_1, \ldots, M_{10}$:*

$$\begin{cases} Q = 0 \\ L_1 = 0 \\ L_2 = 0 \\ M_1 = \displaystyle\sum_{i=2}^{10} \alpha_i M_i \\ M_5 = M_7 = 1 \\ M_2 = M_3 = M_6 = M_8 = 100 \end{cases} \tag{7.13}$$

Provided the α_i's are positive and very close to zero, the system of equations (7.13) *has a unique solution, with all M_i's positive, whenever the* $\mathrm{Ab}(\mathcal{H}_1)$*-component of the periodic data is close to that given in* (7.1)*. The solutions are close to* (7.9) *evaluated at* $M_1 = 0$.

Remark 7.2 One could also prove this result by an argument showing that the corresponding map is a submersion. The same is true for the case of $\mathcal{H}_n$.

7.2 The case of $\mathcal{H}_n$.

Choose the periodic data in $\mathrm{Lie}(\mathcal{H}_n), n > 1$ [8] as follows: for $1 \le k \le n$ set

$$A_1^k = A_3^k = A_6^k = A_8^k = A_{10}^k = Y_k,\ A_2^k = A_5^k = A_7^k = A_9^k = X_k,\ A_4^k = -(X_k + Y_k). \quad (7.14)$$

That is, the collection $\{A_i^k \mid 1 \le i \le 10\}$ is exactly (7.1) in the directions X_k, Y_k (which generate a copy of $\mathcal{H}_1$).

Denote the corresponding multiplicities by $M_i^k, 1 \le i \le 10, 1 \le k \le n$. The A_ℓ^k-values are visited in the order $(A_1^1, \ldots, A_{10}^1, \ldots, A_1^n, \ldots, A_{10}^n)$, so should be labeled $(A_1, A_2, \ldots, A_{10n})$.

We will prove that there are positive solutions for the system of equations analogous to (7.13).

We compute the quadratic form Q. Note that $[A_i^p, A_j^q] = 0$ if $1 \le p, q \le n, p \ne q$, so Q is essentially a sum of copies of (7.2):

$$\begin{aligned} Q &= \sum_{k=1}^{n} \sum_{1 \le i < j \le k+1} M_i^k M_j^k [A_i^k, A_j^k] \\ &= \sum_{k=1}^{n} Q^{(1)}(M_1^k, M_2^k, \ldots, M_{10}^k) \end{aligned} \quad (7.15)$$

where $Q^{(1)}$ is the quadratic form (7.2) computed for $\mathcal{H}_1$ at the periodic data (7.1).

The $10n \times 10n$ matrix associated to the quadratic form is block diagonal, with 10×10 diagonal blocks given by the matrix obtained for the case $n = 1$. The eigenvalues of each diagonal block are given by (7.3), thus the rank of the quadratic form is $9n$. The rank of the quadratic form cannot decrease under perturbations.

The $2n$ linear forms $L = (L_1, L_2, \ldots, L_{2n})$ are given[9] by $\sum M_i^k A_i^k$. Setting them equal to zero gives, for the periodic data (7.14):

$$\begin{aligned} L_{2k-1}: &\quad M_1^k + M_3^k - M_4^k + M_6^k + M_8^k + M_{10}^k = 0, \\ L_{2k}: &\quad M_2^k - M_4^k + M_5^k + M_7^k + M_9^k = 0, \qquad 1 \le k \le n. \end{aligned} \quad (7.16)$$

[8] We could include $n = 1$ here as well, that being a special case of the construction.

[9] L_{2k-1} is the coefficient of Y_k, L_{2k} of X_k.

One can eliminate $M_9^k, M_{10}^k, 1 \le k \le n$,; this gives the reduced quadratic form:

$$\bar{Q} = \sum_{k=1}^{n} \bar{Q}^{(1)}(M_2^k, M_3^k, \ldots, M_8^k) \tag{7.17}$$

where $\bar{Q}^{(1)}$ is the form (7.5) from the $n = 1$ case; it happens that all $M_1^k, 1 \le k \le n$, are eliminated as well.

As in the $\mathcal{H}_1$-case, we impose the additional conditions:

$$\begin{aligned} &M_5^k = M_7^k = B = 1 \\ &M_2^k = M_3^k = M_6^k = M_8^k = A = 100, \qquad 1 \le k \le n. \end{aligned} \tag{7.18}$$

As we saw in section 7.1, the k-th term in the reduced form $\bar{Q}$ (7.17) of the unperturbed case is a quadratic equation in M_4^k only,

$$(M_4^k)^2 - 6AM_4^k + 6A^2 + 6AB.$$

We require that *each of these n expressions be zero*; this gives

$$M_4^k = 3A + \sqrt{3A^2 - 6AB},$$

and thus before perturbation, a solution is given by

$$\begin{aligned} M_5^k &= M_7^k = 1 \\ M_2^k &= M_3^k = M_6^k = M_8^k = M_{10}^k = 100 \\ M_4^k &= 300 + \sqrt{28200} \\ M_9^k &= -M_2^k + M_4^k - M_5^k - M_7^k \\ &= 198 + \sqrt{28200} \\ M_{10}^k &= -M_1^k - M_3^k + M_4^k - M_6^k - M_8^k \\ &= -M_1^k + \sqrt{28200}, \qquad\qquad 1 \le k \le n. \end{aligned} \tag{7.19}$$

The perturbed equations follow from Theorem 5.1: it produces orbits that have the abelian parts as close as desired (say, within ε) to those in (7.14) and the central part such that in $\widetilde{L}_Z$ the coefficients of $M_1^k, 1 \le k \le n$, are positive, those of the other variables are negative, and the former are much larger (with no bound) than the absolute value of the latter. Then $\widetilde{L}_Z = 0$ is a linear combination of the n equations

$$\widetilde{L}_Z^k : \qquad M_1^k = \sum_{i=2}^{10} \alpha_i^k M_i^k, \tag{7.20}$$

where α_i^k are as close to zero as we wish and positive.

In the perturbed case, we again eliminate the $2n$ variables M_9^k, M_{10}^k, $1 \leq k \leq n$, using the $2n$ linear equations $L_1 = 0, \ldots, L_{2n} = 0$ (that is, the perturbed version of (7.16)) and then impose (7.18). After all these, the form $\bar{Q}$ has extra terms of size $\varepsilon(\sum_{k=1}^n \sum_{i\in\{1,4\}} M_i^k)^2$. We would like to split the equation $\bar{Q} = 0$ as in the unperturbed case, and also split the perturbed equation $\widetilde{L}_Z = 0$. Note that at this point there are two equations left, $\bar{Q} = 0$ and $\widetilde{L}_Z = 0$, in the $2n$ variables M_1^k, M_4^k, $1 \leq k \leq n$. By dividing each of these two equations into n, we produce $2n$ equations for the $2n$ variables. We will split $\bar{Q}$ and $\widetilde{L}_Z = 0$ in a way that allows for solving successively for the $k = 1$, $k = 2$, $\ldots$, $k = n$ variables, following the discussion for $\mathcal{H}_1$ in section 7.1. The difficulty is caused by the fact that the extra terms that appear in the perturbed quadratic form do not separate the variables $M_1^k, 1 \leq k \leq n$.

Namely, we proceed as follows. After eliminating M_9^k, M_{10}^k, $1 \leq k \leq n$, and making the substitutions (7.18), we pick from the perturbed quadratic form $\bar{Q}$ the terms containing only the variables M_4^1 and M_1^1 (so, e.g., a term $M_1^1 M_4^2$ is not selected), and $1/n$ of the constant term. Call this $\bar{Q}^1$; this is a perturbation of (7.7), so of the form (7.8), that is

$$\bar{Q}^1: \quad 1(\varepsilon)(M_4^1)^2 - 6(\varepsilon)AM_4^1 + 6(\varepsilon)A^2 + 6(\varepsilon)AB + C_1^1(\varepsilon)(M_1^1)^2 + C_2^1(\varepsilon)M_1^1 + C_3^1(\varepsilon)M_4^1 M_1^1 = 0.$$

where $C(\varepsilon)$ is a constant ε-close to C, $-.001 < C_1^1, C_2^1, C_3^1 < .001$. From the discussion for $\mathcal{H}_1$ in section 7.1 we know that the (reduced) system $\bar{Q}^1 = 0, \widetilde{L}_Z^1 = 0$ in the variables M_1^1 and M_4^1 admits a unique solution, which translates into M_i^1, $1 \leq i \leq 10$, being all positive, and as close as desired to (7.19) for $k = 1$ evaluated at $M_1^1 = 0$.

Next, we substitute the values obtained for M_1^1 and M_4^1 into $\bar{Q}$, and pick the terms containing only the variables M_4^2 and M_1^2, and $1/n$ of the constant term; denote this by $\bar{Q}^2$. Consider the system $\bar{Q}^2 = 0$, $\widetilde{L}_Z^2 = 0$. As for $k = 1$, this fits in the case discussed in section 7.1 so we obtain positive values for M_i^2, $1 \leq i \leq 10$, again close to (7.19) for $k = 2$ evaluated at $M_1^2 = 0$. We substitute these values in $\bar{Q}$ and continue with the variables having index $k = 3$, and so on, up to $k = n$.

In the end, we obtain a positive solution $M_i^k > 0, 1 \leq i \leq 10, 1 \leq k \leq n$, to the perturbed system.

Finally, considering the hypotheses of Theorem 6.2, we note that the rank of Q is $9n$, bigger than $2(2n + 1) + 3$, and that the existence of the positive solution $M_i^k, 1 \leq i \leq 10, 1 \leq k \leq n$, implies that the quadratic form Q restricted to the intersection of kernels of the linear forms is indefinite.

8 Semigroup problem for the Heisenberg group

By Theorem 16.2.5 in Kargapolov & Merzljakov [4]:

Theorem 8.1 ([4], Theorem 16.2.5) *If G is a nilpotent group and $A \subset G$ a subgroup such that $A[G, G] = G$ then $A = G$.*

Proof We show a complete proof, for the convenience of the reader.

Denote by D_k the *lower central series* given by the sequence of subgroups $D_1 = [G, G], \ldots, D_{k+1} = [G, D_k]$, which are all normal in G. Consider $A_k := AD_k$.

We proceed by induction. Assume that $AD_k = G$. Then $AD_{k+1} = G$. Indeed, since $AD_{k+1} \lhd AD_k$ (for this we need that $[A, D_k] \subset D_{k+1}$) and their quotient is abelian ($[A, A] \subset A$, $[A, D_k] \subset D_{k+1}$, $[D_k, D_k] \subset D_{k+1}$, so all commutators from A_k are in the smaller subgroup A_{k+1}). Since $AD_k = G$, this implies that $D_k \subset [G, G] = [A_k, A_k] \subset AD_{k+1}$. But then $AD_{k+1} \supset AD_k = G$. ∎

Based on this, we have the same for nilpotent Lie algebras:

Theorem 8.2 *If $\mathfrak{g}$ is a nilpotent Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra such that $\mathfrak{h} + [\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ then $\mathfrak{h} = \mathfrak{g}$.*

Proof Denote by d_k the *lower central series*, $d_1 = [\mathfrak{g}, \mathfrak{g}]$, $d_{k+1} = [\mathfrak{g}, d_k]$, which are all ideals in $\mathfrak{g}$. Consider $\mathfrak{h}_k := \mathfrak{h} + d_k$ (which is a Lie subalgebra since d_k is an ideal).

We proceed by induction: if $\mathfrak{h} + d_k = \mathfrak{g}$ then $\mathfrak{h} + d_{k+1} = \mathfrak{g}$. Indeed, since $\mathfrak{h} + d_{k+1} \lhd \mathfrak{h} + d_k$ (for this we need that $[\mathfrak{h}, d_k] \subset d_{k+1}$) and their quotient is abelian ($[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h} \subset \mathfrak{h}_{k+1}$, $[\mathfrak{h}, d_k] \subset d_{k+1}$, $[d_k, d_k] \subset d_{k+1}$, so all commutators of $\mathfrak{h} + d_k$ elements are in the smaller Lie subalgebra $\mathfrak{h} + d_{k+1}$). Since $\mathfrak{h} + d_k = \mathfrak{g}$, this implies that $d_k \subset [\mathfrak{g}, \mathfrak{g}] = [\mathfrak{h}_k, \mathfrak{h}_k] \subset \mathfrak{h} + d_{k+1}$. But then $\mathfrak{h} + d_{k+1} \supset \mathfrak{h} + d_k = \mathfrak{g}$. ∎

The following result of Lawson can be applied to $\mathcal{H}_n$.

Theorem 8.3 ([5], Theorem 12.5) *The maximal subsemigroups M with non-empty interior of a simply connected Lie group G with $G/\mathrm{Rad}G$ compact are in one-to-one correspondence with their tangent objects*

$$L(M) := \{X \in \mathrm{Lie}(G) : \exp(tX) \in M \text{ for } t \geq 0\}$$

and the latter are precisely the closed half-spaces with boundary given by a subalgebra. Furthermore, M is the semigroup generated by $\exp(L(M))$.

Proposition 8.4 *In a nilpotent Lie algebra $\mathfrak{g}$, the codimension-one subalgebras contain the commutator $[\mathfrak{g}, \mathfrak{g}]$ and are preimages of a codimension-one subspace of the abelianization $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$.*

Proof Let $\mathfrak{h}$ be a codimension-one Lie subalgebra. Then, by the above Theorems, $\mathfrak{h} + [\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$ (otherwise $\mathfrak{h} = \mathfrak{g}$), hence $[\mathfrak{g}, \mathfrak{g}]$ must be a subspace of $\mathfrak{h}$ (otherwise the sum is strictly larger then $\mathfrak{h}$). The rest follows by looking at the abelianization. ∎

The results above suggest the following:

Conjecture 8.5 (Semigroup conjecture) Assume that S is a semigroup in the simply-connected nilpotent group G. Then $\overline{S/[G,G]} = G/[G,G]$ implies that $\overline{S} = G$. If, in addition, $S/[G,G]$ is not on one side of a hyperplane, then $\overline{S}$ is a group.

We prove next the semigroup conjecture for the Heisenberg groups, but see Remark 8.7.

Theorem 8.6 *Let $S \subseteq \mathcal{H}_d$ be a semigroup.*

(a) If the projection of S onto $\mathcal{H}_d/[\mathcal{H}_d, \mathcal{H}_d] \cong \mathbb{R}^{2d}$ is not on one side of a hyperplane, then the closure of S is a group.

(b) In particular, if the projection of S is dense in $\mathcal{H}_d/[\mathcal{H}_d, \mathcal{H}_d]$, then the closure of S is $\mathcal{H}_d$.

Remark 8.7 The proof of Theorem 8.6 actually proves the semigroup conjecture for any step-2 nilpotent group whose commutator has dimension one.

We identify $\mathcal{H}_d$ with its Lie algebra $\mathrm{Lie}(\mathcal{H}_d) = \mathbb{R}^{2d+1}$, so the group operation is given by (1.2). For $A_i \in \mathcal{H}_d$

$$A_1 * A_2 * A_3 * \cdots * A_m = \sum_{i=1}^{m} A_i + \frac{1}{2} \sum_{1 \le i < j \le m} [A_i, A_j].$$

In particular, the reversed product has the same "linear" part but opposite "bracket" part:

$$A_m * A_{m-1} * \cdots * A_2 * A_1 = \sum_{i=1}^{m} A_i - \frac{1}{2} \sum_{1 \le i < j \le m} [A_i, A_j].$$

The proof of the following lemma can be found in [8, Lemma 2.12]:

Lemma 8.8 ([8], Lemma 2.12) *Assume that the family $\mathcal{L} \subset \mathbb{R}^n$ does not lie in a half-space. Then the closed semigroup generated by $\mathcal{L}$ is a group.*

Proof of Theorem 8.6 Denote the commutator subgroup $[\mathcal{H}_d, \mathcal{H}_d]$ by $C_{\mathcal{H}_d}$ [10]; by choosing a nonzero element in $C_{\mathcal{H}_d}$, identify it with $\mathbb{R}$. Recall that $\mathcal{H}_d/[\mathcal{H}_d, \mathcal{H}_d]$ is denoted by $\mathrm{Ab}(\mathcal{H}_d)$.

We may assume that S is closed. We follow these steps:

(A) $S \cap C_{\mathcal{H}_d} \neq \emptyset$ and is not on one side of 0 in $\mathbb{R}$.

[10] For $\mathcal{H}_n$ the commutator subgroup coincides with the center $Z_{\mathcal{H}_d}$ whereas for a step-2 nilpotent group $\mathcal{N}$, $C_{\mathcal{N}} \subset Z_{\mathcal{N}}$. We distinguish in this argument the center from the commutator subgroup for this more general case. The induction in Step (C) below could be relevant for proving the Conjecture 8.5.

(B) Therefore, $\pi(S)$ is closed in $\mathrm{Ab}(\mathcal{H}_d)$, where π denotes the projection $\mathcal{H}_d \to \mathrm{Ab}(\mathcal{H}_d)$.

(C) By induction, $S \cap C_{\mathcal{H}_d}$ is a group.

(D) Then S is a group.

These prove (a). To prove (b) – again assuming that S is closed – note that from (A) it follows that $\pi(S) = \mathrm{Ab}(\mathcal{H}_d)$, that is $S[\mathcal{H}_d, \mathcal{H}_d] = \mathcal{H}_d$. Now use Theorem 8.1.

If S consists of commuting elements then (a) follows from [8, Lemma 2.12] (see Lemma 8.8). So we will assume that there are elements in S whose bracket is non-zero.

Proof of (A). Since $\pi(S)$ is not on one side of a hyperplane, there are elements $U_1, U_2, \dots, U_r \in S$ such that the open convex hull of $u_i := \pi(U_i)$ contains $0 \in \mathrm{Ab}(\mathcal{H}_d)$; thus there are $\alpha_i > 0$ such that $\sum_{i=1}^r \alpha_i u_i = 0$. We can also assume that $[U_1, U_2] \neq 0$.

As in the proof of [8, Lemma 2.12], there is a sequence of integers $p_i^{(n)} \to \infty$, such that

$$\|(p_1^{(n)}, \dots, p_r^{(n)}) - t^{(n)}(\alpha_1, \dots, \alpha_r)\| \to 0, \tag{8.1}$$

so $e^{(n)} := \sum_{i=1}^r p_i^{(n)} u_i \to 0$.

We will show that some rearrangement of $W^{(n)} := U_1^{p_1^{(n)}} * U_2^{p_2^{(n)}} * \dots U_r^{p_r^{(n)}}$ converges to an element in $C_{\mathcal{H}_d}$; the limit can be chosen to be either positive of negative.

When possible, we drop for simplicity the superscript (n). Compute, using (8.1):

$$W = (p_1 U_1) * (p_2 U_2) * \dots * (p_r U_r) = \sum_{i=1}^r p_i U_i + \frac{1}{2} \sum_{1 \le i < j \le r} p_i p_j [U_i, U_j]$$

where

$$\sum_{i=1}^r p_i U_i = t \sum_{i=1}^r \alpha_i U_i + o(1) \quad \text{and}$$
$$\sum_{1 \le i < j \le r} p_i p_j [U_i, U_j] = t^2 \sum_{1 \le i < j \le r} \alpha_i \alpha_j [U_i, U_j] + o(t^2) \text{ provided } \sum_{1 \le i < j \le r} \alpha_i \alpha_j [U_i, U_j] \neq 0. \tag{8.2}$$

Then $\pi(W^{(n)}) = e^{(n)} \to 0$ as $n \to \infty$. We will find a rearrangement that converges in the $[\mathcal{H}_d, \mathcal{H}_d]$-component too.

Consider the rearrangements of the factors in the expression W. All these rearrangements have the same π-projection. A computation shows that

$$A * X * Y * B = A * Y * X * B + [X, Y]. \tag{8.3}$$

We can assume without loss of generality that $\sum_{1 \le i < j \le r} \alpha_i \alpha_j [U_i, U_j] \neq 0$; otherwise exchange U_1 and U_2.

Then the $[\mathcal{H}_d, \mathcal{H}_d]$-component of W has an (approximately) linear contribution from the first part of (8.2) and a non-zero (approximately) quadratic contribution from the second part. In particular, by using either W or the expression with factors in reverse order, we can get the $[\mathcal{H}_d, \mathcal{H}_d]$-component to converge, as $n \to \infty$, either to $+\infty$ or to $-\infty$.

By (8.3), changing two consecutive factors modifies the $[\mathcal{H}_d, \mathcal{H}_d]$-component by zero or $[U_i, U_j]$, so "walking" from one order to the reverse order gives elements whose $[\mathcal{H}_d, \mathcal{H}_d]$-component varies by bounded amounts.

Proof of (B). Assume that $X_k \in S$ are such that $\pi(X_k)$ converges. Since in $S \cap C_{\mathcal{H}_d} \subset Z_{\mathcal{H}_d}$ there are both positive and negative elements, we can adjust the $C_{\mathcal{H}_d}$-component of the X_k's to remain bounded, therefore obtaining a convergent subsequence in S.

Proof of (D). Since $\pi(S)$ is closed, we know that it is a group (from the abelian case, [8, Lemma 2.12]). Therefore, for $X \in S$, there is an element $Y \in S$ such that $\pi(X) + \pi(Y) = 0$. Then $X * Y \in C_{\mathcal{H}_d} \cap S$, which is a group by (C), so $X^{-1} = Y * (X * Y)^{-1} \in S$. ∎

9 Dense semigroups in $\mathbb{R}^n$

In this section we show some preparatory results about ε-dense semigroups in $\mathbb{R}^n$.

The following result is standard.

Theorem 9.1 (Carathéodory) *The convex hull of a subset $S \subseteq \mathbb{R}^m$ consists of all convex combinations of $m+1$ points from S.*

Theorem 9.2 *Let $K \geq n+1$ integer. Denote by $\mathcal{F}_K$ the family of K-tuples in $\mathbb{R}^n$ for which the origin is in the interior of their convex hull. Then, for any $\varepsilon > 0$, there exists a dense and open set in $\mathcal{F}_K$ of K-tuples that generate an ε-dense semigroup in $\mathbb{R}^n$.*

Proof As the origin belongs to the convex hull of F, it follows from Carathéodory theorem that there exists a subset of F, say $F' = \{x_1, x_2, \ldots, x_{n+1}\}$, such that the origin is in the convex hull of F'. If, in addition, the origin belongs to the interior of the convex hull of F', we can apply Proposition 9.3 to finish the proof. Otherwise, the origin belongs to a lower dimensional face A of a simplex Δ generated by $n+1$ points in F. As the convex hull of F has non-empty interior, we can assume that Δ has non-empty interior. The extreme points of A belong to F. Moving slightly each extreme point of A in a direction opposite to one that points strictly inside Δ will move the face A away from the origin. So the origin belongs now to the interior of a simplex with vertices belonging to a perturbation of F. Thus we can apply Proposition 9.3 to finish the proof. ∎

Proposition 9.3 *Let $\mathcal{F}$ be the collection of all families $F = \{v_1, \ldots, v_{n+1}\} \subset \mathbb{R}^n$ such that $0 \in \operatorname{int}\ \operatorname{co}\{v_1, \ldots, v_{n+1}\}$. Then, for any $\varepsilon > 0$, there exists an open dense set of families in $\mathcal{F}$ that generate an ε-dense semigroup in $\mathbb{R}^n$.*

The proposition is a consequence of the following result:

Lemma 9.4 *Let $F_0 = \{v_1, \ldots, v_{n+1}\} \subset \mathbb{R}^n$ be such that $0 \in \text{int co}\{v_1, \ldots, v_{n+1}\}$. For each k, pick a small neighborhood U_k of v_k such that for any $w_k \in U_k$, the origin is still in the interior of the convex hull of $\{w_k \mid 1 \leq k \leq n+1\}$. Denote $\mathcal{U} := U_1 \times U_2 \times \cdots \times U_{n+1}$.*

Then, for any $\varepsilon > 0$, there exists an open dense set $\mathcal{W}_\varepsilon$ in $\mathcal{U}$ such that for any $F \in \mathcal{W}_\varepsilon$, the semigroup generated by F is ε-dense in $\mathbb{R}^n$.

We first show that

Lemma 9.5 *For any $\varepsilon > 0$, there is an open and dense set $\mathcal{G}_\varepsilon \subset \mathbb{T}^n$ such that for $\mathbf{v} \in \mathcal{G}_\varepsilon$, the orbit of the origin under the translation by $\mathbf{v}$ is ε-dense in $\mathbb{T}^n$.*

Proof Identify $\mathbb{T}^n$ with $\mathbb{R}^n/\mathbb{Z}^n$, pick $0 < \eta < \varepsilon$. Choose $p_1, p_2, \ldots, p_n$ distinct prime numbers such that the subgroup G_0 generated by $(1/p_1, 0 \ldots, 0)$, $(0, 1/p_2, 0, \ldots, 0)$, $\ldots$, $(0, \ldots, 0, 1/p_n) \in \mathbb{T}^n$ is η-dense in $\mathbb{T}^n$. Denote $m := p_1 p_2 \ldots p_n$, the order of G_0.

For integers $0 < \ell_k < p_k$ set $\mathbf{v} := (\ell_1/p_1, \ldots, \ell_n/p_n)$. Then $\{k\mathbf{v} \mid 0 < k \leq m\}$ coincides with G_0 and thus is η-dense in $\mathbb{T}^n$. Moreover, for $\mathbf{w}$ close to $\mathbf{v}$, $\{k\mathbf{w} \mid 0 < k \leq m\}$ is ε-dense in the torus. ∎

Proof of Lemma 9.4 Since $F_0 = \{v_1, \ldots, v_{n+1}\} \subset \mathbb{R}^n$ is such that $0 \in \text{int co}\{v_1, \ldots, v_{n+1}\}$, one has $\sum_{i=1}^{n+1} \gamma_i v_i = 0$ with $\gamma_i > 0$ (determined uniquely up to a multiplicative constant). For simplicity, assume $\gamma_{n+1} \geq \gamma_i$, $1 \leq i \leq n$. By applying a linear transformation one can assume that $\{v_1, \ldots, v_n\}$ is the canonical basis of $\mathbb{R}^n$; therefore v_{n+1} has all coordinates in $[-1, 0)$. Without loss of generality, it suffices to prove the result for this F_0.

Given $\eta > 0$, define $\mathcal{V}_\eta$ as the $(n+1)$-tuples $w_k \in U_k$ for which $w_{n+1} = -\sum_{k=1}^n (1-\alpha_k) w_k$ with $(\alpha_1, \ldots, \alpha_n) \in \mathcal{G}_\eta$, where $\mathcal{G}_\eta$ is described in Lemma 9.5 (for that, see $\mathcal{G}_\eta \subset \mathbb{T}^n \cong [0,1)^n$; the cases when v_{n+1} has coordinates equal to -1 can be treated similarly).

We claim that $\mathcal{W}_\varepsilon := \mathcal{V}_\eta$ has the desired property if η is small enough compared to ε.

That $\mathcal{V}_\eta$ is dense and open in $\mathcal{U}$ follows from the fact that $\mathcal{G}_\eta$ is open and dense in the torus. To check the ε-density property, notice first that by Lemma 8.8, the closures of the semigroup and group generated by a family $F \in \mathcal{U}$ are the same. Therefore it suffices to prove the ε-density for the generated closed group, so we can assume that $-w_k$, $1 \leq k \leq n$, are also among the generators. Then it suffices to prove the ε-density up to a translation from $\text{span}_{\mathbb{Z}}\{w_k \mid 1 \leq k \leq n\} \cong \mathbb{Z}^n$. Take a parallelepiped at the origin that is a fundamental domain for this quotient; there is a linear change of coordinates in $\mathbb{R}^n$ that maps this to the unit cube $[0,1)^n$. Because the vertices are close to those of the cube, the distances are distorted by a bounded amount, uniformly over $\mathcal{U}$. In these transformed coordinates, Lemma 9.5 gives η-density in the unit cube. Thus, for $\eta > 0$ small enough (uniformly on $\mathcal{U}$), we obtain ε-density in the original fundamental domain. ∎

10 Transitivity for $\Gamma = \mathcal{H}_n$

In this section we prove Theorem 1.5, and consequently the main result Theorem 1.4, for cocycles with values in the Heisenberg group $\Gamma = \mathcal{H}_n$.

Proof of Theorem 1.5 Consider the values (7.14) for A^k_ℓ and the quadratic form Q and the linear forms $L = (L_1, L_2, \ldots, L_{2n})$ of Section 7.2. For this data we showed that:

- Q has rank $9n$ (with $4n$ positive and $5n$ negative squares) in $10n$ variables;
- there is a solution $\mathbf{M}_* = (M^1_1, \ldots, M^1_{10}, \ldots, M^n_1, \ldots, M^n_{10})$ with $M^k_1 = 0, 1 \leq k \leq n$, and all the other M^k_ℓ's strictly positive to

$$Q = 0, L_1 = 0, L_2 = 0, \ldots, L_{2n} = 0.$$

Assume that $\beta : \mathcal{X} \to \mathcal{H}_n$ is a Hölder cocycle such that the quotient cocycle $\widehat{\beta} : \mathcal{X} \to Ab(\mathcal{H}_n)$ is transitive. To prove Theorem 1.5 we need to show that β is transitive as well.

We construct now a quadratic form close to the above using the periodic data of our cocycle, and arrange simultaneously that L_Z is proportional to $\sum_{k=1}^n \left(\alpha^k_1 M^k_1 - \sum_{\ell=2}^{10} \alpha^k_\ell M^k_\ell\right)$ with α^k_1 large and positive and $\alpha^k_\ell, 2 \leq \ell \leq 10$, positive and arbitrarily small. This implies as in Sections 7.1 and 7.2 that there exists a positive solution to the system of equations $Q = 0, L_1 = 0, L_2 = 0, \ldots, L_{2n} = 0, L_Z = 0$. Therefore we can apply the Diophantine approximation result, Theorem 6.2, to obtain an element in $\mathcal{L}_\beta$ with the $Ab(\mathcal{H}_n)$-part as close as we wish to a preset value. Here are the details.

Let $a \in Ab(\mathcal{H}_n)$ be given. Fix $\varepsilon > 0$. We observe that the values (7.14) for A^k_ℓ contain the origins of $Ab(\mathcal{H}_n)$ in the interior of their convex hull. By Theorem 9.2, arbitrarily close to the A^k_ℓ's there are open sets of values that generate an ε-dense semigroup in $Ab(\mathcal{H}_n)$.

Pick a heteroclinic cycle for f starting at the periodic point $p_0 \in \mathcal{X}$ connecting $k = 10n - 1$ periodic points. By Theorem 5.1(a), one can pick close to these points some other periodic points of f with periodic data of $\widehat{\beta}$ in the above open sets. We use now the notation introduced in (3.1) and (3.2). Here $k = 10n-1$ and we relabel $(A^1_1, \ldots, A^1_{10}, \ldots, A^n_1, \ldots, A^n_{10})$ as $(A_1, A_2, \ldots, A_{10n})$ and $(M^1_1, \ldots, M^1_{10}, \ldots, M^n_1, \ldots, M^n_{10})$ as $(M_1, M_2, \ldots, M_{10n})$:

$$\begin{aligned}
A(N) &:= \beta(P_1)^{M_1} H_1\, \beta(P_2)^{2M_2} H_2 \cdots \beta(P_k)^{2M_k} H_k\, \beta(P_1)^{M_{k+1}} \\
&= A_1^{M_1} H_1\, A_2^{M_2} H_2 \cdots A_k^{M_k} H_k\, A_{k+1}^{M_{k+1}} \\
&= \sum_{i=1}^{k+1} M_i A_i + \tfrac{1}{2} \sum_{j=1}^{k} \left(\sum_{1 \leq i \leq j} M_i [A_i, H_j] - \sum_{j < i \leq k+1} M_i [A_i, H_j] \right) + \sum_{i=1}^{k} H_i \\
&+ \tfrac{1}{2} \sum_{1 \leq i < j \leq k+1} M_i M_j [A_i, A_j] + \tfrac{1}{2} \sum_{1 \leq i < j \leq k} [H_i, H_j] \\
&= (L(N) + E, Q(N) + L_Z(N) + e) \in \mathbb{R}^{2n} \oplus \mathbb{R}
\end{aligned}$$

By the ε-density, there are non-negative integers $M_1^{(0)}, \dots, M_{10n}^{(0)}$ such that

$$\| \sum_{k=1}^{10n} M_k^{(0)} A_k - (a - E) \| < \varepsilon.$$

Let $N^{(0)} := (M_1^{(0)}, \dots, M_{10n}^{(0)})$. Then

$$A(N + N^{(0)}) = (\widetilde{a}, 0) + (L(N), Q(N) + 2B(N^{(0)}, N) + L_Z(N) + c_Z) \in \mathbb{R}^{2n} \oplus \mathbb{R},$$

with $\|\widetilde{a} - a\| = O(\varepsilon)$ where B is the symmetric bilinear form corresponding to Q, and in c_Z we incorporated all the terms that do not involve N. Note that the central component of the periodic data A_ℓ, $1 \le \ell \le 10n - 1$, contribute to the coefficients of L_Z.

Denote by $\widetilde{L}_Z$ the linear form that appears above in the central component,

$$\widetilde{L}_Z(N) = 2B(N, N^{(0)}) + L_Z(N)$$

By Theorem 5.1(b), we can now pick new periodic points for f such that:

- they are located in $\mathcal{X}$ as close as desired to the previous periodic points;
- the $Ab(\mathcal{H}_n)$-component of their periodic data is as close as desired to that of the previous ones;
- their central component is arbitrarily large, either positive or negative.

In the above expression for $A(N + N^{(0)})$ this means that $\widetilde{a}$, L and Q change as little as desired, but we can arrange, after going back to the notation $(M_1^1, \dots, M_{10}^1, \dots, M_1^n, \dots, M_{10}^n)$, that

$$\widetilde{L}_Z(N) = \sum_{k=1}^{n} \left(\alpha_1^k M_1^k - \sum_{\ell=2}^{10} \alpha_\ell^k M_\ell^k \right), \tag{10.1}$$

where α_ℓ^k are positive for all $1 \le \ell \le 10, 1 \le k \le n$, and $\alpha_1^k, 1 \le k \le n$, is as large as desired. Therefore, each term in the outer sum in (10.1) can be made as close as desired (after dividing by α_1^k) to $M_1^k = 0$. We label the new coefficients also by α_ℓ^k.

Solve now the system of equations

$$Q = 0, L_1 = 0, L_2 = 0, \dots, L_{2n} = 0,$$
$$M_1^1 - \sum_{\ell=2}^{10} \alpha_\ell^1 M_\ell^1 = 0, \dots, M_1^n - \sum_{\ell=2}^{10} \alpha_\ell^n M_\ell^n = 0,$$

using the method presented in Section 7.2. It follows that we can find a solution with all M_ℓ^k strictly positive.

Now use Theorem 6.2 to keep $L, \widetilde{L}_Z$ at most ε and Q bounded while N goes to ∞ along a "good" vector v. This implies $a(\varepsilon) \in \pi(\mathcal{L}_\beta(p_0))$. As this can be done for any $a \in Ab(\mathcal{H}_n)$, we have that $\pi(\mathcal{L}_\beta(p_0))$ contains an ε dense set in $Ab(\mathcal{H}_n)$. Repeating the procedure above for a sequence $\varepsilon_n > 0$ with $\varepsilon_n \to \infty$, we obtain that $\pi(\mathcal{L}_\beta(p_0))$ contains a dense set in $Ab(\mathcal{H}_n)$. Using now Theorem 8.6 we conclude that $\mathcal{L}_\beta(p_0) = \mathcal{H}_n$. ∎

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