A–1 Note that if \( r(x) \) and \( s(x) \) are any two functions, then
\[
\max(r, s) = (r + s + |r - s|)/2.
\]
Therefore, if \( F(x) \) is the given function, we have
\[
F(x) = \max\{ -3x - 3, 0 \} - \max\{ 5x, 0 \} + 3x + 2
= (3x - 3 + |3x - 3|)/2
= (5x + |5x|)/2 + 3x + 2
= |(3x - 3)/2 - |5x/2 - x + 1/2,
\]
so we may set \( f(x) = (3x - 3)/2, g(x) = 5x/2, \) and \( h(x) = -x + 1/2. \)

A–2 First solution: First factor \( p(x) = q(x)r(x) \), where \( q \) has all real roots and \( r \) has all complex roots. Notice that each root of \( q \) has even multiplicity, otherwise \( p \) would have a sign change at that root. Thus \( q(x) \) has a square root \( s(x) \).

Now write \( r(x) = \prod_{j=1}^k (x - a_j)(x - \overline{a_j}) \) (possible because \( r \) has roots in complex conjugate pairs). Write \( \prod_{j=1}^k (x - a_j) = t(x) + iu(x) \) with \( t, x \) having real coefficients. Then for \( x \) real,
\[
p(x) = q(x)r(x)
= s(x)^2(t(x) + iu(x))(t(x) + iu(x))
= (s(x)t(x))^2 + (s(x)u(x))^2.
\]
(Alternatively, one can factor \( r(x) \) as a product of quadratic polynomials with real coefficients, write each as a sum of squares, then multiply together to get a sum of many squares.)

Second solution: We proceed by induction on the degree of \( p \), with base case where \( p \) has degree 0. As in the first solution, we may reduce to a smaller degree in case \( p \) has any real roots, so assume it has none. Then \( p(x) > 0 \) for all real \( x \), and since \( p(x) \to \infty \) for \( x \to \pm \infty, p \) has a minimum value \( c \). Now \( p(x) - c \) has real roots, so as above, we deduce that \( p(x) - c \) is a sum of squares. Now add one more square, namely \( (\sqrt{c})^2 \), to get \( p(x) \) as a sum of squares.

A–3 First solution: Computing the coefficient of \( x^{n+1} \) in the identity \( (1 - 2x - x^2) \sum_{m=0}^\infty a_n x^m = 1 \) yields the recurrence \( a_{n+1} = 2a_n + a_{n-1} \); the sequence \( \{a_n\} \) is then characterized by this recurrence and the initial conditions \( a_0 = 1, a_1 = 2. \)

Define the sequence \( \{b_n\} \) by \( b_{2n} = a_{n}^2 + a_{n+1}^2, b_{2n+1} = a_{n}(a_{n+1} + a_{n+2}). \)
Then
\[
2b_{2n+1} + b_{2n} = 2a_{n}a_{n+1} + 2a_{n-1}a_{n} + a_{n-1}^2 + a_{n}^2
= 2a_{n}a_{n+1} + a_{n-1}a_{n+1} + a_{n}^2
= a_{n+1}^2 + a_{n}^2 = b_{2n+2},
\]
and similarly \( 2b_{2n} + b_{2n-1} = b_{2n+1} \), so that \( \{b_n\} \) satisfies the same recurrence as \( \{a_n\}. \) Since further \( b_0 = 1, b_1 = 2 \) (where we use the recurrence for \( \{a_n\} \) to calculate \( a_{-1} = 0 \), we deduce that \( b_n = a_n \) for all \( n \). In particular, \( a_n + a_{n+1} = b_{2n+2} = a_{2n+2}. \)

Second solution: Note that
\[
\frac{1}{1 - 2x - x^2}
= \frac{1}{2\sqrt{2}} \left( \sqrt{\frac{\sqrt{2} + 1}{2}} \frac{\sqrt{\frac{\sqrt{2} + 1}{2}} + \sqrt{\frac{\sqrt{2} - 1}{2}}}{1 - (1 + \sqrt{2})x + 1 - (1 - \sqrt{2})x} \right)
\]
and that
\[
\frac{1}{1 + (1 \pm \sqrt{2})x}
= \sum_{n=0}^\infty (1 \pm \sqrt{2})^n x^n,
\]
so that
\[
a_n = \frac{1}{2\sqrt{2}} ((\sqrt{2} + 1)^{n+1} - (1 - \sqrt{2})^{n+1}).
\]

A simple computation (omitted here) now shows that \( a_n^2 + a_{n+1}^2 = a_{2n+2}. \)

Third solution (by Richard Stanley): Let \( A \) be the matrix \( \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix} \). A simple induction argument shows that
\[
A^{n+2} = \begin{pmatrix} a_n & a_{n+1} \\ a_{n+1} & a_{n+2} \end{pmatrix}.
\]
The desired result now follows from comparing the top left corner entries of the equality \( A^{n+2}A^{n+2} = A^{2n+4} \).

A–4 Denote the series by \( S \), and let \( a_n = 3^n/n. \) Note that
\[
S = \sum_{m=1}^\infty \sum_{n=1}^m \frac{1}{a_m(a_m + a_n)}
= \sum_{m=1}^\infty \sum_{n=1}^\infty \frac{1}{a_m(a_m + a_n)}.
\]
where the second equality follows by interchanging $m$ and $n$. Thus

$$2S = \sum_{m} \sum_{n} \left( \frac{1}{a_{m}a_{m} + a_{n}} + \frac{1}{a_{n}a_{m} + a_{n}} \right)$$

$$= \sum_{m} \sum_{n} \frac{1}{a_{m}a_{n}}$$

$$= \left( \sum_{n=1}^{\infty} \frac{n}{3^{n}} \right)^{2}.$$  

But

$$\sum_{n=1}^{\infty} \frac{n}{3^{n}} = \frac{3}{4},$$

since, e.g., it’s $f'(1)$, where

$$f(x) = \sum_{n=0}^{\infty} \frac{x^{n}}{3^{n}} = \frac{3}{3 - x},$$

and we conclude that $S = 9/32$.

A-5 First solution: (by Reid Barton) Let $r_{1}, \ldots, r_{1999}$ be the roots of $P$. Draw a disc of radius $\epsilon$ around each $r_{i}$, where $\epsilon < 1/3998$; this disc covers a subinterval of $[-1/2, 1/2]$ of length at most $2\epsilon$, and so of the 2000 (or fewer) uncovered intervals in $[-1/2, 1/2]$, one, which we call $I$, has length at least $\delta = (1 - 3998 \epsilon)/2000 > 0$. We will exhibit an explicit lower bound for the integral of $|P(x)|/P(0)$ over this interval, which will yield such a bound for the entire integral.

Note that

$$\frac{|P(x)|}{|P(0)|} = \prod_{i=1}^{1999} \frac{|x - r_{i}|}{|r_{i}|}.$$  

Also note that by construction, $|x - r_{i}| \geq \epsilon$ for each $x \in I$. If $|r_{i}| \leq 1$, then we have $\frac{|x - r_{i}|}{|r_{i}|} \geq \epsilon$. If $|r_{i}| > 1$, then

$$\frac{|x - r_{i}|}{|r_{i}|} = |1 - x/r_{i}| \geq 1 - |x/r_{i}| \geq 1/2 > \epsilon.$$  

We conclude that $\int_{I} |P(x)/P(0)| \, dx \geq \delta \epsilon$, independent of $P$.

Second solution: It will be a bit more convenient to assume $P(0) = 1$ (which we may achieve by rescaling unless $P(0) = 0$, in which case there is nothing to prove) and to prove that there exists $D > 0$ such that $\int_{-1}^{1} |P(x)| \, dx \geq D$, or even such that $\int_{0}^{1} |P(x)| \, dx \geq D$.

We first reduce to the case where $P$ has all of its roots in $[0, 1]$. If this is not the case, we can factor $P(x)$ as $Q(x)R(x)$, where $Q$ has all roots in the interval and $R$ has none. Then $R$ is either always positive or always negative on $[0, 1]$; assume the former. Let $k$ be the largest positive real number such that $R(x) - kx \geq 0$ on $[0, 1]$; then

$$\int_{-1}^{1} |P(x)| \, dx = \int_{-1}^{1} |Q(x)R(x)| \, dx$$

$$> \int_{-1}^{1} |Q(x)(R(x) - kx)| \, dx,$$

and $Q(x)(R(x) - kx)$ has more roots in $[0, 1]$ than does $P$ (and has the same value at 0). Repeating this argument shows that $\int_{0}^{1} |P(x)| \, dx$ is greater than the corresponding integral for some polynomial with all of its roots in $[0, 1]$.

Under this assumption, we have

$$P(x) = c \prod_{i=1}^{1999} (x - r_{i})$$

for some $r_{i} \in (0, 1]$. Since

$$P(0) = -c \prod_{i=1}^{1999} r_{i} = 1,$$

we have

$$|c| \geq \prod_{i=1}^{1999} |r_{i}^{-1}| \geq 1.$$  

Thus it suffices to prove that if $Q(x)$ is a monic polynomial of degree 1999 with all of its roots in $[0, 1]$, then $\int_{0}^{1} |Q(x)| \, dx \geq D$ for some constant $D > 0$. But the integral of $\int_{0}^{1} \prod_{i=1}^{1999} |x - r_{i}| \, dx$ is a continuous function for $r_{i} \in (0, 1]$. The product of all of these intervals is compact, so the integral achieves a minimum value for some $r_{i}$. This minimum is the desired $D$.

Third solution (by Abe Kunin): It suffices to prove the stronger inequality

$$\sup_{x \in [-1, 1]} |P(x)| \leq C \int_{-1}^{1} |P(x)| \, dx$$

holds for some $C$. But this follows immediately from the following standard fact: any two norms on a finite-dimensional vector space (here the polynomials of degree at most 1999) are equivalent. (The proof of this statement is also a compactness argument: $C$ can be taken to be the maximum of the L1-norm divided by the sup norm over the set of polynomials with L1-norm 1.)

Note: combining the first two approaches gives a constructive solution with a constant that is better than that given by the first solution, but is still far from optimal. I don’t know offhand whether it is even known what the optimal constant and/or the polynomials achieving that constant are.
A–6 Rearranging the given equation yields the much more tractable equation
\[ \frac{a_n}{a_{n-1}} = 6 \frac{a_{n-1}}{a_{n-2}} - 8 \frac{a_{n-2}}{a_{n-3}}. \]
Let \( b_n = a_n/a_{n-1} \); with the initial conditions \( b_2 = 2, b_3 = 12 \), one easily obtains \( b_n = 2n^{-1}(2^n - 2) \), and so
\[ a_n = 2^{n(n-1)/2} \prod_{i=1}^{n-1} (2^i - 1). \]
To see that \( n \) divides \( a_n \), factor \( n \) as \( 2^k m \), with \( m \) odd. Then note that \( k \leq n \leq n(n-1)/2 \), and that there exists \( i \leq m - 1 \) such that \( m \) divides \( 2^i - 1 \), namely \( i = \phi(m) \) (Euler’s totient function; the number of integers in \( \{1, \ldots, m\} \) relatively prime to \( m \)).

B–1 The answer is 1/3. Let \( G \) be the point obtained by reflecting \( C \) about the line \( AB \). Since \( \angle ADC = \frac{\pi}{2} \), we find that \( \angle BDE = \pi - \theta \) and \( \angle ADC = \frac{\pi}{2} - \theta \) is a multiple root, so that \( E, D, G \) are collinear. Hence
\[ \frac{|EF|}{|BC|} = \frac{|BE|}{|BG|} = \frac{\sin(\theta/2)}{\sin(3\theta/2)}, \]
where we have used the law of sines in \( \triangle BDG \). By l’Hôpital’s Rule,
\[ \lim_{\theta \to 0} \frac{\sin(\theta/2)}{\sin(3\theta/2)} = \lim_{\theta \to 0} \frac{\cos(\theta/2)}{3\cos(3\theta/2)} = 1/3. \]

B–2 First solution: Suppose that \( P \) does not have \( n \) distinct roots; then it has a root of multiplicity at least 2, which we may assume is \( x = 0 \) without loss of generality. Let \( x^k \) be the greatest power of \( x \) dividing \( P(x) \), so that \( P(x) = x^k R(x) \) with \( R(0) \neq 0 \); a simple computation yields
\[ P''(x) = (k^2 - k)x^{k-2} R(x) + 2kx^{k-1} R'(x) + x^k R''(x). \]
Since \( R(0) \neq 0 \) and \( k \geq 2 \), we conclude that the greatest power of \( x \) dividing \( P''(x) \) is \( x^{k-2} \). But \( P(x) = Q(x)P''(x) \), and so \( x^2 \) divides \( Q(x) \). We deduce (since \( Q \) is quadratic) that \( Q(x) \) is a constant \( C \) times \( x^2 \); in fact, \( C = 1/(n(n-1)) \) by inspection of the leading-degree terms of \( P(x) \) and \( P''(x) \).

Now if \( P(x) = \sum_{j=0}^n a_j x^j \), then the relation \( P(x) = Cx^2 P''(x) \) implies that \( a_j = Cj(j-1)a_j \) for all \( j \); hence \( a_j = 0 \) for \( j \leq n-1 \), and we conclude that \( P(x) = a_n x^n \), which has all identical roots.

Second solution (by Greg Kuperberg): Let \( f(x) = P''(x)/P(x) = 1/Q(x) \). By hypothesis, \( f \) has at most two poles (counting multiplicity).
Recall that for any complex polynomial \( P \), the roots of \( P' \) lie within the convex hull of \( P \). To show this, it suffices to show that if the roots of \( P \) lie on one side of a line, say on the positive side of the imaginary axis, then \( P' \) has no roots on the other side. That follows because if \( r_1, \ldots, r_n \) are the roots of \( P \),
\[ \frac{P'(z)}{P(z)} = \sum_{i=1}^n \frac{1}{z - r_i} \]
and if \( z \) has negative real part, so does \( 1/(z - r_i) \) for \( i = 1, \ldots, n \), so the sum is nonzero.

The above argument also carries through if \( z \) lies on the imaginary axis, provided that \( z \) is not equal to a root of \( P \). Thus we also have that no roots of \( P' \) lie on the sides of the convex hull of \( P \), unless they are also roots of \( P \).

From this we conclude that if \( r \) is a root of \( P \) which is a vertex of the convex hull of the roots, and which is not also a root of \( P' \), then \( f \) has a single pole at \( r \) (as \( r \) cannot be a root of \( P'' \)). On the other hand, if \( r \) is a root of \( P \) which is also a root of \( P' \), it is a multiple root, and then \( f \) has a double pole at \( r \).

If \( P \) has roots not all equal, the convex hull of its roots has at least two vertices.

B–3 We first note that
\[ \sum_{m,n>0} x^m y^n = \frac{xy}{(1-x)(1-y)}. \]
Subtracting \( S \) from this gives two sums, one of which is
\[ \sum_{m \geq 2n+1} x^m y^n = \sum_n y^n \frac{x^{2n+1}}{1-x} = \frac{x^3 y}{(1-x)(1-x^2 y)} \]
and the other of which sums to \( x y^3/[(1-y)(1-x y^2)] \).

Therefore
\[ S(x, y) = \frac{xy}{(1-x)(1-y)} - \frac{x^3 y}{(1-x)(1-x^2 y)} - \frac{x y^3}{(1-y)(1-x y^2)} \]
and the desired limit is \( \lim_{x \to 1, y \to 1} x y(1 + x + y + x y - x^2 y^2) = 3 \).

B–4 (based on work by Daniel Stronger) We make repeated use of the following fact: if \( f \) is a differentiable function on all of \( \mathbb{R} \), \( \lim_{x \to -\infty} f(x) \geq 0 \), and \( f'(x) > 0 \) for all \( x \in \mathbb{R} \), then \( f(x) > 0 \) for all \( x \in \mathbb{R} \). (Proof: if \( f(y) < 0 \) for some \( x \), then \( f(x) < f(y) \) for all \( x < y \) since \( f' > 0 \), but then \( \lim_{x \to -\infty} f(x) \leq f(y) < 0 \).)

From the inequality \( f'''(x) \leq f(x) \) we obtain
\[ f'''(x) \leq f''(x)f(x) < f''(x)f(x) + f'(x)^2 \]
since \( f'(x) \) is positive. Applying the fact to the difference between the right and left sides, we get
\[ \frac{1}{2}(f''(x))^2 < f(x)f'(x). \]
On the other hand, since \( f(x) \) and \( f'''(x) \) are both positive for all \( x \), we have
\[
2f'(x)f'''(x) < 2f'(x)f''(x) + 2f(x)f'''(x).
\]
Applying the fact to the difference between the sides yields
\[
f'(x)^2 \leq 2f(x)f''(x).
\]
Combining (1) and (2), we obtain
\[
\frac{1}{2} \left( \frac{f'(x)^2}{2f(x)} \right)^2 < \frac{1}{2} \left( \frac{f''(x)}{2f'(x)} \right)^2 \leq f(x)f'(x),
\]
or \((f'(x))^3 < f(x)^3\). We conclude \(f'(x) < 2f(x)\), as desired.

Note: one can actually prove the result with a smaller constant in place of 2, as follows. Adding \( \frac{1}{2} f'(x)f'''(x) \) to both sides of (1) and again invoking the original bound \( f'''(x) \leq f(x) \), we get
\[
\frac{1}{2} f'(x)f'''(x) + (f''(x))^2 < f(x)f'(x) + \frac{1}{2} f'(x)f'''(x) \leq \frac{3}{2} f(x)f'(x).
\]
Applying the fact again, we get
\[
\frac{1}{2} f'(x)f''(x) < \frac{3}{4} f(x)^2.
\]
Multiplying both sides by \( f'(x) \) and applying the fact once more, we get
\[
\frac{1}{6} (f'(x))^3 < \frac{1}{4} f(x)^3.
\]
From this we deduce \( f'(x) < (3/2)^{1/3} f(x) < 2f(x) \), as desired.

I don’t know what the best constant is, except that it is not less than 1 (because \( f(x) = e^x \) satisfies the given conditions).

B–5 We claim that the eigenvalues of \( A \) are 0 with multiplicity \( n - 2 \), and \( n/2 \) and \( -n/2 \), each with multiplicity 1. To prove this claim, define vectors \( v^{(m)} \), \( 0 \leq m \leq n - 1 \), componentwise by \((v^{(m)})_k = e^{ikm}\), and note that the \( v^{(m)} \) form a basis for \( \mathbb{C}^n \). (If we arrange the \( v^{(m)} \) into an \( n \times n \) matrix, then the determinant of this matrix is a Vandermonde product which is nonzero.) Now note that
\[
(Av^{(m)})_j = \sum_{k=1}^{n} \cos(j\theta + k\theta)e^{ikm\theta} = \frac{1}{2} \sum_{k=1}^{n} e^{ik(m+1)\theta} + \frac{1}{2} \sum_{k=1}^{n} e^{ik(m-1)\theta}.
\]
Since \( \sum_{k=1}^{n} e^{ik\theta} = 0 \) for integer \( \ell \) unless \( n \mid \ell \), we conclude that \( Av^{(m)} \neq 0 \) for \( m = 0 \) or for \( 2 \leq m \leq n - 1 \). In addition, we find that \( (Av^{(1)})_j = \frac{1}{2} e^{ij\theta} = \frac{1}{2} (v^{(n-1)})_j \) and \( (Av^{(n-1)})_j = \frac{1}{2} e^{ij\theta} = \frac{1}{2} (v^{(1)})_j \), so that \( A(v^{(1)} \pm v^{(n-1)}) = \pm \frac{n}{2} (v^{(1)} \pm v^{(n-1)}) \). Thus \{\( v^{(0)}, v^{(2)}, \ldots, v^{(n-2)}, v^{(1)} \pm v^{(n-1)} \)\} is a basis for \( \mathbb{C}^n \) of eigenvectors of \( A \) with the claimed eigenvalues.

Finally, the determinant of \( I + A \) is the product of \((1 + \lambda)\) over all eigenvalues \( \lambda \) of \( A \); in this case, \( \det(I + A) = (1 + n/2)(1 - n/2) = 1 - n^2/4 \).

B–6 First solution: Choose a sequence \( p_1, p_2, \ldots \) of primes as follows. Let \( p_1 \) be any prime dividing an element of \( S \). To define \( p_{j+1} \) given \( p_1, \ldots, p_j \), choose an integer \( N_j \in S \) relatively prime to \( p_1 \cdots p_j \) and let \( p_{j+1} \) be a prime divisor of \( N_j \), or stop if no such \( N_j \) exists.

Since \( S \) is finite, the above algorithm eventually terminates in a finite sequence \( p_1, \ldots, p_k \). Let \( m \) be the smallest integer such that \( p_1 \cdots p_m \) has a divisor in \( S \). (By the assumption on \( S \) with \( n = p_1 \cdots p_k, m = k \) has this property, so \( m \) is well-defined.) If \( m = 1 \), then \( p_1 \in S \), and we are done, so assume \( m \geq 2 \). Any divisor \( d \) of \( p_1 \cdots p_m \) in \( S \) must be a multiple of \( p_m \), or else it would also be a divisor of \( p_1 \cdots p_{m-1} \), contradicting the choice of \( m \). But now \( \gcd(d, N_{m-1}) = p_m \), as desired.

Second solution (from sci.math): Let \( n \) be the smallest integer such that \( \gcd(s, n) > 1 \) for all \( s \) in \( n \); note that \( n \) obviously has no repeated prime factors. By the condition on \( S \), there exists \( s \in S \) which divides \( n \).

On the other hand, if \( p \) is a prime divisor of \( s \), then by the choice of \( n, n/p \) is relatively prime to some element \( t \) of \( S \). Since \( n \) cannot be relatively prime to \( t, t \) is divisible by \( p \), but not by any other prime divisor of \( n \) (as those primes divide \( n/p \). Thus \( \gcd(s, t) = p \), as desired.